SECOND ORDER DUAL FOR NON-DIFFERENTIABLE MULTIOBJECTIVE FRACTIONAL PROGRAMMING PROBLEM UNDER η-BONVEXITY

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Abstract: This paper presents second order multiobjective dual and derived weak and strong duality theorems under generalized type 1 functions and obtained various duality results involving a new class of generalized second order (F, ρ)-convex functions and establishmultiobjective duality results for a mixed type dual.

Keywords: I order dual formulation, II order dual formulation, η- quasi bonvex η- pseudo bonvex, η-bonvex.

Introduction : Second order duality was first introduced by Mangasarian [3] for a non-linear programming problem which involves second order derivatives of the objective and the constraint functions. Mond [6] reproved the second order duality results under different and less restricted assumptions than these previously considered in [4]. Zhang and Mond [7] discussed duality results for non-differentiable programms under generalized invexity. Mishra [5] formulated a second-order Mon-Weir type multiobjective dual and derived weak and strong duality theorems under generalized type-I functions. Aghezzaf [1] formulated a second-order multiobjective mixed type dual and obtained various duality results involving a new class of generalized second order (F, ρ)-convex functions. In [1], Hachimi and Aghezzaf proposed a new class of second order generalized type-I functions and establish multiobjective duality results for a mixed type dual. Recently, Ahmad & Husain [2] studied a Mond-Weir type multiobjective dual and derived duality results by defining second-order $\left(F, \alpha, \rho, d\right)$-convex functions and its generalizations.

Formulation: Consider the following minimax multiobjective fractional programming problem.

(FP) minimize $\psi(x) = \sup_{y\in Y} \dfrac{f_i(x, y)}{h_i(x, y)} \rightarrow (FP), \quad i = 1, 2, \ldots\ldots s$

subject $g_{j(x)} \le 0, \quad x \in R^n$,

Where Y is a compact subject of R^l, $f(\cdot,\cdot): R^n \times R^l \rightarrow R$, $h(\cdot,\cdot): R^n \rightarrow R^m$ are C^2 on $R^n \times R^l$ and $g(\cdot,\cdot): R^n \times R^l \rightarrow R$ in C^2 on R^n. It is assumed that for each (x, y) in $R^n \times R^l$, $f(x, y) \ge 0$ and $h_i(x, y) > 0$.

Let $S = \{x \in R^n, g_{j(x)} \le 0\}$ denote the set of all feasible solutions of (FP) for each $(x, y) \in R^n \times R^l$, we define

$$J(x) = \{j \in M = \{1, 2, \ldots\ldots m\} \;:\; g_{j(x)} = 0\},$$

$$Y(x) = \{y \in Y : f_i\;(x, y) + (x^J Bx)^{\frac{1}{2}} = \sup_{z\in Y} f_i\;(x, z) + (x^J Bx)^{\frac{1}{2}},$$

and $k(x) = \{(s, t, y) \in N \times R^s_T \times R^{l_s} : 1 \le s \le n+1,$

$t = (t_1, t_2, \ldots\ldots, t_s) \in R^s_+$ with

$\sum_{i=1}^{s} t_i = 1,\; \tilde{y} = (\bar{y}_1, \bar{y}_2, \ldots\ldots \bar{y}_s)$ with $\bar{y}_i \in y(x) = i = 1, 2, \ldots.. s$ let $\dfrac{f_i}{h_i} : R^n \rightarrow R$ be a twice differentiable function.

