COMMON FIXED POINT THEOREM FOR SIX MAPPINGS IN PARTIAL ORDERED CONE METRIC SPACE

ANUSHRI A.ASERKAR, MANJUSHA P.GANDHI, KAVITA B. BAJPAI

Abstract: In the present paper we have proved a unique common fixed point theorem for six mappings in cone metric space satisfying a contractive condition with increasing function. The mappings in pairs satisfy the weakly compatibility condition and only one mapping satisfy the condition of continuity. It is more generalized form of the result of S.V.R.Naidu and J.Rajendra Prasad [16].

Key words: Cone metric space,Partial order, Increasing function,weakly compatibility condition.

Introduction: Fixed point theory occupies a prominent place in the study of metric spaces. Many eminent Mathematicians T. G. Rogers and G. E. Hardy [17],R. Kannan [13], G.Jungck, B.E.Rhoades [5] ,M. A. Khamsi and W. A. Kirk [10] etc. discussed and proved fixed point theorems in metric spaces.In 2007, Huang and Zhang [9] introduced the concept of cone metric spaces, which is a generalization of metric spaces, byreplacing the real numbers with ordered Banach spaces.Huang and Zhang [9] proved some fixed point theorems with some contractive conditions in normal cone metric spaces.

In other recently published works [6, 7, 12], fixed point theorems with different contractive conditions in cone metric spaces were deduced. Alber et al.[18] generalized Banach's contraction principle and defined weak contraction principle in Hilbert spaces and which was subsequently extended to metric space by Rhoades [4].Binayak S. Choudhury, N. Metiya[3]extendedthe weak contraction principle in conemetric spaces.

The use of control function in fixed point theory was initiated by Khan et.al. [11] which they called Altering distance function. This function has beenused in obtaining fixed point results in metric spaces [8, 14]

In this paper we have proved some fixed point results in partial order cone metric spaceby using a increasing function with six mapping It is improved and extension of S.V.R.Naidu and J.Rajendra Prasad results[16] to partial ordered cone metric space .

2. **Preliminary**:Weneed to use the following fundamental concepts throughout this paper.

Definition

2.1: Cone: Let E be a real Banach space and $P \subset E$ .Then the set

P is called a cone if and only if

(i) P is closed, non empty and $P \neq \phi$;

(ii) $a, b \in R, a, b \geq 0, x, y \in P$ imply that $ax + by \in P$

(iii) $P \cap (-P) = \phi$;

2.2: Partial Ordered Cone: For given cone $P \subset E$, we define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. We shall write x y for y - $x \in P_0$, where P_0 stands for interior of P. Also we will use $x < y$ to indicate that $x \leq y$ and $x \neq y$

2.3: Cone Metric Space: Let X be a non empty set. Suppose that the mappings $d: X \times X \to E$ satisfies:

(i) $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(iii) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a cone metric on X and (X ,d) is called a cone metric space.

Example 1:

Let $E = R^2$, $P = \{(x, y) \in E : x, y \geq 0\} \subset R^2$, X = R and $d : X \times X \to E$ such that $d(x, y) = (|x-y|, \beta|x-y|)$, here$\beta \geq 0$ is a constant. Then (X, d) is a cone metric space.

2.4: Convergent Sequence: Let (X, d) be a cone metric space. The sequence $\{x_n\}$ in X is said to be a convergent sequence if for every c ϵ E with 0 < c, there is n_0 ϵ N such that for all $n \geq n_0$, $d(x_n, x) < c$ for some x ϵ X. We denote this by $\lim_{n\to\infty} x_n = x$

2.5: Cauchy Sequence: Let (X, d) be a cone metric space. The sequence $\{x_n\}$ in X is said to be a Cauchy sequence if for all c ϵ E with $0 \ll c$, there is n_oϵ N such that $d(x_m, x_n) \ll c$, for all $m, n \geq n_0$.

2.6: Complete cone metric space: A cone metric space (X, d) is said to be complete if every Cauchy sequence in X is convergent in X.

