A CLASS OF 3 – STEP BLOCK METHOD FOR SOLVING ORDINARY DIFFERENTIAL EQUATIONS

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Abstract : In this paper, linear multistep technique using power series as the basis function is used to develop a block method which is suitable for generating direct solution of the special second order ordinary differential equations of the form $y'' = f(x, y),\ a \le x \le b$ with associated initial or boundary conditions. The continuous hybrid formulations enable us to differentiate and evaluate at some grids and off – grid points to obtain four discrete schemes of order $(5,5,5,5)^T$, which were used in block form for parallel or sequential solutions of the problems. The computational burden and computer time wastage involved in the usual reduction of second order problem into system of first order equations are avoided by this approach. Further more, a stability analysis and efficiency of the block method are tested on linear and non-linear ordinary differential equations whose solutions are oscillatory or nearly periodic in nature, and the results obtained compared favourably with the exact solution.

Keywords: Block Method, Hybrid, Linear Multistep Method, Self – starting, Special Second Order

Introduction

Let us consider the numerical solution of the special second order ordinary differential equation of the form $y'' = f(x, y),\ \ a \le x \le b$ (1)

with associated initial or boundary conditions. The mathematical models of most physical phenomena especially in mechanical systems without dissipation leads to special second order initial value problem of type (1). Solutions to initial value problem of type (1) "according to Fatunla (1991:1994)" are often highly oscillatory in nature and thus, severely restrict the mesh size of the conventional linear multistep method. Such system often occurs in mechanical systems without dissipation, satellite tracking and celestial mechanics.

Lambert (1973) and several authors such as Aladeselu (2007), Awoyemi (1998), Fatunla et al. (1999), Fudziah et al. (2009), Onumanyi et al (1994), Yahaya and Adegboye (2008), and Yahaya and Mohammed (2010), have written on conventional linear multistep method:

$$\sum_{j=0}^{k} \alpha_j\, y_{n+j} = h^2 \sum_{j=0}^{k} \beta_j f_{n+j}\,, k \ge 2 \tag{2}$$

or compactly in the form

$$\rho(E) y_n = h^2 \delta(E) f_n \tag{3}$$

where E is the shift operator specified by $E^j y_n = y_{n+j}$ while $\rho\ and\ \delta$ are characteristics polynomials and are given as $\rho(\xi) = \sum_{j=0}^{k} \alpha_j\, \xi^j,\ \delta(\xi) = \sum_{j=0}^{k} \beta_j\, \xi^j$ (4)

y_n is the numerical approximation to the theoretical solution y(x) and $f_n = f(x_n, y_n)$.

In the present consideration, our motivations for the study of this approach is a further advancement in efficiency, i.e. obtaining the most accuracy per unit of computational effort, that can be secured with the group of methods proposed in this paper over Awoyemi (1998), and Yahaya and Mohammed (2010).

Definition 1.1: Consistent Lambert (1973)

The linear multistep method (2) is said to be consistent if it has order p ≥ 1, that is, if

$$\sum_{j=0}^{k} \alpha_j = 0 \text{ and } \sum_{j=0}^{k} j\, \alpha_j - \sum_{j=0}^{k} \beta_j = 0 \tag{5}$$

Introducing the first and second characteristics polynomials (4), we have from (5) LMM type (2) is consistent if $\rho(1) = 0,\ \rho^1(1) = \delta(1)$

Definition 1.2: Zero stability Lambert (1973)

A linear multistep method type (2) is zero stable provided the roots $\xi_j, j = 0(1)k$ of first characteristics polynomial $\rho(\xi)$ specified as $\rho(\xi) = det\left|\sum_{j=0}^{k} A(i) \xi^{(k-i)}\right| = 0$ satisfies $|\xi_j| \le 1$ and for those roots with $|\xi_j| = 1$ the multiplicity must not exceed two. The principal root of $\rho(\xi)$ is denoted by $\xi_1 = \xi_2 = 1$.