I order dual formulation : Consider the following I order dual to (FP)

(FD$_1$) max sup λ_i

$$(s, t, \bar{y}) \in k(z) \qquad (z, \mu, \lambda, p) \in H_1(s, t, \bar{y})$$

Where $H_1(s, t, \bar{y})$ denotes the set of all $(z, \mu_j, \lambda_i, p) \in R^n \times R^m_+ \times R_t \times R^n$ Satisfying

$$\nabla \sum_{i=1}^{s} t_i \left(f_i\ (z, \bar{y}_i)\right) - \lambda_i h_i(z, \bar{y}_i) + \nabla^2 \sum_{i=1}^{s} t_i \left(f\ \ (z, \bar{y}_i)\right) - \lambda_i h_i(z, \bar{y}_i) p$$

$$+\ \nabla \sum_{j=1}^{m} \mu_i g_i(z)\ +\ \nabla^2 \sum_{j=1}^{m} \mu_j g_j(z) p = 0 \qquad (2.1.1)$$

$$\sum_{i=1}^{s} t_i \left(f_i\ (z, \bar{y}_i)\right) - \lambda_i h_i(z, \bar{y}_i) - \frac{1}{2} p^T \nabla^2 \sum_{i=1}^{s} t_i \left(f_i\ (z, \bar{y}_i)\right) - \lambda_i h_i(z, \bar{y}_i) p \geq 0 \qquad (2.1.2)$$

$$\sum_{j=1}^{m} \mu_j\ g_j(z) - \frac{1}{2} p^T\ \nabla^2 \sum_{j=1}^{m} \mu_j g_j(z) p \geq 0 \qquad (2.1.3)$$

If, for a triplet $(s, t, \bar{y}) \in k(z))$, the set

$H_1(s, t, \bar{y}) = \phi$ then we define the supremum over it to be $-\infty$

II order dual formulation: Consider the following II order dual to (FP)

$$(FD_2) \qquad \max_{(s, t, \bar{y}) \in k(z)} \qquad \sup_{(z, \mu, \lambda, p) \in H_2(s, t, \bar{y})} \qquad \lambda_i$$

Where $H_2(s, t, \bar{y})$ denotes the set of all

$\left(z, \mu_j, \lambda_i, p\right) \in R^n \times R^m_+ \times R_+ \times R^n$ Satisfying

$$\nabla \sum_{i=1}^{s} t_i \left(f_i\ (z, \bar{y}_i)\right) - \lambda\lambda_i(z, \bar{y}_i) + \nabla^2 \sum_{i=1}^{s} t_i \left(f_i\ (z, \bar{y}_i)\right) - \lambda_i h_i(z, \bar{y}_i) p$$

$$+\ \nabla \sum_{j=1}^{m} \mu_j g_j(z)\ +\ \nabla^2 \sum_{j=1}^{m} \mu_j g_j(z) p = 0 \qquad (2.2.1)$$

$$\sum_{i=1}^{s} t_i \left(f_i\ (z, \bar{y}_i)\right) - \lambda_i h_i(z, \bar{y}_i) + \sum_{j \in J_0} \mu_j g_j(z) - \frac{1}{2} p^T \nabla^2$$

$$\left(\sum_{i=1}^{m} t_i \left(f_i\ (z, \bar{y}_i)\right) - \lambda_i h_i(z, \bar{y}_i) + \sum_{j=J_0}^{m} \mu_j g_i(z)\right) p \geq 0 \qquad (2.2.2)$$

$$\sum_{j \in J_\alpha} \mu_j g_j(z) - \frac{1}{2} p^T \nabla^2 \sum_{j \in J_\alpha} \mu_j g_j(z) p \geq 0,\ \alpha = 1, 2, \ldots. r \qquad (2.2.3)$$

Where $J_\alpha \subseteq M, \alpha = 1, 2, \ldots. r$ with $\bigcup_{\alpha=0}^{r} J_\alpha = M\ \&\ J_\alpha \cap J_B = \phi$, if $\alpha \neq \beta$.

If, for a triplet $(s, t, \bar{y}) \in k(z)$, the set $H_2(s, t, \bar{y}) = \phi$, then we define the supermum over it to be $-\infty$.

