EDGE TOPOGENIC GRAPHS

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Abstract: The notion of Topogenic graphs was introduced by B.D Acharya ,K.A.Germina and Jisha Elizabeth Joy. A graph G = (V,E) is said to be topogenic with respect to a non empty ground set X if it admits a topogenic set indexer,which is a function $f : V \rightarrow 2^X$ such that $f(V) \cup f^{\oplus}(E)$ is a topology on X. This motivated us to define edge topogenic and edge gracefully edge topogenic set indexers. An edge topogenic indexer is an indexer

$f : E \rightarrow 2^X$ such that $f^{\oplus}(V) \cup f(E)$ is a topology on X where

$$f^{\oplus}(V) = \left\{ f^{\oplus}(v) \big/ v \in V \right\} and$$

$$f^{\oplus}(v) = \left\{ \bigcup_{e_{i \in E(G)}} f(e_i) \big/ e_i \text{ is incident at } v \right\}$$

A graph G = (V,E) is said to be edge topogenic if it admits an edge topogenic set indexer.In this paper we aim to establish some interesting results on edge topogenic and edge gracefully edge topogenic set indexers of graphs.

Keywords: Topogenic graphs, edge topogenic graphs, edge gracefully edge topogenic graphs .

Introduction: In graph theory every concept defined for vertices has been viewed for edges also. Giving edge analog for any concept is a natural one .The notion of Topogenic graphs was introduced by B.D Acharya ,K.A.Germina and Jisha Elizabeth Joy in [1], [2] and now we try to give an edge analog of this concept. Section 1 deals the topogenic graphs. In section 2 we introduce edge topogenic graphs and in section 3 edge gracefully edge topogenic graphs are dealt.

Topogenic Graphs:

Definition 1.1A graph G=(V,E) is said to be topogenic if it admits a topogenic set indexer,which is a set indexer $f : V \rightarrow 2^X$ such that $f(V) \cup f^{\oplus}(E)$ is a topology on X.

The following problem has been posed in

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results on topogenic and graceful topogenic set-indexers of graphs. We have partially answered the above problem.

Problem:Given a complete graph K_p determine the minimum number of edges $\{e_i\}$ to be deleted so that K_p-$\{e_i\}$ is a topogenic graph.

Theorem 1.1When n is odd , $m(K_n) \leq \left(\frac{n-1}{2}\right)\left(\frac{n-3}{2}\right)$ wherem(K_n) denotes the minimum number of edges to be removed from K_n so that the resulting graph is topogenic.

Proof:Draw K_n as usual so that the vertices form a regular polygon. Let $\left\{ v_1, v_2, \ldots, v_{\frac{n+1}{2}}, v_{\frac{n+3}{2}}, \ldots, v_n \right\}$ be the vertex set of $K_n$taken in the anticlockwise direction. Now delete the following edges from K_n.

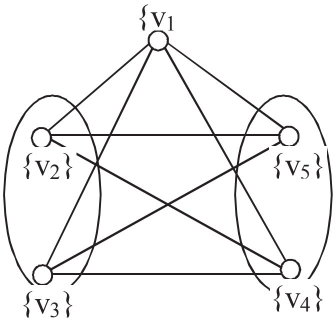
$$v_i v_j \; , 2 \leq i \neq j \leq \frac{n+1}{2} \qquad\qquad v_l v_k, \; \frac{n+3}{2} \leq l \neq k \leq n$$

Number of edges deleted

$$= \begin{pmatrix} \frac{n-1}{2} \\ 2 \end{pmatrix} + \begin{pmatrix} \frac{n-1}{2} \\ 2 \end{pmatrix} \qquad = 2 \times \frac{1}{2}\left(\frac{n-1}{2}\right)\left(\frac{n-3}{2}\right) \qquad = \left(\frac{n-1}{2}\right)\left(\frac{n-3}{2}\right)$$

After removing these edges from K_n, the resulting graph becomes a complete tripartite graph $K_{1,\frac{n-1}{2},\frac{n-1}{2}}$.

Illustration:Consider K_5.Let $V(K_5)=\{v_1,v_2,v_3,v_4,v_5\}$. Remove v_2v_3 and v_4v_5. $K_5-\{v_2v_3, v_4v_5\}$ is



It has been proved in [1] that any complete tripartite graph $K_{1,m,n}$ is topogenic for all positive integers m and n.

