COMMON FIXED POINT THEOREMS IN 2- METRIC SPACES WITH SUB COMPATIBLE AND SUB SEQUENTIALLY CONTINUOUS SIX SELF MAPS

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Abstract: The Purpose of this paper is to establish a common fixed point theorem for six self mapping using sub compatible and sub sequentially continuity. Our results extend and generalize the result of S. Ranjit Kumar et al. [30] , H. Bouhadjera et al. [6], Jungck and Rhoades [20].

Keywords: compatible maps, Weak Compatible maps, Sub compatible maps, Sub sequential continuity, fixed points and 2 metric space.

Introduction: Ghaler [18] gave the concept of 2-metric space whose abstract properties were suggested by the area function in Euclidean space. A 2-metic space is on which finds its wide range of applications in the fields of military, medicine and economics. Employing various contractive conditions Iseki[23] set out the tradition of proving fixed point theorems in 2-metric spaces. Later on , Naidu and Prasad [25] contributed few fixed point theorems in 2-metric space introducing the concept of weak commutativity. Cho et al. [9] introduced the notion of semi-compatible maps n d-topological space. Various authors like Saliga [32], Sharma et al. [31] and Popa [26] proved some interesting fixed point results using implicit real function and semi-compatibility in d-complete topological spaces.

Recently, B. Singh and S. Jain [35,36,37,38] introduced the concept of semi-compatibility in fuzzy metric space, D-metric space, 2-metric space and proved fixed point results using implicit relations in these spaces.

Various authors have discussed and studied extensively various results on coincidence, existence and uniqueness of fixed and common fixed points by using the concept of weak commutativity, compatibility, non-compatibility and weak compatibility for single and set valued maps satisfying certain contractive conditions in 2-metric spaces and they have been applied to diverse problems.

Recently, Al-Thagafi and N. Shahzad [2] weakened the concept of compatibllity by giving a new notion of occasionally weakly compatible (owc) maps which is more general among the commutativity concepts.

Most recently, H. Bouhadjera and C. Godet-Thobie [6], weakened the concept of occasionally weak compatibility and reciprocal continuity in the form of sub compatibility and sub sequential continuity respectively and proved some interesting results with these concepts in metric spaces.

The main purpose of our paper is to prove a common fixed point theorem for six self maps by sub compatibility and sub sequential continuity in 2-metric spaces with extends the results of [6], [30].

Preliminaries

Definition 2.1. Let X be non-empty set with real valued function d on X x X x X satisfying the followings:

(i) d(x,y,z) = 0 if at least tow of x, y, z are equal,

(ii) d(x,y,z) = d(p(x,y,z)) for all x, y, z ∈X and each permutation p(x,y,z) of x, y, z,

(iii) d(x,y,z) ≤ d(x,y,w) + d(x,w,z) + d(w,y,z) for all x, y, z, w ∈X

The function d is call ed a 2-metric on X the pair (X, d) is called a 2-metric space.

Definition 2.2. Let S and T be mappings from a 2-metric space (X, d) into itself. The mappings S and T are said to be compatible if lim d(STx_n, TSx_n, z) = 0 for all z∈ X, whenever $\{x_n\}$ is a sequence in X such that lim Sx_n = lim Tx_n = t for some t ∈ X.

Definition 2.3. A pair of self mappings S and T of a 2-metric space (X, d) is said to be weakly compatible if they commute at their coincidence points i.e., if Sx = Tx (for some x∈ X) implies STx = TSx.

Definition 2.4. Let S and T be mappings from a 2-metric space (X, d) into itself. The mappings S and T are said to be semi-compatible if lim d(STx_n, Tx, z) = 0 for all z∈ X, whenever $\{x_n\}$ is a sequence in X such that lim Sx_n = lim Tx_n = t for some t ∈ X.

Now, we give some examples to show the relationship of compatible, weakly compatible and semi-compatible maps.