2.7: Weakly Compatible: Let f and g be two self-maps defined on a set X. Then f and g are said to be weakly compatible if they commute at coincidence points. That is, if fu = gu for some $u \in X$, then fgu = gfu.

2.8: Coincidence Point: Let f and g be self-maps on a set X. If w = fx = gx, for some x in X , then w is called coincidence point of f and g.

2.9 :Let (X,d) be a cone metric space over a solid cone *P*. Denote by Φ the class of functions ϕ : int P $\bigcup$ { } →

int P $\bigcup$ { } satisfying the following conditions:

(1) ϕ (t) = t if and only if t = 0

(2) ϕ (t) t for t $\in$ int P

(3) for all t $\in$ int P and x,y $\in$ X, either ϕ (t) $\leq$ d(x,y) or d(x,y) ϕ(t).

Lemma 2.10 [1] Let (X, d) be a cone metric space with regular cone P such that d(x, y) $\in$ intP, for x, y $\in$ X with x $\neq$ y. Let ϕ: intP $\bigcup$ {0} →intP $\bigcup$ {0} be a function with the following properties:

(i) ϕ(t) = 0 if and only if t = 0,

(ii) ϕ(t) $\leq$ t, for t $\in$ intP and

(iii) either ϕ(t) $\leq$ d(x, y) or d(x, y) ϕ(t), for t $\in$ intP $\bigcup$ {} and x, y $\in$ X.

Let $\{x_n\}$ be a sequence in X for which $\{d(x_n, x_{n+1})\}$ is monotonic decreasing. Then $\{d(x_n, x_{n+1})\}$ is convergent to either r = 0 or r $\in$ int P.

Lemma 2.11[7]. Let (X, d) be a cone metric space over a regular cone P such that 2.3(i) holds and suppose that there exists $\phi \in \Phi$ (2.9). Let $\{y_n\}$ be a sequence in X such that $\{d(y_n,y_{n+1})\}$ is decreasing w.r.t. $\leq$ and that $\lim_{n\to\infty} d(y_{2n}, y_{2n+1}) = 0$. If $\{y_n\}$ is not a Cauchy sequence, then there exists c $\in$ int P and two $\{m_i\}$ and $\{n_i\}$ of positive integers such that the following five sequences tend to ϕ(c) as i→∞ for all c 0.

$$d(y_{2m_i}, y_{2n_i}), d(y_{2m_i-1}, y_{2n_i+1}),$$

$$d(y_{2m_i}, y_{2n_i+1}),$$

$$d(y_{2m_i-1}, y_{2n_i}), d(y_{2m_i+1}, y_{2n_i+1})$$

3. **Main Result:** The main theorem of this paper is as follows.

Theorem.3.1: Let (X,d) be a complete cone metric space with respect to a cone in a real Banach space P. Let P, Q, R, S, T, U be self maps in X.

(i) $T(X) \subseteq RS(X)$, $U(X) \subseteq PQ(X)$

(ii) Pair (T, PQ) and (U, RS) is weakly compatible.

(iii) One of T, PQ is continuous.

(iv) Let ξ (t): intP $\bigcup$ {0} → intP $\bigcup$ {0} be a function such that

(a) ξ (t) = 0 if and only if t = 0 and

(b) ξ (t) $\leq$ t, for t $\in$ intP.

(v) For x,y $\in$ X $d(Tx,Uy) \leq \xi(\mu(x,y))$ where

$\mu(x,y)$ = max{d(Tx,PQx), d(Uy,RSy), d(PQx,RSy), d(Tx,RSy)}

Then P, Q, R, S, T, U have a unique common fixed point in X.

Proof: Let $x_0 \in X$ be any point in X. Let $\{x_n\}$, $\{y_n\}$ are in X such that $Tx_{2n}=RSx_{2n+1}=y_{2n}$ and $Ux_{2n+1} = PQx_{2n+2} = y_{2n+1}$ for all n.