Definition 1.3: Convergence Lambert (1973)

The necessary and sufficient conditions for the linear multistep method type (2) is said to be convergent if it is consistent and zero stable.

Definition 1.4: Order and Error Constant Lambert (1973)

The linear multistep method type (2) is said to be of order p if $c_0 = c_1 = \cdots c_{P+1} = 0$ $but\ c_{p+2} \ne 0$ and c_{p+2} is called the error constant,

where $c_0 = \sum_{j=0}^{k} \alpha_j = \alpha_0 + \alpha_1 + \alpha_2 + \ \dots\ + \alpha_k$

$$c_1 = \sum_{j=0}^{k} j\, \alpha_j = (\alpha_1 + 2\,\alpha_2 + 3\,\alpha_3 + \ \dots\ + k\,\alpha_k) - (\beta_0 + \beta_1 + \beta_2 + \cdots + \beta_k)$$

$$c_2 = \sum_{j=0}^{k} \frac{1}{2!} j^2\, \alpha_j - \sum_{j=0}^{k} \beta_j = \begin{cases} \frac{1}{2!}(\alpha_1 + 2^2\,\alpha_2 + 3^3\,\alpha_3 + \ \dots\ + k^2\,\alpha_k) \\ -(\beta_1 + 2\beta_2 + 3\beta_3 + \ \dots + k\beta_k) \end{cases}$$

$$c_q = \sum_{j=1}^{k} \{\frac{1}{q!} j^q\, \alpha_j - \frac{1}{(q-2)!} j^{q-2} \beta_j\}$$

$$=\left\{\begin{array}{c}\frac{1}{q!}(\alpha_1+2^q\alpha_2+3^q\alpha_3+\ldots+k^q\alpha_k)\\-\frac{1}{(q-1)!}\left(\beta_1+2^{(q-1)}\beta_2+3^{(q-1)}\beta_3+\cdots+k^{(q-1)}\beta_k\right)\end{array}\right\} \quad (6)$$

Theorem 1.1: Lambert (1973)

Let f (x, y) be defined and continuous for all points (x, y) in the region D defined by{(x, y) : a ≤ x ≤ b, −∞ < y < ∞} where a and b finite, and let there exist a constant L such that for every x, y, y* such that (x, y) and (x, y*) are both inD:

$$|f(x, y) - f(x, y^*)| \le L|y - y^*| \quad (7)$$

Then if η is any given number, there exist a unique solution y(x) of the initial value problem (1), where y(x) is continuous and differentiable for all (x, y) in D. The inequality (7) is known as a Lipschitz condition and the constant L as a Lipschitz constant.

Consequently, this paper is organized as follows: in first section shows the introduction, this lead to second section which shows how the method was derived, third section presents stability analysis of the method, fourth section show some numerical experiments, while the fifth and last section of this paper concludes the work and references respectively.

Derivation Of The Proposed Method

We proposed an approximate solution to (1) in the form

$$y(x)=\sum_{j=0}^{t+m-1}a_jx^i=y_{n+j},i=0(1)m+t-1$$

$$y''(x)=\sum_{j=0}^{t+m-1}i(i-1)a_jx^{i-2}=f_{n+j},\ i=2(3)m+t-1$$

with m = 4, t = 3 and p = m+t-1

where the $a_j, j=0,1,(m+t-1)$ are the parameters to be determined, t and m are points of interpolation and collocation respectively. Where P, is the degree of the polynomial interpolant of our choice.

