ACYCLIC COLOURING AND MATCHING OF SOME GRAPHS

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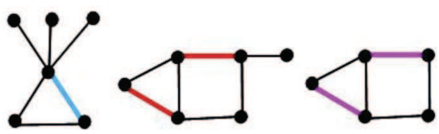
Abstract: The aim of this paper is to discuss acyclic vertex colouring of the complete graph, Harary graph and other two special graphs and to find a relation between acyclic chromatic number and the maximum matching.

Keywords: Harary graph, Goldner- Harary graph, Herschel graph, Matching, acyclic colouring, acyclic chromatic number.

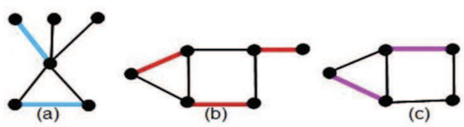
Introduction: Graph considered in this paper are simple and finite. Terms not defined here are used in the sense of Harary [6]. Graph colouring [3,6] is an assignment of labels traditionally called "colours" to elements of a graph subject to certain constraints. The most common types of colourings are vertex colouring, edge colouring and face colouring. The vertex colouring is proper, if no two adjacent vertices are assigned the same colour.A proper vertex colouring of a graph is acyclic if every cycle uses at least three colours [8]. The acyclic chromatic number of G, denoted by $a(G)$, is the minimum colours required for its acyclic colouring.

2 Acyclic Colouring of Complete Graph And Matching

2.1 Definitions[2]**:** A subset M of E is called a matching in G, if its elements are edges and no two are adjacent in G. The two ends of an edge in M are said to be matched (or saturated) under$M$. The vertex v is said to be M -saturated, if some edge of M is incident with v. Otherwise the vertex is unmatched. M is maximal if it is not a proper subset of any other matching in graph G. See the examples.



M is a maximum matching, if G has no matching N with $|N| > |M|$. The matching number of a graph is the size of a maximum matching.



Note that every maximum matching is maximal, but not every maximal matching is a maximum matching. If every vertex of G is M -saturated, the matching M is perfect. Figure (b) above is an example of a perfect matching. Note that every perfect matching is maximum and hence maximal.

A near-perfect matching is one in which exactly one vertex is unmatched and such a matching must be maximum. See figure (c). Every near- perfect matching is maximum.

2.2 Theorem For a complete graph, $a(K_n) = \begin{cases} 2|M|, & if\ n\ is\ even \text{ and} \\ 2|M|+1, & if\ n\ is\ odd \end{cases}$, where M is the maximum matching.

Proof: It is clear that there exist no proper k -colouring for which $k < n$ in K_n. Hence $a(K_n) = n$. Since given graph is a complete graph with n -vertices, there exist a unique cycle C of length n. Let it be $C = e_1, e_2, e_3, \ldots, e_n$.

Case 1: If n is even Let $M = \{e_i / \ i\ is\ even\} = \{e_2, e_4, e_6, \ldots\}$. Since every vertices of K_n is saturated by M, M is a perfect matching which is also maximum with $|M| = \frac{n}{2}$. Thus $a(K_n) = n = 2|M|$.

Case 2: If n is oddLet $M = \{e_i / \ i\ is\ even\}$.Since only one vertex is unmatched by M, M is a near-perfect matching, which is also maximum with $|M| = \left\lfloor \frac{n}{2} \right\rfloor$. Thus $a(K_n) = 2|M| + 1$.

3 Acyclic Colouring of Harary Graphs

3.1 Definition[10]**:** Let $k \geq 1$ and $n \geq k$ be any two integers. The k -connected Harary graph on n vertices, denoted by $H_{k,n}$, has vertex set $\{v_0, v_1, v_3, \ldots, v_{n-1}\}$ and the following edges.

- If $k = 2r$ is even, then two vertices v_i and v_j are linked if and only if $i - r \leq j \leq i + r$.
- If $k = 2r + 1$ is odd and n is even, then$H_{k,n}$ is obtained by joining v_i and $v_{i+\frac{n}{2}}$ in $H_{2r,n}$ for every $i \in \left[0, \frac{n}{2} - 1\right]$.
- If $k = 2r + 1$ and n is odd, then $H_{k,n}$ is obtained from $H_{2r,n}$ by first linking v_0 to both $v_{\left\lfloor \frac{n}{2} \right\rfloor}$ and $v_{\left\lceil \frac{n}{2} \right\rceil}$,

and then each vertex v_i to $v_{i+\lceil\frac{n}{2}\rceil}$ for every $i \in \left[1, \left\lfloor\frac{n}{2}\right\rfloor - 1\right]$.