Example 2.5. (example 1 of [30]) Let $X = \{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots.\}$. Define d: X x X x X → [0, ∞) by

d(x, y, z) = $\begin{cases} 0 \; if x, y, z are disticnt and \left\{\frac{1}{n}, \frac{1}{n+1}\right\} \subset \{x, y, z\} \\ 1 \; otherwise \end{cases}$

Then (X,d) is a 2-metric space as proved in [12].

Define A,B : X → X as

Ax = 1-x, Bx = $\begin{cases} x if \; 0 \le x \le \frac{1}{2} \\ 1, \; \frac{1}{2} < x \le 1 \end{cases}$. Consider a sequence x_n = $\left(\frac{1}{2} - \frac{1}{n}\right)$ for all n and (A, B) is semi-compatible and (B, A) is not semi-compatible.

The semi-compatibility of the pair (A, B) does not imply the semi-compatbility of the pair (B, A).

Moreover, weak-compatibility need not imply the semi-compatibility. Here B and A are weak compatible as they commute at their coincident point $\frac{1}{2}$ but they are not semi-compatible.

Also, semi-compatible maps need not be compatible. Here (A, B) is semi-compatible but not compatible as, $\lim d(ABx_n, BAx_n, z) = \lim d\left(\frac{1}{2}-\frac{1}{n}, 1, z\right) \neq 0$ for all z in X and t > 0.

Again, weak compatibility does not imply compatibility as the maps B and A are weak compatible but not compatible.

The following example shows that compatible maps need not be semi-compatible.

Example 2.6. (example 2 of [30]) Let $X = \{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots\}$. Define d: X x X x X → [0, ∞) by

$$d(x, y, z) = \begin{cases} 0 \; if x, y, z are disticnt and \left\{\frac{1}{n}, \frac{1}{n+1}\right\} \subset \{x, y, z\} \\ \quad 1 \; otherwise \end{cases}$$

Then (X,d) is a 2-metric space. Define A,B : X → X as

$Ax = 1-x$, $Bx = \begin{cases} x if \; 0 \le x \le \frac{1}{2} \\ 1, \; x \ge \frac{1}{2} \end{cases}$. Consider a sequence $x_n = \left(\frac{1}{2}-\frac{1}{n}\right)$ for all n

Definition 2.7. Two self maps S and T on a set X are said to be occasionally weakly compatible(owc) if and only if there is a point x ∈ X which is a coincidence point of S and T at which S and T commute. i.e., there exists a point x ∈ X such that Sx=Tx and STx = TSx.

Definition 2.8. Let S and T be mappings from a 2-metric space (X, d) into itself. The mappings S and T are said to be subcompatible if there exists $\{x_n\}$ a sequence in X such that $\lim Sx_n = \lim Tx_n = t$ for some t ∈ X and which satisfy $\lim d(STx_n, TSx_n, z) = 0$ for all $z \in X$.

Obviously two occcasionally weakly compatible maps are su compatible maps, however, the converse if not true in general as shown in the following example.

Example 2.9. (example 3 of [30]) Let X=R and

$d(x,y,z) = \min[\,|x-y|, |y-z|, |z-x|\,]$, for all x, y, z ∈X

Then clearly (X,d) is a 2-metric space.

Define A,B : X → X by setting

$Ax = \begin{cases} x^2, & x < 1 \\ 2x-1, & x \ge 1 \end{cases}$, $Bx = \begin{cases} 3x-2, & x < 1 \\ x+3, & x \ge 1 \end{cases}$.

Difine a sequence $x_n = \left(1-\frac{1}{n}\right)$, then $Ax_n = \left(1-\frac{1}{n}\right)^2 \to 1$, $Bx_n = 3\left(1-\frac{1}{n}\right) - 2 = \left(1-\frac{3}{n}\right) \to 1$, as n→ ∞

A new notion callled sub sequential continuity [30] by weakening the concept of reciprocal continuity introduced by Pant[28].