Specifically, we collocate equation (9) at $x=x_{n+j}, j=0(1)k$ and interpolate equation (8) at $x=x_{n+j},\ j=0(1)k-\frac{5}{3}$ using the method described above. Putting in the matrix equation form and then solved to obtain the values of parameters α_j's, j = 0, 1, … which is substituted in (8) yields, after some algebraic manipulation, the new continuous form for the solution

$$y(x)=\sum_{j=0}^{k-\frac{5}{3}}\alpha_j(x)y_{n+j}+\sum_{j=0}^{k}\beta_j(x)f_{n+j}$$

We set $\mu=(x-x_{n+1})$

If we let k = 3, after some algebraic manipulations we obtain a continuous form of solution

$$\begin{aligned}y(x)=&\left\{\frac{-486(\mu)^6+1458h(\mu)^5+1215h^2(\mu)^4-4860h^3(\mu)^3+479h^5\mu}{3652h^6}\right\}y_n\\&+\left\{\frac{486(\mu)^6-1458h(\mu)^5-1215h^2(\mu)^4+4860h^3(\mu)^3-3218h^5(\mu)+913h^6}{913h^6}\right\}y_{n+1}\\&+\left\{\frac{-1458(\mu)^6+4374h(\mu)^5+3645h^2(\mu)^4-14580h^3(\mu)^3+12393h^5\mu}{3652h^6}\right\}y_{n+\frac{4}{3}}\\&+\left\{\frac{4410(\mu)^6-15969h(\mu)^5+2670h^2(\mu)^4+25840h^3(\mu)^3-2791h^5\mu}{328680h^4}\right\}f_n\\&+\left\{\frac{9180(\mu)^6-2480h(\mu)^5-32080h^2(\mu)^4+82670h^3(\mu)^3+54780h^4(\mu)^2-25989h^5\mu}{109560h^4}\right\}f_{n+1}\\&+\left\{\frac{-1170(\mu)^6+771h(\mu)^5+7490h^2(\mu)^4+6560h^3(\mu)^3-1011h^5\mu}{109560h^4}\right\}f_{n+2}\\&+\left\{\frac{720(\mu)^6+579h(\mu)^5-1800h^2(\mu)^4-1930h^3(\mu)^3+271h^5\mu}{328680h^4}\right\}f_{n+3}\end{aligned}\quad (11)$$

Evaluating equation (11) at some selected points and its second derivative at an off - grid point respectively yield the following schemes:

$$\begin{aligned}&(a)\ y_{n+2}-\frac{2187}{1826}y_{n+\frac{4}{3}}-\frac{368}{913}y_{n+1}+\frac{1097}{1826}y_n=\frac{h^2}{2739}\left\{\begin{array}{c}118f_n+1594f_{n+1}\\+316f_{n+2}-18f_{n+3}\end{array}\right\}\\&(b)\ y_{n+3}-\frac{6561}{1826}y_{n+\frac{4}{3}}+\frac{1635}{913}y_{n+1}+\frac{1465}{1826}y_n=\frac{h^2}{10956}\left\{\begin{array}{c}503f_n+10911f_{n+1}\\+12009f_{n+2}+697f_{n+3}\end{array}\right\}\\&(c)\ \frac{5400}{913}y_{n+\frac{4}{3}}-\frac{7200}{913}y_{n+1}+\frac{1800}{913}y_n=\frac{h^2}{73953}\left\{\begin{array}{c}10135f_n+146580f_{n+1}\\-73953f_{n+\frac{4}{3}}+15690f_{n+2}\\-1252f_{n+3}\end{array}\right\}\end{aligned}\quad (12)$$

Taking the first derivative of equation (11), thereafter, evaluate the resulting continuous polynomial solution at $x = x_0$ yields

$$(d)\ hz_0 + \frac{15309}{3652} y_{n+\frac{4}{3}} - \frac{6016}{913} y_{n+1} + \frac{8755}{3652} y_n = \frac{h^2}{41085} \begin{Bmatrix} -5282f_n + 23136f_{n+1} \\ -156f_{n+2} \\ +32f_{n+3} \end{Bmatrix} \qquad (13)$$

Equations (12) and (13) constitute the member of a zero stable block integrators of order $(5,5,5,5)^T$ with $c_7 = \left(\frac{101}{82170}, -\frac{7}{14608}, \frac{2351}{665577}, -\frac{32}{95865}\right)$. The application of the block integrators with n = 0 gives the accurate values as shown in tables 1 and 2 of fourth section of this paper.