$\therefore K_{1,\frac{n-1}{2},\frac{n-1}{2}}$ is topogenic. Hence $m(K_n) \le \left(\frac{n-1}{2}\right)\left(\frac{n-3}{2}\right)$.

Theorem 1.2:When n is even, $m(K_n) \le \frac{1}{4}\left[n^2-4n+4\right]$.

Proof:Draw K_n as usual so that the vertices form a regular polygon. Let $\left\{v_1, v_2, \ldots, v_{\frac{n+2}{2}}, v_{\frac{n+4}{2}}, \ldots, v_n\right\}$ be the vertex set of $K_n$taken in the anticlockwise direction.Delete the following edges from K_n.

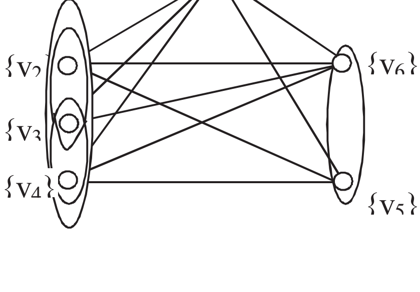
$$v_iv_j\ ,2 \le i \ne j \le \frac{n+2}{2} \qquad v_lv_k,\ \frac{n+4}{2} \le l \ne k \le n$$

Number of edges to be deleted $= \binom{\frac{n}{2}}{2} + \binom{\frac{n-2}{2}}{2}$

$$= \frac{1}{2}\left[\frac{n}{2}\left(\frac{n}{2}-1\right) + \left(\frac{n-2}{2}\right)\left(\frac{n-4}{2}\right)\right] \qquad = \frac{1}{4}\left[n^2-4n+4\right]$$

After removing these edges from K_n, the resulting graph becomes a complete tripartite graph $K_{1,\frac{n}{2},\frac{n}{2}-1}$.

Illustration:
Consider K_6.Let $V(K_6)=\{v_1,v_2,v_3,v_4,v_5,v_6\}$. Remove $v_2v_3,v_2v_4,v_3v_4$and v_5v_6.
$K_6-\{v_2v_3, v_2v_4, v_3v_4,v_5v_6\}$ is



It has been proved in [1] that the complete tripartite graph $K_{1,m,n}$ is topogenic for all positive integers m and n.

$\therefore K_{1,\frac{n}{2},\frac{n}{2}-1}$ is topogenic. Hence, $m\left(K_n\right) \le \frac{1}{4}\left[n^2 - 4n + 4\right]$.

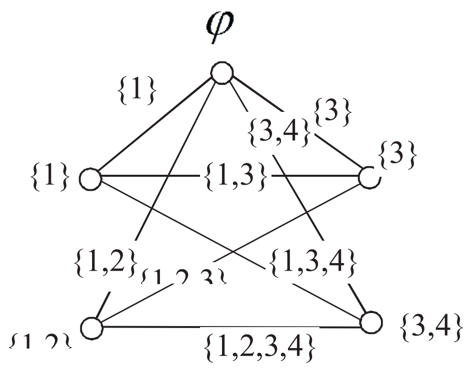
Corollary 1: $m(k_3) = 0$. K_3 is topogenic. So $m(k_3) = 0$.

Corollary 2 : $m(k_4) = 1$. By theorem1.2,

$$m\left(K_n\right) \le \frac{1}{4}\left[16 - 16 + 4\right] = 1.$$

K_4 not topogenic $\Rightarrow m\left(K_4\right) \neq 0$ $\therefore m(k_4) = 1$.

Example: Topogenic labeling of K_5-{e_1,e_2}.



$T_f = f^{\oplus}(V) \cup f(E) = \left\{\phi,\{1\},\{3\},\{1,2\},\{1,3\},\{3,4\}\{1,2,3\},\{1,3,4\},\{1,2,3,4\}\right\}$

Clearly T_f is a topology on X.

2. Edge Topogenic Graphs:

Definition 2.1: A graph G = (V,E) is said to be edge topogenic if it admits an edge topogenic set indexer which is an indexer $f : E \to 2^X$ such that $f^{\oplus}(V) \cup f(E)$ is a topology on X.