Definition 2.10. Two self maps S and T on a 2-metric space (X,d) are called reciprocal continuous if $\lim STx_n = St$ and $\lim TSx_n = Tt$ for some t ∈ X whenever $\{x_n\}$ is a sequence in X such that $\lim Sx_n = \lim Tx_n = t$ ∈ X as n→ ∞.

Definition 2.11. Two self maps S and T on a 2-metric space (X,d) are said to be sub sequentially continuous if and only if there exists a sequence $\{x_n\}$ in X such that $\lim Sx_n = \lim Tx_n = t$ ∈ X as n→ ∞ for some t ∈ X and satisfy $\lim STx_n = St$ and $\lim TSx_n = Tt$ as n→ ∞.

Remark 2.12. If S and T are both continuous and reciprocally continous than they are obviously sub sequentially continuous.

The next example shows that there exist sub sequentially continous pairs of maps which are neither continuous nor reciprocally continuous.

Example 2.13. (example 3 of [30]) Let X=R be the set of real numbers. We define 'd' as

$$d(x,y,z) = \begin{cases} 0, \; if at least two of the three points are equal \\ 2, \quad otherwise \end{cases}$$

Then clearly (X,d) is a 2 metric space.

Define A,B : X → X as

$Ax = \begin{cases} 2, & x < 3 \\ x, & x \ge 3 \end{cases}$, $Bx = \begin{cases} 2x-4, & x \le 1 \\ 3, & x > 1 \end{cases}$.

Consider a sequence $x_n = \left(3+\frac{1}{n}\right)$ then A and B are not reciprocally continous but if we consider a sequence $x_n = \left(3-\frac{1}{n}\right)$ then A and B are sub sequenctially continous.

Implicit Relations: Let F be the set of all continuous functions $F : R_+^6 \to R$ satisfying the following conditions:

(3.1) F is non-increasing in variables t_5 and t_6.

(3.2) there exists h ∈ (0,1) such that for u, v ≥ 0 with

(3.3) F (u, v, v, u, u + v, 0) ≤ 0 or

(3.4) F (u, v, u, v, 0, u + v) ≤ 0 implies u ≤ h.v.

(3.5) F (u, u, 0,0, u, u) >0 for all u >0.

The following examples of such functions F satisfying (3.1), (3.2) and (3.5) are available in [27] with verifications and other details.

Example 3.6: Define $F(t_1, t_2,\ldots,t_6) : R_+^6 \to R$ as

$F(t_1, t_2,\ldots,t_6) = t_1 - k \max\{ t_2, t_3, t_4, \frac{1}{2}(t_5 + t_6) \}$, where k ∈ (0,1).

Example 3.7:. Define $F(t_1, t_2,\ldots,t_6) : R_+^6 \to R$ as

$F(t_1, t_2,\ldots,t_6) = t_1^2 - t_1(\alpha t_2 + \beta t_3 + \gamma t_4) - \eta t_5 t_6$,

where α > 0; β, γ, η ≥ 0 ; α + β + γ < 1 and α + η < 1.

Example 3.8: Define $F(t_1, t_2,\ldots,t_6) : R_+^6 \to R$as

$F(t_1, t_2,\ldots,t_6) = t_1^3 - \alpha t_1^2 t_2 - \beta t_1 t_3 t_4 - \gamma t_5^2 t_6 - \eta t_5 t_6^2$,

where α > 0; β, γ, η ≥ 0 ; α + β < 1 and α + γ + η <1.

Example 3.9: Define $F(t_1, t_2,\ldots,t_6) : R_+^6 \to R$ as

$F(t_1, t_2,\ldots,t_6) = t_1^3 - \alpha \frac{t_3^2 t_4^2 + t_5^2 t_6^2}{1+t_2+t_3+t_4}$, where α ∈ (0,1).