To start the IVP integration on the sub interval $[X_0, X_3]$, we combine equations (12) and (13), when n = 0 i.e. the 1-block 4-point method are given in equation (14). Thus produces simultaneously values for y_1, y_2, y_3 *and* $y_{\frac{4}{3}}$ without recourse to any predictor like Awoyemi (1998) to provide y_1 *and* y_2 in the main method. Hence this is an improvement over these reported works. Though, this does not becloud the contribution of these authors.

Stability Analysis

Recall, that, it is a desirable property for a numerical integrator to produce solution that behave similar to the theoretical solution to a problem at all times. Thus, several definitions, which call for the method to posses some "adequate" region of absolute stability can be found in several literatures. See Lambert (1973), Fatunla (1991:1994) e.t.c

Following Fatunla (1991:1994), the four integrator proposed in this report in equation (12) and (13) are put in the matrix equation form and for easy analysis the result was normalized to obtain;

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_{n+1} \\ y_{n+\frac{4}{3}} \\ y_{n+2} \\ y_{n+3} \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1097}{1826} & -\frac{2399139}{1460800} & \frac{3291}{1826} \\ 0 & -\frac{1465}{1826} & -\frac{640791}{292160} & \frac{4395}{1826} \\ 0 & -\frac{1800}{913} & -\frac{19683}{3652} & \frac{5400}{913} \\ 0 & -\frac{8755}{3652} & -\frac{3829437}{584320} & \frac{26265}{3652} \end{bmatrix} \begin{bmatrix} y_{n-2} \\ y_{n-\frac{5}{3}} \\ y_{n-1} \\ y_n \end{bmatrix}$$

$$+ h^2 \left\{ \begin{bmatrix} \frac{316}{2739} & -\frac{6}{913} & \frac{26541}{45650} & -\frac{724}{913} \\ \frac{4003}{3652} & \frac{697}{10956} & \frac{387909}{146080} & -\frac{3085}{913} \\ \frac{5230}{24651} & -\frac{1252}{73953} & \frac{72491}{123255} & -\frac{2800}{2739} \\ -\frac{52}{13695} & \frac{32}{41085} & \frac{9018}{22825} & -\frac{1528}{2739} \end{bmatrix} \begin{bmatrix} f_{n+1} \\ f_{n+\frac{4}{3}} \\ f_{n+2} \\ f_{n+3} \end{bmatrix} + \begin{bmatrix} 0 & \frac{118}{2739} & \frac{43011}{365200} & -\frac{118}{913} \\ 0 & \frac{503}{10956} & \frac{366687}{2921600} & -\frac{503}{3652} \\ 0 & \frac{10135}{73953} & \frac{54729}{146080} & -\frac{10135}{24651} \\ 0 & -\frac{5282}{41085} & -\frac{641763}{1826000} & \frac{5282}{13695} \end{bmatrix} \begin{bmatrix} f_{n-2} \\ f_{n-\frac{5}{3}} \\ f_{n-1} \\ f_n \end{bmatrix} \right\} \qquad (14)$$

with $y_0 = \begin{pmatrix} y_0 \\ hz_0 \end{pmatrix}$ usually giving along the initial value problem. Equation (14) is the 1- block 4 – point method. The first characteristics polynomial of the proposed 1- block 4 – point method is given by $\rho(\lambda) = det[\lambda I - A_1^{(1)}]$ (15)

$$= det \begin{bmatrix} \lambda & \frac{1097}{1826} & \frac{2399139}{14608} & -\frac{3291}{1826} \\ 0 & \lambda + \frac{1465}{1826} & \frac{640791}{292160} & -\frac{4395}{1826} \\ 0 & \frac{1800}{913} & \lambda + \frac{19683}{3652} & -\frac{5400}{913} \\ 0 & \frac{8755}{3652} & \frac{3829437}{584320} & \lambda - \frac{26265}{3652} \end{bmatrix} \qquad (16)$$