Putting $x = x_{2n}$ and $y = x_{2n+1}$ in (v) we get

$d(Tx_{2n}, Ux_{2n+1}) \leq \xi(\mu(x_{2n}, x_{2n+1}))$

where

$$\mu(x_{2n}, x_{2n+1}) = \max(d(Tx_{2n}, PQx_{2n}), d(Ux_{2n+1}, RSx_{2n+1}), d(PQx_{2n}, RSx_{2n+1}), d(Tx_{2n}, RSx_{2n+1}))$$

$$= \max(d(y_{2n}, y_{2n-1}), d(y_{2n+1}, y_{2n}), d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n}))$$

$$= \max(d(y_{2n}, y_{2n-1}), d(y_{2n+1}, y_{2n}))$$

If $\max(d(y_{2n}, y_{2n-1}), d(y_{2n+1}, y_{2n})) = d(y_{2n+1}, y_{2n})$

$$\therefore d(Tx_{2n}, Ux_{2n+1}) \leq \xi(d(y_{2n}, y_{2n+1})) < d(y_{2n}, y_{2n+1}) \quad (\because \xi(t) < t)$$

which is a contradiction.

If $\max(d(y_{2n}, y_{2n-1}), d(y_{2n+1}, y_{2n})) = d(y_{2n}, y_{2n-1})$

$$\therefore d(Tx_{2n}, Ux_{2n+1}) \leq \xi(d(y_{2n}, y_{2n-1}))$$

$$d(y_{2n}, y_{2n+1}) \leq \xi(d(y_{2n}, y_{2n-1})) < d(y_{2n}, y_{2n-1})$$

$\because \xi(t)$ is a non-decreasing sequence.

$\therefore d(y_{2n}, y_{2n+1})$ is a monotone decreasing sequence of positive numbers.

And so by (2.10) it implies that it converges to some r where r=0 or r $\in$ int P.

If r $\in$ int P then $\lim_{n\to\infty} d(y_{2n}, y_{2n+1}) \to r$

$$d(y_{2n}, y_{2n+1}) = d(Tx_{2n}, Ux_{2n+1}) \leq \xi(\mu(x_{2n}, x_{2n+1}))$$

where $\mu(x_{2n}, x_{2n+1}) = \max(d(Tx_{2n}, PQx_{2n}), d(Ux_{2n+1}, RSx_{2n+1}), d(PQx_{2n}, RSx_{2n+1}), d(Tx_{2n}, RSx_{2n+1}))$

$$= \max(d(y_{2n}, y_{2n-1}), d(y_{2n+1}, y_{2n}), d(y_{2n-1}, y_{2n}) d(y_{2n}, y_{2n}))$$

$$= \max\{r, r, r, 0\} \qquad n \to \infty$$

$\therefore r \le \xi(r)$

which is possible if and only if $r = 0$

$\therefore \lim_{n\to\infty} d(y_{2n}, y_{2n+1}) = 0$

Now we show that $\{y_n\}$ is a Cauchy sequence.

Suppose on the contrary $\{y_n\}$ is not a Cauchy sequence. Then Lemma (2.11) implies that there exists sequences $\{m_i\}$ and $\{n_i\}$ of positive integers such that the following five sequences tend to $\xi(c)$ when $i \to\infty$ for c o.

$d(y_{2m_i}, y_{2n_i}), d(y_{2m_i-1}, y_{2n_i+1}), d(y_{2m_i}, y_{2n_i+1}),$
$d(y_{2m_i-1}, y_{2n_i}), d(y_{2m_i+1}, y_{2n_i+1})$

$$\therefore d(y_{2m_i}, y_{2n_i+1}) = d(Tx_{2m_i}, Ux_{2n_i}) \le \xi\ \mu(x_{2m_i}, x_{2n_i+1})$$