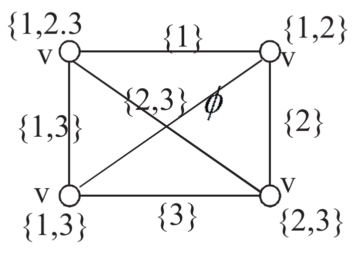
$$f^{\oplus}(V) = \left\{ f^{\oplus}(v) \big/ v \in V \right\} and$$

where

$$f^{\oplus}(v) = \left\{ \bigcup_{e_{i \in E(G)}} f(e_i) \Big/ e_i \text{ is incident at } v \right\}$$

Example:

K_4 is edge topogenic.



$f^{\oplus}(V) \cup f(E) = \left\{\phi,\{1\},\{2\},\{3\}\{1,2\}\{1,3\}\{2,3\}\{1,2,3\}\right\}$ which is a topology on X. Therefore K_4 is edge topogenic.

Note: K_4 is not topogenic but edge topogenic.

Definition 2.2:Let G = (V,E) be a (p,q) graph. Let $f:E(G)\to 2^X$ be an edge topogenic set indexer of G. Let $T_f = f^{\oplus}(V)\cup f(E)$. Then the number of distinct f-open sets, $|T_f|$ is called the edge topogenic strength of f over G.If G is finite, the minimum (respectively maximum) of $|T_f|$ is taken over all possible edge topogenic set indexers f of G is denoted by $\rho^{\min}(G)$ (respectively $\rho^{\max}(G)$). Because of the injectivity of f and $f^{\oplus}$, we must have

$\rho^{\min}(G)\le |f(E)\cup f^{\oplus}(V)|\le \rho^{\max}(G)\le p+q$

Also for an edge topogenic graph p $\le \rho^{\min}(G)$ or q $\le \rho^{\min}(G)$.So we have the following observations:

1. For any topogenic (p,q) graph G,max{p,q} $\le \rho^{\min}(G)\le \rho^{\max}(G)\le p+q$.
2. For an edge topogenic path $p_n, n\le \rho^{\min}(G)\le 2n-1$.
3. For an edge topogenic cycle $C_n, n\le \rho^{\min}(G)\le 2n$.
4. For an edge topogenic complete bipartite graph $K_{m,n}$, $m\le n, mn\le \rho^{\min}(G)\le m+n+mn$.
5. If K_p is edge topogenic then $\dfrac{p^2-p}{2}\le \rho^{\min}(G)\le \dfrac{p^2+p}{2}$.

Theorem 2.1:For any positive integer n, there exists a connected edge topogenic graph of order n.

Proof:Let G = $K_{1,n}$ be the n-star whose vertices are $u, v_1, v_2\cdots v_n$ where u is the central vertex of the star. Let X {1, 2,...n-1}. Define f:E($K_{1,n}$)$\to 2^X$ as follows:

Let $f(uv_n)=\begin{cases}\phi & for\ i=1\\ \{1,2,\cdots i-1\} & for\ \ i=2,3,\ldots n-1\end{cases}$

$f(uv_n)=\{1,2,3,4,\ldots,n-3,n-1\}$

$f(E(G))=\{\phi,\{1\},\{1,2\},\{1,2,3\},\ldots,\{1,2,\ldots,n-2\},\{1,2,3\ldots,n-3,n-1\}\}$ $f^{\oplus}(u)$ ={1,2,...n-1}

$f^{\oplus}(v_i)=f(uv_i)$ ={1,2,.... i-1},i= 2,3,...n-1

$f^{\oplus}(v_1)=\phi$ $f^{\oplus}(v_n)=\{1,2,3,4,\ldots,n-3,n-1\}$

It can be easily verified that $f(E)\cup f^{\oplus}(V)$ is a topology on X

Theorem 2.2:For every positive integer n$\ge$ 4, P_n is edge topogenic.

Proof:Let G = P_n be the path whose vertices are $v_1,v_2,\ldots v_n$.

Let x = {1,2, ..., n-2} Define $f:E(P_n)\to 2^X$ as

$f(v_1v_2)=\phi$ and $f(v_i,v_{i+1})$={1,2,3, ..., i-1},i = 2, 3, ... n-2.

$f(v_{n-1}v_n)$ = {1, 2,...n-4, n-2} $f^{\oplus}(v_1)=\phi$, $f^{\oplus}(v_n)=f(v_{n-1}v_n)$ = {1, 2,...n-4, n-2}

Itcanbe easily verified that $f(E)\cup f^{\oplus}(V)=\{\phi,\{1\},\{1,2\},\{1,2,3\}$
$\ldots\{1,2,3\ldots n-2\},\{1,2,\ldots n-4,n-2\}\}$
is a topology on X.Therefore, P_n is edge topogenic.