Definition Function $\frac{f_i}{h_i}$ is said to be η-bonvex at $x^* \in R^n$, if there exists a certain mapping $\eta : R^n \times R^n \to R^n$ such that for all, $x, p \in R^n$, we have

$$\frac{f_i(x)}{h_i(x)}-\frac{f_i(x^*)}{h_i(x^*)}+\frac{1}{2}p^T\nabla^2\frac{f_i(x^*)}{h_i(x^*)}\geq\eta^T(x,x^*)\left[\frac{\nabla f_i^*(x)}{h_i(x)}+\nabla^2\frac{f_i(x^*)}{h_i(x^*)}\right]$$

Definition Function $\frac{f_i}{h_i}$ is said to be η-pseudo bonvex at $x^*\in R^n$, if there exists a certain mapping $\eta:R^n\times R^n\to R^n$ such that for all, $x,p\in R^n$, we have

$$\eta^T(x,x^*)\left[\nabla\frac{f_i(x^*)}{h_i(x^*)}+\nabla^2\frac{f_i(x^*)}{h_i(x^*)}p\right]\geq 0$$

$$\Rightarrow\frac{f_i(x)}{h_i(x)}\geq\frac{f_i(x^*)}{h_i(x^*)}-\frac{1}{2}p^T\nabla^2\frac{f_i(x^*)}{h_i(x^*)}p$$

Definition Function $\frac{f_i}{h_i}$ is said to be η-quasi bonvex at $x^*\in R^n$, if there exists a certain mapping $\eta:R^n\times R^n\to R^n$ such that for all, $x,p\in R^n$, we have

$$\frac{f_i(x)}{h_i(x)}\leq\frac{f_i(x^*)}{h_i(x^*)}-\frac{1}{2}p^T\nabla^2\frac{f_i(x^*)}{h_i(x^*)}p\Rightarrow\ \eta^T(x,x^*)\left[\nabla\frac{f_i(x^*)}{h_i(x^*)}+\nabla^2\frac{f_i(x^*)}{h_i(x^*)}p\right]\leq 0$$

Necessary Condition Let x^* be a solution of (FP) and Let $\nabla gj(x^*),\ j\in J(x^*)$ be linearly independent. Then there exist $(s^*,t^*,\bar{y}^*)\in k(x^*),\ \lambda^*\in R_+$, and $\mu^*\in R_+^m$ such that

$$\nabla\sum_{i=1}^{s^*}t_i^*\left(f_i(x^*,\bar{y}_i^*)\right)-\lambda_i^*hi(x^*,\bar{y}_i^*)+\nabla\sum_{j=1}^{m}\mu_j^*g_j(x^*)=0$$

$$f_i(x^*,\bar{y}_i^*)-\lambda_i^*hi(x^*,\bar{y}_i^*)=0,\ i=1,2,\ldots\ldots.. s^*,$$

$$\sum_{j=1}^{m}\mu_j^*g_j(x^*)=0,\qquad t_i^*\geq 0,\ \ \sum_{i=1}^{s^*}t_i^*=1,\ \ \bar{y}_i^*\in y(x^*),\ \ i=1,2,\ldots\ldots.. s^*$$

Weak Duality Theorems & Strong Duality Theorem

Weak Duality Theorem: Let x and $(z,\mu_j,\lambda_i,s,t_i,\bar{y},p)$ be feasible solutions of (Fp) and (FD_1), respectively. Assume that (i) $\sum_{i=1}^{s}t_i\left(f_i(\cdot,\bar{y}_i)\right)-\lambda_i h(\cdot,\bar{y}_i)$ is $\eta-$pseudo bonvex at z; and

(ii) $\sum_{j=1}^{m}\mu_j g_j(\cdot)$ is $\eta-$quasi bonvex at z. Then $\sup_{y\in Y}\frac{f_i(x,y)}{h_i(x,y)}\geq\lambda_i$

Proof: By the feasibility of x for (FP), $\mu_j\geq 0$ and (2.2.3), we get

$$\sum_{j=1}^{m}\mu_j g_j(x)\leq 0\leq\sum_{j=1}^{m}\mu_j g_j(z)-\frac{1}{2}p^T\nabla^2\sum_{j=1}^{m}\mu_j g_j(z)p$$