3.2 Some Properties of Harary Graph

1. $H_{k,n}$ is k - connected
2. The number of edges of Harary graph $H_{k,n}$ is $\left\lceil\frac{nk}{2}\right\rceil$, for $k \geq 2$.
3. The distance between i and j is the smaller of the two values $|i-j|$ and $|n-(i-j)|$.
4. The adjacency between two vertices i and j in the graph $H_{k,n}$ is determined by the distance between i and j along the perimeter of the n - cycle.

3.3 Theorem: The acyclic chromatic number, $a(H_{k,n}) = k+1$, for odd $k > 2$.

Proof: Let $G = H_{k,n}$ be the given graph. To prove that $a(G) = k+1$, if possible assume that let $C = \{c_1, c_2, c_3, \ldots, c_p\}$ with $p \leq k$ be an acyclic colouring for G. Then by definition of acyclic colouring there exist no pair (v_x, v_y) such that they induce a bi-chromatic cycle. That means there exist a $k-1$ vertex cut in $H_{k,n}$, which is a contradiction to the fact that G is k-connected. Hence there exist no acyclic colouring C with $p \leq k$. Thus $a(G) = k+1$.

Note: $a(H_{k,n}) = k+1$ for $k = 2$.

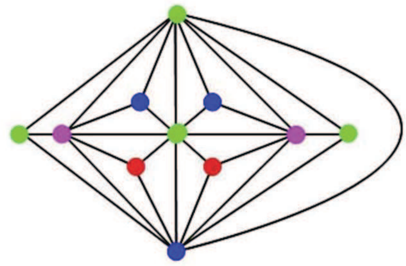
3.4 Theorem$a(H_{k,n}) = \begin{cases} 2|M| - (d-2), \text{if } n \text{ is odd} \text{ and} \\ 2|M| - (d-1), \text{if } n \text{ is even} \end{cases}$, where M is the maximum matching and $d = n-k$.

4 Acyclic Colouring of Goldner- Harary Graph

4.1 Definition[7]**:** The Goldner–Harary graph is a simple undirected graph with 11 vertices and 27 edges.

4.2 Some Properties of Goldner–Harary graph

1. $|V| = 11$ and $|E| = 27$.
2. Goldner–Harary graph is planar, all its faces are triangular and is 3-vertex-connected as well as 3-edge-connected.
3. Goldner–Harary graph is the smallest non-Hamiltonian maximal planar graph.
4. Chromatic number 4, chromatic index 8, girth 3, radius 2 and diameter 2.
5. Goldner–Harary graph is also a 3-tree, and therefore it has treewidth 3.
6. Like any k-tree, Goldner–Harary graph is a chordal graph.
7. Goldner–Harary graph is perfect.



Goldner–Harary graph

4.3 Theorem

The acyclic chromatic number of Goldner-Harary graph is 4.

Proof:

Consider the colour class $C = \{c_1, c_2, c_3, c_4\}$. The colouring is acyclic, because no bi- chromatic cycles are induced by C. The minimality is obvious, as $a > \frac{|E|}{|V|} + 1$. [11]

4.4 Theorem

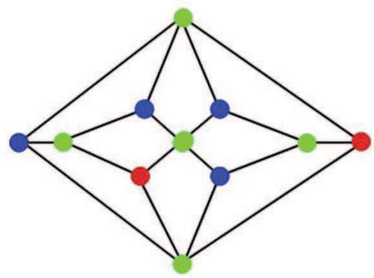
For Goldner- Harary graph G, $a(G) = |M| - 1$, where M is the maximum matching.

5 Acyclic Colouring of Herschel Graph

5.1 Definition[9]: The Herschel graph is a bipartite undirected graph with 11 vertices and 18 edges.

5.2 Some Properties of Herschel graph

1. $|V| = 11$ and $|E| = 28$.
2. Herschel graph is planar, perfect, bi-partite and 3-vertex-connected graph.
3. Herschel graph has chromatic number 2, chromatic index 4, girth 4, radius 3 and diameter 4.
4. Herschel graph is the smallest non-Hamiltonian polyhedral graph.



The Herschel graph.

5.3 Theorem The acyclic chromatic number of Herschel graph is 3.

Proof: Consider the colour class $C = \{c_1, c_2, c_3\}$. The colouring is acyclic, because no bi- chromatic cycles are induced by C. The minimality is obvious, as minimum three colours are required for the acyclic colouring of any graph containing cycles.

5.4 Theorem For Herschel graph G, $a(G) = |M| - 2$, where M is the maximum matching.

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Conclusion: In this paper we derived acyclic chromatic number of some families and established a relation between acyclic chromatic number and its matching number

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