Solving the determinant of equation (16), yields $\rho(\lambda) = \lambda^3(\lambda - 1)$, which implies, $\lambda_1 = \lambda_2 = \lambda_3 = 0 \; or \; \lambda_4 = 1$

By definition of zero stable and equation (16), the 1 - block 4 - point method is zero stable and is also consistent as its order $(5,5,5,5)^T > 1$, thus, it is convergent following Henrici (1962) and Fatunla (1994).

Numerical Experiments : In what follows, we present some numerical results on some problems.

Problem 1: Consider a Non-Linear IVP; $y'' = 2y^3; y(1) = 1, y'(1) = -1$, whose exact solution is $y(x) = 1/x$

Table 1: Results for the Proposed Method

N	x	Exact Value	Approximate Value	Awoyemi (1998)	Error of Proposed Method
0	1	1	1	0	0
1	1.1	0.909090109	0.9090914832	2.8483722E-03	1.37420E-06
2	1.2	0.833333333	0.8333348886	2.26883436E-01	1.55560E-06
3	1.3	0.769230769	0.7692330281	7.3968630E+00	2.25910E-06
4	1.4	0.714285714	0.7142880973	2.1168783E-01	2.38330E-06
5	1.5	0.666666667	0.6666693038	3.3156524E-01	2.63680E-06
6	1.6	0.625	0.6250029082	4.3968593E-01	2.90820E-06
7	1.7	0.588235294	0.5882382539	5.3903097E-01	2.95990E-06
8	1.8	0.555555556	0.5555586407	6.3121827E-01	3.08470E-06
9	1.9	0.526315789	0.5263190456	7.1723621E-01	3.25660E-06
10	2.0	0.5	0.5000032878	7.9776590E-01	3.28780E-06

Problem 2: Consider the BVP $y'' = 3x + 4y; \; y(0) = 0, \; y(1) = 1$, whose exact solution is

$$y(x) = \frac{7}{4(e^2 - e^{-2})}[e^{2x} - e^{-2x}] - \frac{3}{4}x$$

Table 2: Results for the Proposed Method

x	Exact Value	Approximate Value	Yahaya and Mohammed (2010)	Error of Proposed Method
0	0.00000000000	0.00000000000	0.000000000E+00	0.000000000E+00
0.2	0.04819251100	0.04819008513	1.828560000E-04	2.425870000E-06
0.4	0.12852089500	0.1285138496	3.931120000E-04	7.045400000E-06
0.6	0.27833169000	0.2783207232	6.672570000E-04	1.096680000E-05
0.8	0.54623764000	0.5463115949	1.050930000E-03	7.395490000E-05
1.0	1.00000000000	1.000042276	0.000000000E+00	4.227600000E-05

Conclusion : Onumanyi etal (1994), and Awoyemi (1998) discussed in some detail theoretical and practical aspects of collocation with piecewise polynomial function. Roughly, their results particularly Awoyemi (1998) indicate that the solution of a second order non linear problem can be approximated with linear multistep methods. In this paper we developed a uniform order 1 – block 4 – point integrators of order $(5,5,5,5)^T$. The resultant numerical integrators posses the following desirable properties:

i. Zero stability i.e. stability at the origin
ii. Convergent schemes
iii. An addition of equation from the use of first derivative
iv. Being self – starting as such it eliminate the use of predictor – corrector method
v. Facility to generate solutions at 4 points simultaneously
vi. Produce solution over sub intervals that do not overlaps
vii. Apply uniformly to both IVP_s and BVP_s with adjustment to the boundary conditions

In addition, the new schemes compares favourably with the theoretical solution and the results are more accurate than Awoyemi (1998), and Yahaya and Mohammed (2010), see table 1 and 2. Hence, this work is an improvement over other cited works.

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