The above inequality together with hypothesis ii) implies

$$\eta^T(x,z)\left[\nabla\sum_{j=1}^{m}\mu_j g_j(z)+\nabla^2\sum_{j=1}^{m}\mu_j g_j(z)\right]p\leq 0 \qquad (2.1.4)$$

from (2.1.2) and (2.1.4), we have

$$\eta^T(x,z)\left[\nabla\sum_{i=1}^{s}t_i\ f_i\ (z,\overline{y}_i)-\lambda_i h_i(z,\overline{y}_i)+\nabla^2\sum_{i=1}^{s}t_i f_i\ (z,\overline{y}_i)-\lambda_i h_i(z,\overline{y}_i)\right](p)\geq 0$$

which by the of hypothesis (i) yields

$$\sum_{i=1}^{s}t_i\left(f_i\ (x,\overline{y}_i)-\lambda_i h_i(x,\overline{y}_i)\right)\geq\sum_{i=1}^{s}t_i\left(f_i\ (z,\overline{y}_i)-\lambda_i h(z,\overline{y}_i)\right)-$$

$$\frac{1}{2}p^T\nabla^2\sum_{i=1}^{s}t_i\left(f_i\ (z,\overline{y}_i)-\lambda_i h_i(z,\overline{y}_i)\right)p\geq 0 \qquad \text{(by 2.1.2)}$$

Therefore, there exists a certain i_o such that $f_i\ (x,\overline{y}_{i_0})-\lambda_i h_i(x,\overline{y}_{i_0})\geq 0$

$$\sup_{y\in Y}\frac{f_i\ (x,y)}{h_i(x,y)}\geq\frac{f_i\ (x,\overline{y}_{i_0})}{h_i(x,\overline{y}_{i_0})}\geq\lambda_i$$

Strong duality theorem Assume that x^* is an optimal solution of (FP) and $\nabla gj\,(x^*),\ j\in J(x^*)$, are linearly independent. Then there exists $(s^*,t^*,\overline{y}^*)\in k(x^*)$, such that $\left(x^*,\mu_j^*,\lambda_i^*,s^*,t_i^*,\overline{y}^*,p^*=0\right)$ $\in H_1(s^*,t_i^*,\overline{y}^*)$ such that $\left(x^*,\mu_j^*,\lambda_i^*,s^*,t_i^*,\overline{y}^*,p^*=0\right)$ is feasible solution of (FD_1) and the two objectives have the same values. If, in addition, the assumptions of weak duality (5.1 theorem) hold for all feasible solutions. $\left(z,\mu_j,\lambda_i,s,t_i,\overline{y},p\right)$ (FD_1), them $\left(x^*,\mu_j^*,\ \lambda_i^*,s^*,t_i^*,\overline{y}^*,p^*=0\right)$ is an optimal solution of (FD_1).

Proof: Since x^* is an optimal solution of (FP) and $\nabla gj\,(x^*),\ j\in J(x^*)$, are linearly independent, then by (necessary condition), there exist $(s^*,t_i^*,\overline{y}^*)\in k(x^*)$, and $\left(x^*,\mu_j^*,\lambda_i^*,p^*=0\right)\in H_1(s^*,t_i^*,\overline{y}^*)$ such that $\left(x^*,\mu_j^*,\lambda_i^*,s^*,t_i^*,\overline{y}^*,p^*=0\right)$ is feasible solution of (FD_1) and the two objectives have same values. Optimality of $\left(x^*,\mu_j^*,\lambda_i^*,s^*,t_i^*,\overline{y}^*,p^*=0\right)$ for (FD_1), thus follows from weak duality theorem (5.2).

Strict Converse duality Let x^* and $\left(z^*,\mu_j^*,s^*,t_i^*,\overline{y}^*,p^*\right)$ be optimal solutions of (FP) and (FD_1), respectively. Suppose that (i) $\nabla g_j(x^*),\ \ j\in J(x^*)$ are linearly independent,

(ii) $\left[\sum_{i=1}^{s^*}t_i^*\left(f_i\ (\cdot,\overline{y}_i^*)\right)-\lambda_i^* h_i\ \ (\cdot,\overline{y}_i^*)\right]$ is strictly η -quasibonvex at z^*.