Theorem 2.3: For n$\ge$ 3, C_n is edge topogenic.

Proof:Let $v_1, v_2,\ldots v_n$be the vertices of C_n.

Let x = {1,2, ..., n-1} Define $f:E(C_n)\to 2^X$ as follows.

$f(v_1v_2)=\phi$ and $f(v_iv_{i+1})$={1,2,3,...,i-1},i=2,3, ... n-1.

$f(v_nv_1)$={1,2,....n-3,n-1}. $f^{\oplus}(v_1)=f(v_nv_1)$

$f^{\oplus}(v_i) = f(v_i v_{i+1})$ for i= 2,3,...n-1. $f^{\oplus}(v_n) = \{1,2,...,n-1\}$

$f(E) \cup f^{\oplus}(V) = \{\phi, \{1\}, \{1,2\}, \{1,2,3\} ... \{1,2,3...n-1\}, \{1,2,...,n-3,n-1\}\}$

is a topology on X. Therefore C_n is edge topogenic.

Theorem 2.4:The ladder graph G = $k_2 \square P_n$ is edge topogenic where P_n is path graph.

Proof:Let x = {1,2, ...,2n-2}.

Let $v_1, v_2, ... v_{2n}$ be the vertices of G.

Define $f: E(G) \to 2^X$ as $f(v_1 v_2) = \phi$ and $f(v_i, v_{i+1})$ ={1,2,3, ..., i-1}, i = 2,3, ... 2n-2. $f(v_{2n-1} v_{2n})$ = {1,2, ...2n-4, 2n-2}.

$f(v_{n-i} v_{n+i+1}) = \{1,2,...n+i-3, n+i-1\}, i = 1,2,...n-2$ $f(v_1 v_{2n})$ = {1,2, ... 2n-6,2n-4, 2n-2}.

$f^{\oplus}(v_i) = f(v_i v_{2n-(i-1)})$,i = 1,2,...,n $f^{\oplus}(v_i) = f(v_i v_{i+1})$

fori= $n+1, n+2, ... 2n-3, 2n-2$

$f^{\oplus}(v_{2n-1})$ = {1,2,...2n-3,2n-2} = X $f^{\oplus}(v_{2n}) = \{1,2,...,2n\text{-}6,2n\text{-}4,2n\text{-}2\} = f(v_1 v_{2n})$

Let $T_f = f(E) \cup f^{\oplus}(V)$. Let $T_f = f(E) \cup f^{\oplus}(V)$.

Claim: T_f is a topology. $f(v_n v_{n+1}) \supset f(v_i v_{i+1})$ for i = 1,2,...,n-1 and

$f(v_n v_{n+1}) \subset f(v_i v_{i+1})$ for i = n+1,n+2,...,2n-1 so their union and intersection will be in T_f.

Also $f(v_n v_{n+1}) \subset f(v_i v_{2n-i+1})$,i = 1,2,...,n-2 $\therefore f(v_n v_{n+1}) \cap f(v_i v_{2n-i+1}) = f(v_n v_{n+1}) \in T_f$,i= 1,2,...,n-2

$f(v_n v_{n+1}) \cup f(v_i v_{2n-i+1}) = f(v_i v_{2n-i+1}) \in T_f$,i= 1,2,...,n-2 $f(v_n v_{n+1}) \cap f(v_{n-1} v_{n+2}) = f(v_{n-1} v_n) \in T_f$.

Hence T_f is a topology on X. $\therefore k_2 \square P_n$ is edge topogenic.

<u>Theorem 2.5:</u> The peterson graph G = p(n,1) is edge topogenic for all n $\geq$ 5.

<u>Proof:</u> Let x = {1,2, ..., 2n-2}. Define $f: E(G) \to 2^X$ as

$f(v_1 v_2) = \phi$ and $f(v_i, v_{i+1})$ = {1,2,3, ..., i-1}, i = 2,3,... n-1.

$f(v_n v_1)$ = {1,2,3, ...,n-3, n-1}

$f(v_i, v_{i+1})$ = {1,2,3, ..., i-2}, i = n+1, n+2,....2n-1.