Example 3.10: Define $F(t_1, t_2,\ldots,t_6) : R_+^6 \to R$ as

$F(t_1, t_2,\ldots,t_6) = t_1^2 - \alpha t_2^2 - \beta \frac{t_5 t_6}{1+t_3^2+t_4^2}$

Example 3.11: Define $F(t_1, t_2,\ldots,t_6) : R_+^6 \to R$ as

$$F(t_1, t_2,\ldots,t_6) = \begin{cases} t_1 - a_1 \frac{t_3^2+t_4^2}{t_3+t_4} - a_2 t_2 - a_3(t_5+t_6), & \text{if } t_3+t_4 \neq 0 \\ \quad t_1, & \text{if } t_3+t_4 = 0 \end{cases}$$

Where $a_i \geq 0$ with at least one a_i non zero and $a_1 + a_2 + 2a_3 < 1$. (3.1): Obvious.

(3.2)((3.3)): Let u >0, $F(u,v,v,u,u+v,0) = u - a_1(v^2 + u^2)/(v+u) - a_2 v - a_3(u+v) \leq 0$. If $u \geq v$, then $u \leq (a_1 + a_2 + 2a_3)u < u$ which is a contradiction. Hence $u < v$ and $u \leq hv$ where $h \in (0, 1)$.

(3.4): Similar argument as in (3.3).

(3.5): $F(u, u, 0, 0, u, u) = u > 0$ for all u >0.

We also add the following examples [26] without verification.

Example 3.12: Define $F(t_1, t_2,...,t_6) : R_+^6 \to R$ as

$$F(t_1, t_2,...,t_6) = \begin{cases} t_1 - \alpha t_2 - \frac{\beta t_3 t_4 + \gamma t_5 t_6}{t_3 + t_4}, & \text{if } t_3 + t_4 \neq 0 \\ t_1, & \text{if } t_3 + t_4 = 0 \end{cases}$$

Where $\alpha, \beta, \gamma \geq 0$ such that $1 < 2\alpha + \beta < 2$.

Example 3.13: Define $F(t_1, t_2,...,t_6) : R_+^6 \to R$ as

$F(t_1, t_2,...,t_6) = t_1 - a_1 t_2 - a_2 t_3 - a_3 t_4 - a_4 t_5 - a_5 t_6$ where $\sum_{i=1}^{5} a_i < 1$

Example 3.14: Define $F(t_1, t_2,...,t_6) : R_+^6 \to R$ as

$F(t_1, t_2,...,t_6) = t_1 - \alpha[\beta \max\{t_2, t_3, t_4, \frac{1}{2}(t_5+t_6)\} + (1-\beta)[\max\{t_2^2, t_3 t_4, t_5 t_6, \frac{t_3 t_6}{2}, \frac{t_4 t_5}{2}\}]^{\frac{1}{2}}]$, where $\alpha \in (0, 1)$ and $0 \leq \beta \leq 1$.

Example 3.15: Define $F(t_1, t_2,...,t_6) : R_+^6 \to R$ as

$F(t_1, t_2,...,t_6) = t_1^2 - \alpha \max\{t_2^2, t_3^2, t_4^2\} - \beta \max\{\frac{t_3 t_5}{2}, \frac{t_4 t_6}{2}\} - \gamma t_5 t_6$.

Where $\alpha, \beta, \gamma \geq 0$ and $\alpha + \beta + \gamma < 1$.

Popa et al[26], slightly modified(3.1) as follows:

(3.1)' F is decreasing in variables $t_2, ..., t_6$.

Hereafter, let $F : R_+^6 \to R$ be a continuous function which satisfy the conditions (3.1)',(3.2) and (3.5) and let ψ be the family of such functions F. Some examples of [27].