where $\mu(x_{2m_i}, x_{2n_i+1}) = \max\{d(Tx_{2m_i}, PQx_{2m_i}),$

$d(Ux_{2n_i+1}, RSx_{2n_i+1}), d(PQx_{2m_i}, RSx_{2n_i+1}),$

$d(Tx_{2m_i}, RSx_{2n_i+1})\}$

$= \max\{d(y_{2m_i}, y_{2m_i-1}), d(y_{2n_i+1}, y_{2n_i}),$

$d(y_{2m_i-1}, y_{2n_i}), d(y_{2m_i}, y_{2n_i})\}$

$= \max\{0, 0, \xi(c), \xi(c)\} = \xi(c) \quad \text{as } i\to\infty$

$\therefore \xi(c) \le \xi(\xi(c))$

This is possible only if $\xi(c) = 0 \Rightarrow c = 0$

which is a contradiction.

Hence $\{y_n)$ is a Cauchy sequence.

As (X,d)is a complete cone metric space

$y_{2n} \to u \in X$

$\therefore Tx_{2n} \to Tu, Ux_{2n+1} \to Uu, RSx_{2n+1} \to RSu,$
$PQx_{2n+2} \to PQu$

Case - I :

As T is continuous

$\therefore T^2x_{2n} \to Tu$

$TPQx_{2n} \to Tu$

As(T, PQ) is compatible $PQTx_{2n} \to Tu$

$d(Tu,u) \le d(Tu, T^2x_{2n}) + d(T^2x_{2n}, Ux_{2n+1}) + d(Ux_{2n+1}, u)$

$\le d(Tu, T^2x_{2n}) + d(y_{2n+1}, u) + d(T^2x_{2n}, Ux_{2n+1})$

$\le d(Tu, T^2x_{2n}) + d(y_{2n+1}, u) + \xi(\mu(Tx_{2n}, x_{2n+1}))$

where $\mu(Tx_{2n}, x_{2n+1}) = \max\{d(T^2x_{2n}, PQTx_{2n}),$

$d(Ux_{2n+1}, RSx_{2n+1}),$

$d(PQTx_{2n}, RSx_{2n+1}), d(T^2x_{2n}, RSx_{2n+1})\}$

as $n \to \infty$

$$\mu(Tx_{2n}, x_{2n+1}) = \max\{d(Tu,Tu), d(u,u), d(Tu,u), d(Tu,u)\}$$

$$= d(Tu,u)$$

$\therefore d(Tu,u) \le \xi d(Tu,u) = d(Tu,u)$

$\therefore d(Tu,u) = 0 \Rightarrow Tu = u.$

As $T(X) \in RS(X)$ there exists $v \in X$

such that $u = Tu = RSv$.

$$d(u,Uv) \le d(Uv,Tx_{2n}) + d(Tx_{2n},u)$$

$$\therefore d(u,Uv) \le d(Uv,Tx_{2n}) + d(y_{2n},u)$$

$$\therefore d(Uv,Tx_{2n}) \le \xi\mu(x_{2n},v)$$

$$\text{where } \mu(x_{2n},v) = \max\{d(Tx_{2n},PQx_{2n}),$$
$$d(Uv,RSv), d(PQx_{2n},RSv), dTx_{2n},RSv)\}$$

as $n \to \infty$

$$\mu(x_{2n},v) = \max\{d(u,u), d(Uv,u), d(u,u), d(Uv,u)\}$$
$$= d(Uv,u)$$

$$d(Uv,u) \le \xi(d(Uv,u)) \le d(Uv,u)$$

$$\therefore Uv = u$$

Thus RSv=Uv=u

As U,RS is weakly compatible URSv=RSUv

$\therefore$ Uu=RSu

$$\therefore d(u,Uu) \le d(Tx_{2n},u) + d(Tx_{2n},Uu)$$
$$\le d(y_{2n},u) + \xi(\mu(x_{2n},u))$$