$f(v_{2n}, v_{n+1})$ = {1,2,....,2n-4, 2n-2}

$$f(v_i v_{n+i}) = \begin{cases} \{i+1\}, \; i = 1,2,3 \\ \{1, i-1\}, \; i = 4,5 \\ \{2, i-3\}, \; i = 6,7 \\ \{1, i-6, 4\}, \; i = 8,9 \\ \{2, i-7, 4\}, \; i = 10 \\ \{5\}, \; i = 11 \end{cases}$$

Assign sets to the edges in this way upto the edge $v_n v_{2n}$

$f^{\oplus}(v_1) = f(v_n v_1)$

$f^{\oplus}(v_2)$ = {1,3}

$f^{\oplus}(v_3)$ = {1,2,4}

$f^{\oplus}(v_i) = f(v_i v_{i+1})$ for i = 4,5,...n-1

$f^{\oplus}(v_n)$ ={1,2,...n-1}

$f^{\oplus}(v_{n+1})$ = {1,2,...2n-4,2n-2} = $f(v_{2n}, v_{n+1})$

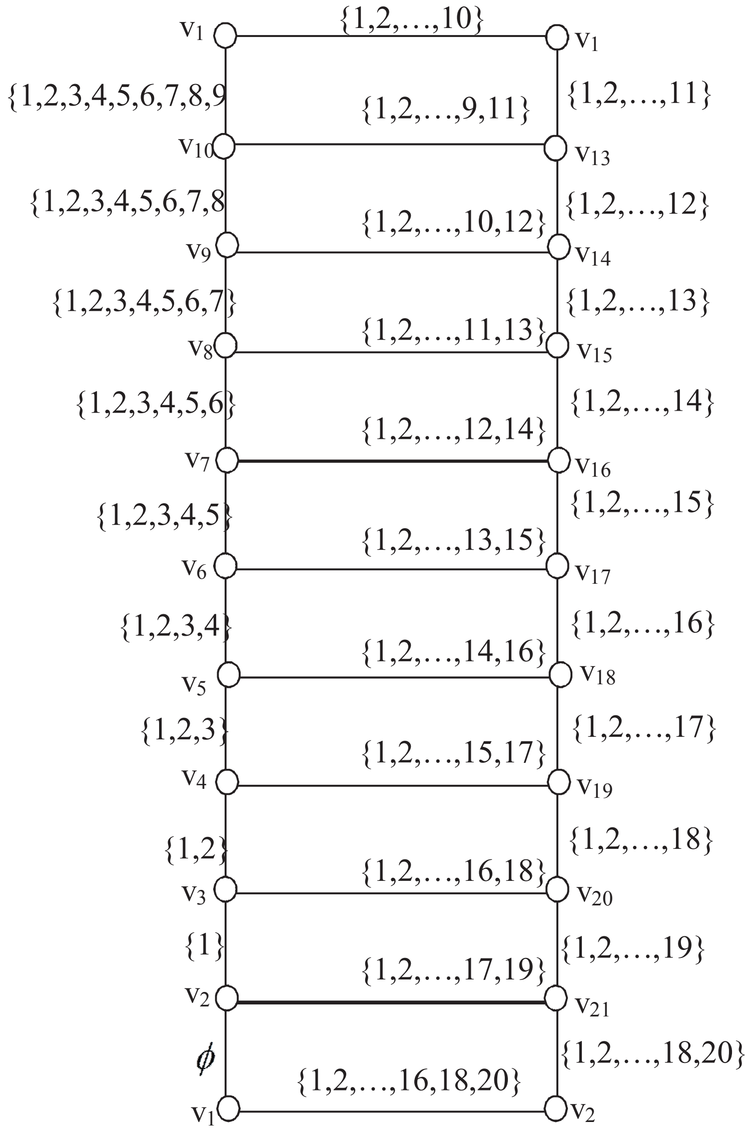
$f^{\oplus}(v_{n+2})$ = {1,2,...n}, $f^{\oplus}(v_{n+3})$ = {1,2,...n+1}

$f^{\oplus}(v_{n+4})$ = {1,2,...n+2}, $f^{\oplus}(v_{n+5})$ = {1,2,...n+3}

$f^{\oplus}(v_{2n})$ = {1,2,...,2n-2}

It can be easily verified that $f^{\oplus}(V) \cup f(E)$ is a topology on X. Hence Peterson graph is edge topogenic.