Example 3.16: Define $F(t_1, t_2,...,t_6) : R_+^6 \to R$ as

$F(t_1, t_2,...,t_6) = t_1 - \phi(\max\{t_2, t_3, t_4, \frac{1}{2}(t_5+t_6)\})$

Where $\phi : R^+ \to R^+$ is an increasing upper semi continuous function with $\phi(0) = 0$ and $\phi(t) < t$ for each t >0.

(3.1)': Obvious.

(3.2)((3.3)): Let u >0. $F(u, v, v, u, u + v, 0) = u - \phi(\max\{v, v, u, (u+v)/2\}) \leq 0$.

If $u \geq v$, then $u \leq \phi(u) < u$ which is a contradiction. Hence $u < v$ and $u \leq hv$

Where $h \in (0, 1)$.

(3.4): Similar argument as in (3.3).

(3.5): $F(u, u, 0, 0, u, u) = u - \phi(\max\{u, 0, 0, (u+u)/2\}) = u - \phi(u) > 0$ for all u >0.

Example 3.17: Define $F(t_1, t_2,...,t_6) : R_+^6 \to R$ as

$F(t_1, t_2,...,t_6) = t_1 - \phi(t_2, t_3, ..., t_6)$

Where $\phi : R_+^5 \to R^+$ is an upper semi continuous and non decreasing function in each coordinate variable such that $\phi(t, t, \alpha t, \beta t, \gamma t) < t$ for each t >0 and $\alpha, \beta, \gamma \geq 0$ with $\alpha + \beta + \gamma \leq 3$.

Example 3.18: Define $F(t_1, t_2,...,t_6) : R_+^6 \to R$ as

$F(t_1, t_2,...,t_6) = t_1^2 - \phi(t_2^2, t_3 t_4, t_5 t_6, t_3 t_6, t_4 t_5)$

Where $\phi : R_+^5 \to R^+$ is an upper semi continuous and non decreasing function in each coordinate variable such that $\phi(t, t, \alpha t, \beta t, \gamma t) < t$ for each t >0 and $\alpha, \beta, \gamma \geq 0$ with $\alpha + \beta + \gamma \leq 3$.

Results and Discussion

Now, we prove our main result.

4.1Theorem: Let L, M, A, B, S and T be self mappings of a 2-metric space (X, d) satisfying the conditions

(4.1.1) $F(d(Lx,My, a), d(ABx, STy, a), d(ABx, Lx, a), d(STy,My, a), d(ABx,My, a), d(Sty, Lx, a)) \leq 0$

for all $x,y \in X$ and for all $a \in X$, where F enjoys the property (3.5) and

(4.1.2) $L(X) \subseteq ST(X)$ and $M(X) \subseteq AB(X)$.

If one of L(X),M(X),AB(X) or ST(X) is a complete subspace of X, then

(4.1.3) the pair (L,AB) has a point of coincidence,

(4.1.4) the pair (M, ST) has a point of coincidence.

Moreover, L,M,AB and ST have a unique common fixed point provided (L, AB) and (M, ST) are sub compatible and sub sequentially continuous mappings.

Further if (A, B), (S,T), (M,T), (L,T),(M,B) are commuting maps, then L, M, A,B,S and T have a unique common fixed point in X.

Proof. Since $L(X) \subseteq ST(X)$, for arbitrary point $x_0 \in X$ there exists a point $x_1 \in X$ such that $Lx_0 = STx_1$.

Since $M(X) \subseteq AB(X)$, for the point x_1, we can choose a point $x_2 \in X$ such that $Mx_1 = ABx_2$ and so on.

Inductively, we can define a sequence $<y_n>$ in X such that(4.1.5) $y_{2n} = Lx_{2n} = STx_{2n+1}$ and $y_{2n+1} = Mx_{2n+1} = ABx_{2n+2}$; n = 0,1,2, ...

Now we will prove

(4.1.6) $d(y_n, y_{n+1}, y_{n+2}) = 0$ for every $n \in N$;

(4.1.7) $d(y_i, y_j, y_k) = 0$ for $i, j, k \in N$,

where $\langle y_n \rangle$is a sequence described by (4.1.5).