$$\mu(x_{2n},u) = \max\{d(Tx_{2n},PQx_{2n}), d(Uu,RSu),$$
$$d(PQx_{2n},Uu), d(Tx_{2n},Rsu)\}$$
$$= \max\{d(u,u), d(Uu,Uu), d(u,Uu),$$
$$d(u,Uu)\} = d(u,Uu)$$

$$\therefore d(u,Uu) \le \xi d(u,Uu) \Rightarrow d(u,Uu) = 0$$

$$\therefore Uu = u$$

$$\therefore u = Uu = RSu = Tu.$$

$$\text{Now } U(X) \in PQ(X)$$

$\therefore$ There is $w \in X$ such that $u = Uu = PQw$

$$\therefore d(Tw,u) = d(Tw,Uu) \le \xi(\mu(w,u))$$

$$\text{where } \mu(w,u) = \max\{d(Tw,PQw), d(Uu,RSu),$$
$$d(PQw,Uu), d(Tw,RSu)\}$$
$$= \max\{d(Tw,u), d(u,u), d(u,u),$$
$$d(Tw,u)\} = d(Tw,u)$$

$$\therefore d(Tw,u) \le \xi(\mu(w,u)) = d(Tw,u)$$

$$\therefore d(Tw,u) = 0 \Rightarrow Tw = u$$

Thus Tw=PQw=u

As (T, PQ) are compatible (T, PQ) is weakly compatible.

$\therefore$ Tu=PQu

$\therefore$ u=Tu=Uu=PQu=RSu

Thus u is a fixed point of six self maps in X.

<u>Uniqueness.</u>

Let θ=Tθ=Uθ=PQθ=RSθ be anther fixed point

(Put x=u, y= θ)

$d(Tu, U\theta) \le \xi(\mu(u,\theta))$

where $\mu(u,\theta) = \max\{d(Tu,PQ\theta),$
$d(U\theta,RS\theta), d(PQu,RS\theta), d(Tu,RS\theta)\}$
$= \max\{d(u,\theta), d(\theta,\theta),$
$d(u,\theta), d(u,\theta)\} = d(u,\theta)$

$\therefore d(u,\theta) \le \xi(\mu(u,\theta))$

$d(u,\theta)=0 \Rightarrow u=\theta$ .Thus it has a unique fixed point.

Corrollary: 3.2 Let (X,d) be a complete cone metric space with respect to a cone in a real Banach space P.Let P, R,T,U be self maps in X.

(i)$T(X) \subseteq R(X), U(X) \subseteq P(X)$
(ii)Pair (T,P) is compatible and (U,R) is weakly compatible.
(iii)One of T, P is continuous.
(iv) Let ξ (t): intP $\bigcup$ {0} →intP $\bigcup$ {0} be a function such that

(a) ξ (t) = 0 if and only if t = 0 and

(b) ξ (t) ≤t, for t ∈intP.

(v)For $x,y \in X$ $d(Tx,Uy) \le \xi(\mu(x,y))$ Where $\mu(x,y)$ =max{d(Tx,Px),d(Uy,Ry), d(Px,Ry),d(Tx,Ry)}
Then P, R, T, U have a unique common fixed point in X.
Proof: In the main theorem if we consider S=Q=I we get the proof of the corollary: 3.2.
Corrollary: 3.3 Let (X,d) be a complete cone metric space with respect to a cone in a real Banach space P.Let P,T be self maps in X.
(i)$T(X) \subseteq P(X)$
(ii)Pair (T, P) is compatible
(iii)One of T, P is continuous.
(iv) Let ξ (t): intP $\bigcup$ {0} →intP $\bigcup$ {0} be a function such that

(a) ξ (t) = 0 if and only if t = 0 and

(b) ξ (t) ≤t, for t ∈intP.

(v)For $x,y \in X$ $d(Tx,Ty) \le \xi(\mu(x,y))$ Where $\mu(x,y)$ =max{d(Tx,Px),d(Ty,Py), d(Px,Py)}
Then P,T have a unique common fixed point in X.
Proof: If P=Q, T=U in the Corollary 3.2, we get proof of Corollary 3.3.
Acknowledgment: The authors are thankful to the affiliated college authorities for financial support given by them.