Illustration: Edge topogenic set indexer of $k_2 \square P_{22}$



3. **Edge gracefully edge topogenic graphs**

Definition 3.1: A graph G=(V,E) is edge gracefully edge topogenic, if it admits an edge gracefully edge topogenic set indexer which is a set indexer $f:E \rightarrow 2^X$ of G such that $f^{\oplus}(V) \cup f(E) = T_f$ generates a discrete topology on X.

Example: K_4 is edge gracefully edge topogenic

Theorem 3.1: The star $k_{1,2^n-1}$ is edge gracefully edge topogenic for any positive integer n.

Proof: Let $V(k_{1,2^n-1}) = \{u, v_1, v_2, \ldots, v_{2^n-1})$ be the vertex set with u as the central vertex of the star. Let $X = \{1,2,\ldots,n\}$. P(X) has 2^n elements.Assign all subsets of X except X to the spokes of the star. Then the central vertex u receives X. All other members of P(X) will appear as labels of $v_1, v_2, \ldots v_{2^n-1}$. $f^{\oplus}(V) \cup f(E)$ = P(X) forms a discrete topology on X. Therefore, $k_{1,2^n-1}$ is edge gracefully edge topogenic.

Conclusion: A study of relation between topogenic graphs and edge topogenic graphs may lead to many useful results. This work can be extended in many ways.

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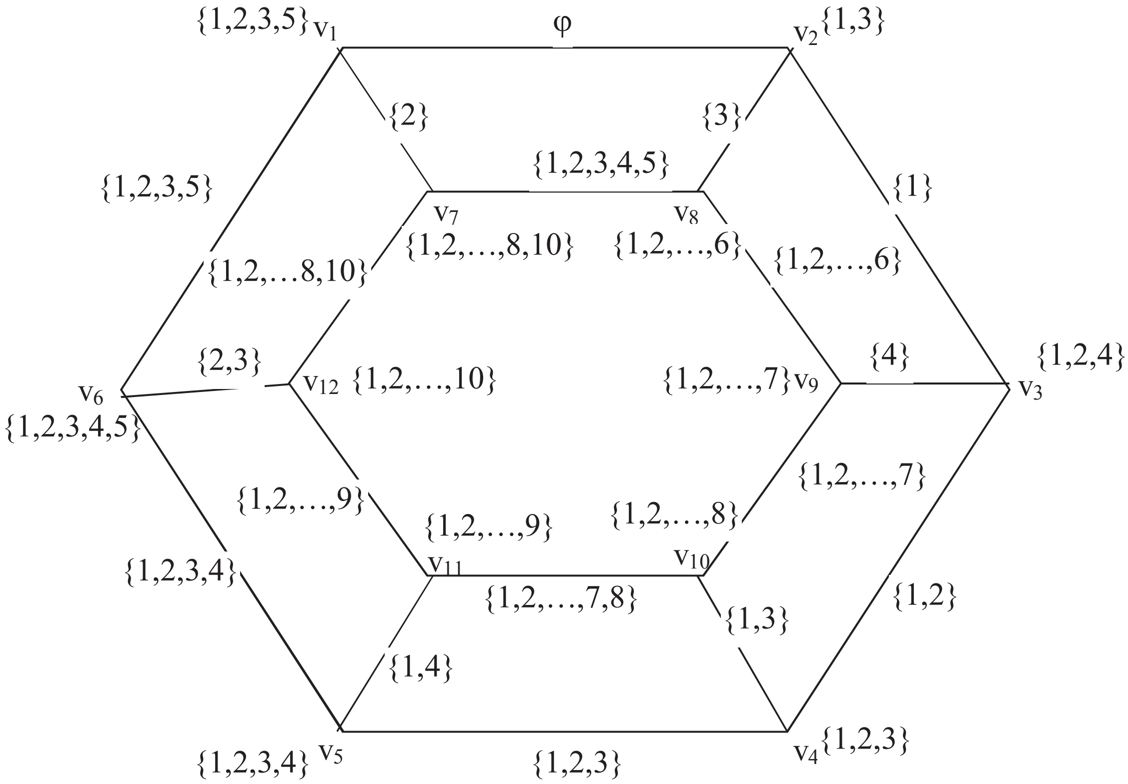
Illustration: Edge topogenic set indexer of P(6,1).



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