(iii) $\sum_{j=1}^{m}\mu_j^* g_j(\cdot)$ is η -quasibonvex at z^*. Then $z^*=x^*$.

Then $z^*=x^*$,

Proof: Suppose to the contrary that $z^*\neq x^*$, and well derive a contradiction. Since x^* and $\left(z^*,\mu_j^*,\lambda_i^*,s^*,t_i^*,\overline{y}^*,p^*\right)$ are optimal solutions of (FP) and (FD_1) respectively and $\nabla g_j(x^*),\ \ j\in J(x^*)$ are linearly independent therefore by 5.2. Theorem, we have $\sup_{y^*\in Y}\frac{f_i\ (x^*,y^*)}{h_i(x^*,y^*)}=\lambda^*$ (2.1.5). The feasibility of x^* for (FP), $\mu_j^*\geq 0$ and (2.1.3) imply

$\sum_{j=1}^{m}\mu_j^* g_j(x^*) \le 0 \le 0 \sum_{j=1}^{m}\mu_j^* g_j(z^*) - \frac{1}{2}p^{*T}\nabla^2 \sum_{j=1}^{m}\mu_j^* g_j(z^*)p^*$, which along with hypothesis (iii) gives $\eta^T(x^*, z^*)\left[\nabla \sum_{j=1}^{m}\mu_j^* g_j(z^*) + \nabla^2 \sum_{j=1}^{m}\mu_j^* g_j(z^*)p^*\right] \le 0$ (2.1.6)

Therefore, inequality (2.1.1) along with (2.1.6) yields

$$\eta^T(x^*, z^*)\left[\nabla \sum_{j=1}^{m} t_i^*\left(f_i(x^*, \bar{y}_i^*)\right) - \lambda_i^* h_i\left(x^*, \bar{y}_i^*\right)p^*\right] \ge 0$$

which by hypothesis (ii) and inequality (2.1.2) gives $\sum_{i=1}^{s^*} t_i^*\left(f_i(x^*, \bar{y}_i^*)\right) - \lambda_i^*\left(hi(x^*, \bar{y}_i^*)\right) \ge 0$

For a certain i_o, this implies

$$\sup_{y^*\in Y} \frac{f_i(x^*, y^*)}{h_i(x^*, y^*)} \ge \frac{f_i(x^*, \bar{y}_{i_0}^*)}{h_i(x^*, \bar{y}_{i_0}^*)} > \lambda^*,$$

Which is contradiction to (2.1.5). Hence $z^*=x^*$

Weak and Strong Duality using 2nd Order Dual

Weak Duality Theorem: Let x and $\left(z, \mu_j, \lambda_i, s, t_i^*, \bar{y}, p\right)$ be feasible solutions of (FP) and (FD_2), respectively. Assume that

(i) $\left[\sum_{i=1}^{s^*} t_i^*\left(f_i(\cdot, \bar{y}_i^*)\right) - \lambda_i^* h_i(\cdot, \bar{y}_i^*)\right] + \sum_{j=1}^{m}\mu_j^* g_j(\cdot)$ is η-pseudo bonvex at z; and

(ii) $\sum_{j=1}^{m}\mu_j^* g_j(\cdot)$, $\alpha = 1, 2......r$ is η-quasi bonvex at z. Then $\sup_{y\in Y}\frac{f_i(x, y)}{h_i(x, y)} \ge \lambda_i$

Proof: By the feasibility of x for (FP), $\mu_j \ge 0$ and (2.4.3), we have

$$\sum_{j\in J_\alpha}\mu_j g_j(x) \le 0 \le \sum_{j\in J_\alpha}\mu_j g_j(z) - \frac{1}{2}p^T\nabla^2 \sum_{j\in J_\alpha}\mu_j g_j(p),\ \alpha = 1, 2, ... r \quad (2.2.4)$$

The inequality (2.4.4) and hypothesis (ii) give

$$\eta^T(x, z)\left[\nabla \sum_{j\in J_\alpha}\mu_j g_j(z) + \nabla^2 \sum_{j\in J_\alpha}\mu_j g_j(z)p\right] \le 0,\quad \alpha = 1, 2, ... r$$