References:

1. B. S. Choudhury, N. Metiya, The point of coincidence and common fixed point for a pair of mappings in cone metric spaces, Comput. Math. Appl.60 (2010), 1686-1695.
2. Binayak S. Choudhury, N. Metiya Fixed point and common fixed point results in ordered cone metric spaces An. S_t. Univ. Ovidius Constant_a Vol. 20(1), 2012, 55{72}
3. B. S. Choudhury, N. Metiya, Fixed points of weak contractions in cone metric spaces, Nonlinear Anal. 72 (2010), 1589-1593.
4. B. E. Rhoades, Some theorems on weakly contractive maps, Nonlinear Anal. 47(4) (200l), 2683-2693.
5. G. Jungck and B. E. Rhoades, "Fixed points for set valued functions without continuity," Indian Journal of Pure and Applied Mathematics, vol. 29, no. 3,pp. 227–238, 1998.
6. G. Jungck, S. Radenovi´c, S. Radojevi´c, V. Rakoˇcevi´c, Common fixed point theorems for weakly compatible pairs on cone metric spaces, Fixed Point Theory Appl. Volume 2009 (2009), Article ID 643840, 13 pages.
7. Hui-Sheng Ding et.alHindawi Publishing CorporationAbstract and Applied AnalysisVolume 2012, Article ID 793862, 18 pages doi:10.1155/2012/793862
8. J. O. Olaleru, Some generalizations of fixed point theorems in cone metric spaces, Fixed Point Theory Appl. Volume 2009 (2009), Article ID 657914, 10 pp.
9. K. P. R. Sastry, G. V. R. Babu, Some fixed point theorems by altering distances between the points, Ind. J. Pure. Appl. Math. 30(6) (1999),641-647.
10. L.G. Huang and X. Zhang, Cone metric spaces and fixed point theorems of contractivemappings, J. Math. Anal. Appl., 332(2007), 1468-1476.
11. M. A. Khamsi and W. A. Kirk, An Introduction to Metric Spaces and Fixed Point Theory, Pure and Applied Mathematics,Wiley-Interscience, New York,NY, USA, 2001.
12. M. S. Khan, M. Swaleh, S. Sessa, Fixed points theorems by altering distances between the points, Bull. Austral. Math. Soc. 30 (1984), 1-9.
13. R. H. Haghi, Sh. Rezapour, Fixed points of multifunctions on regular cone metric spaces, Expo. Math. 28 (2010), 71-77.
14. R. Kannan, Some remarks on fixed points, Bull. Calcutta Math. Soc.,60(1960), 71-76.
15. S. V. R. Naidu, Some fixed point theorems in metric spaces by altering distances, Czechoslovak Math. Jr. 53(1) (2003), 205-212.
16. S.V.R.Naidu and J.Rajendra Prasad Common Fixed Points For Four Self Maps On A Metric

space.Indian J.pure appl.Math.,16(10):1089-1103,October 1985

17. T. G. Rogers and G. E. Hardy, A generalization of a fixed point theorem of Reich, Canad. Math. Bull., 16(1973), 201-206
18. Ya. I. Alber, S. Guerre-Delabriere, Principles of weakly contractive maps in Hilbert spaces, in :I. Gohberg, Yu. Lyubich(Eds.), New Results in Operator Theory, in : Advances and Appl. 98, Birkhuser, Basel, 1997,7-22.

***

Assistant Professor,Rajiv Gandhi College of Engineering and Research, Nagpur,
email:aserkar_aaa@rediffmail.com
Assistant Professor,Yeshwantrao Chavan College of Engineering, Nagpur
E-mail:manjusha_g2@rediffmail.com
Assistant Professor,Karmavir Dadasaheb Kannamwar College of Engineering, Nagpur
E-mail:kavi_baj@rediffmail.com