(a) From (4.1.1)and (4.1.5), we have

(4.1.8) $F(d(Lx_{2n+2}, Mx_{2n+1}, y_{2n}), d(ABx_{2n+2}, STx_{2n+1}, y_{2n}), d(ABx_{2n+2}, Lx_{2n+2}, y_{2n}), d(STx_{2n+1}, Mx_{2n+1}, y_{2n}), d(ABx_{2n+2}, Mx_{2n+1}, y_{2n}), d(STx_{2n+1}, Lx_{2n+2}, y_{2n})) \leq 0$

or $F(d(y_{2n+2}, y_{2n+1}, y_{2n}), d(y_{2n+1}, y_{2n}, y_{2n}), d(y_{2n+1}, y_{2n+2}, y_{2n}), d(y_{2n}, y_{2n+1}, y_{2n}), d(y_{2n+1}, y_{2n+1}, y_{2n}), d(y_{2n}, y_{2n+2}, y_{2n})) \leq 0$ or $F(d(y_{2n+2}, y_{2n+1}, y_{2n}), 0, d(y_{2n+2}, y_{2n+1}, y_{2n}), 0, 0, 0) \leq 0$ or $F(d(y_{2n+2}, y_{2n+1}, y_{2n}), 0, d(y_{2n+2}, y_{2n+1}, y_{2n}), 0, 0, d(y_{2n+2}, y_{2n+1}, y_{2n})) \leq 0$

yielding thereby $d(y_{2n+2}, y_{2n+1}, y_{2n}) = 0$ (due to (3.4)).

Similarly, using (3.3) we can show that $d(y_{2n+1}, y_{2n}, y_{2n-1}) = 0$.

Thus it follows that $d(y_n, y_{n+1}, y_{n+2}) = 0$ for every $n \in N$.

(b) For all $a \in X$, let us write

$d_n = d(y_n, y_{n+1}, a)$, n = 0, 1, 2,....

First we shall prove that $<d_n>$ is a non-decreasing sequence in R^+. From (4.1.1) for $x = x_{2n}$, $y = x_{2n+1}$and (4.1.5), we have

(4.1.9) $F(d(Lx_{2n}, Mx_{2n+1}, a),$
$d(ABx_{2n}, STx_{2n+1}, a),$
$d(ABx_{2n}, Lx_{2n}, a),$
$d(STx_{2n+1}, Mx_{2n+1}, a),$
$d(ABx_{2n}, Mx_{2n+1}, a),$
$d(STx_{2n+1}, Lx_{2n}, a)) \le 0,$

or $F(d(y_{2n}, y_{2n+1}, a), d(y_{2n-1}, y_{2n}, a),$
$d(y_{2n-1}, y_{2n}, a), d(y_{2n}, y_{2n+1}, a),$
$d(y_{2n-1}, y_{2n+1}, a), d(y_{2n}, y_{2n}, a)) \le 0,$

or $F(d(y_{2n}, y_{2n+1}, a), d(y_{2n-1}, y_{2n}, a),$
$d(y_{2n-1}, y_{2n}, a), d(y_{2n}, y_{2n+1}, a),$
$d(y_{2n-1}, y_{2n+1}, y_{2n}) + d(y_{2n-1}, y_{2n}, a) + d(y_{2n+1}, y_{2n}, a), 0) \le 0$

or $F(d(y_{2n}, y_{2n+1}, a), d(y_{2n-1}, y_{2n}, a),$
$d(y_{2n-1}, y_{2n}, a), d(y_{2n}, y_{2n+1}, a),$
$d(y_{2n-1}, y_{2n}, a) + d(y_{2n}, y_{2n+1}, a), 0) \le 0$