Which together (2.4.1) yields $\eta^T(x, z)\left[\nabla \sum_{i=1}^{s} t_i\left(f_i(z, \bar{y}_i)\right) - \lambda_i h_i(z, \bar{y}_i) + \nabla^2 \sum_{i=1}^{s} t_i\left(f_i(z, \bar{y}_i)\right)\right] -$

$$\lambda_i h_i(z, \bar{y}_i)p + \nabla \sum_{j\in J_0}\mu_j g_j(z) + \nabla^2 \sum_{j\in J_0}\mu_j g_j(z)p \ge 0$$

By using hypothesis (i), the above inequality implies

$$\sum_{i=1}^{s} t_i\left(f_i(x, \bar{y}_i)\right) - \lambda_i h_i(x, \bar{y}_i) + \sum_{j\in J_0}\mu_j g_j(x) \ge \sum_{i=1} t_i\left(f_i(z, \bar{y}_i)\right) -$$

$$\lambda_i h_i(z, \bar{y}_i) + \sum_{j\in J_0}\mu_j g_j(z) - \frac{1}{2}p^T\nabla^2\left[\sum_{i=1}^{s} t_i\left(f_i(z, \bar{y}_i) - \lambda_i h_i(z, \bar{y}_i) + \sum_{j\in J_0}\mu_j g_j(z)\right)\right]p$$

By $\mu_j \ge 0,\ g_j(x) \le 0$ and (2.3.2) it follows that $\sum_{i=1}^{s} t_i\left(f_i(x, \bar{y}_i) - \lambda_i h_i(x, \bar{y}_i)\right) \ge 0$

Therefore, there exists a certain i_o such that $f_i(x, \bar{y}_{i_0}) - \lambda_i h_i(x, \bar{y}_{i_0}) \geq 0$

$$\text{Hence } \sup_{y \in Y} \frac{f_i(x, y)}{h_i(x, y)} \geq \frac{f_i(x, \bar{y}_{i_0})}{h_i(x, \bar{y}_{i_0})} \geq \lambda_i$$

Strong duality theorem: Assume that x^* is an optimal solution of (FP) and $\nabla gj(x^*), j \in J(x^*)$, are linearly independent. Then there exists $(s^*, t^*, \bar{y}^*) \in k(x^*)$, and $\left(x^*, \mu_j^*, \lambda_i^*, p^* = 0\right)$ $\in H_1(s^*, t_i^*, \bar{y}^*)$ such that $\left(x^*, \mu_j^*, \lambda_i^*, s^*, t_i^*, \bar{y}^*, p^* = 0\right)$ is a feasible solution of (FD_2) and the two objectives have the same values. If, in addition, the assumptions of weak duality (5.5 theorem) hold for all feasible solutions $\left(z, \mu_j, \lambda_i, s, t_i^*, \bar{y}, p\right)$ (FD_2), them $\left(x^*, \mu_j^*, \lambda_i^*, s^*, t_i^*, \bar{y}^*, p^* = 0\right)$ is an optimal solution of (FD_2).

Theorem (5.6): (strict converse duality) Let x^* and $\left(z^*, \mu_j^*, \lambda_i^*, s^*, t_i^*, \bar{y}^*, p^*\right)$ be optimal solutions of (FP) and (FD_2), respectively. Suppose that (i) $\nabla gj(x^*), j \in J(x^*)$ are linearly independent,

(ii) $\left[\sum_{i=1}^{s^*} t_i^*\left(f_i(\cdot, \bar{y}_i^*)\right) - \lambda_i^* h_i(\cdot, \bar{y}_i^*)\right] + \sum_{j=1}^{m} \mu_j^* g_j(\cdot)$ is strictly η-pseudo bonvex at z^*; and

(iii) $\sum_{j=1}^{m} \mu_j^* g_j(\cdot)$, $\alpha = 1, 2......r$ is η-quasi bonvex at z^*. Then $z^*=x^*$.

Proof: It can be proved similarly to theorem (5.4).

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