implying thereby $d_{2n} \le hd_{2n-1} < d_{2n-1}$ (due to (3.3)).
Similarly using (3.4), we have $d_{2n+1} \le hd_{2n}$.
Thus $d_{n+1} < d_n$ for n = 0, 1, 2, ...
Now proceeding on the lines of the proof of Lemma 3.2 [29, p.355], we can show that
$d(y_i, y_j, y_k) = 0$ for $i, j, k \in N$.
we have $d_{2n+1} \le hd_{2n}$ and
$d_{2n} \le hd_{2n-1}$.
Therefore, we obtain $d_n \le h^n d_0$.
Hence $\lim d(y_n, y_{n+1}, a) = \lim d_n = 0$.
Since $\lim d(y_n, y_{n+1}, a) = 0$,it is sufficient to show that a subsequence $<y_{2n}>$ of $<y_n>$ is a Cauchy sequence in X.
Suppose that $<y_{2n}>$ is not a Cauchy sequence in X.
Then for every $\in > 0$ there exists $a \in X$ and strictly increasing sequences $<m_k>$, $<n_k>$ of positive integers such that $k \le n_k < m_k$ with
$d(y_{2n_k-1}, y_{2m_k}, a) \ge \in$ and
$d(y_{2n_k}, y_{2m_k-2}, a) < \in$.
Now proceeding on the lines of the proof of Lemma 1.3[11] (or Lemma 3.3[29]), we obtain (4.1.10)
$\lim d(y_{2n_k}, y_{2m_k}, a) = \in,$
$\lim d(y_{2n_k}, y_{2m_k-1}, a) = \in,$
$\lim d(y_{2n_k+1}, y_{2m_k}, a) = \in$ and
$\lim d(y_{2n_k+1}, y_{2m_k-1}, a) = \in.$
Now using (4.1.1) for $x = x_{2m_k}$,
$y = x_{2n_k+1}$, and (4.1.5), we have

(4.1.11) $F(d(Lx_{2m_k}, Mx_{2n_k+1}, a),$
$d(ABx_{2m_k}, STx_{2n_k+1}, a),$
$d(ABx_{2m_k}, Lx_{2m_k}, a),$
$d(Mx_{2n_k+1}, STx_{2n_k+1}, a),$
$d(ABx_{2m_k}, Mx_{2n_k+1}, a),$
$d(STx_{2n_k+1}, Lx_{2m_k}, a) \le 0$

or $F(d(y_{2m_k}, y_{2n_k+1}, a),$
$d(y_{2m_k-1}, y_{2n_k}, a),$
$d(y_{2m_k-1}, y_{2m_k}, a),$
$d(y_{2n_k}, y_{2n_k+1}, a),$
$d(y_{2m_k-1}, y_{2n_k+1}, a),$
$d(y_{2n_k}, y_{2m_k}, a) \le 0.$

Letting $n \to \infty$ and using (4.1.10), we have
$F(\in,\in,0,0,\in,\in) \le 0$, which is a contradiction to (3.5).
Therefore $<y_{2n}>$ is a Cauchy sequence.
Now we have $<y_n>$ is a Cauchy sequence in X.
Suppose that AB(X) is a complete subspace of X, then the subsequence $<y_{2n+1}>$which is contained in AB(X) must have a limit z in AB(X).
As $<y_n>$is a Cauchy sequence containing a convergent subsequence $<y_{2n+1}>$, therefore $<y_n>$also converges implying thereby the convergence of the subsequence $<y_{2n}>$, i.e.,

(4.1.12) $\lim_{n\to\infty} Lx_{2n} = \lim_{n\to\infty} Mx_{2n+1} = \lim_{n\to\infty} STx_{2n+1} = \lim_{n\to\infty} Bx_{2n+2} = z.$

Let $u \in (AB)^{-1}(z)$, then ABu= z.
If Lu $\ne$z, then using (4.1.1) for x = u and y = $_{2n+1}$and (4.1.5), we have

(4.1.13) $F(d(Lu, Mx_{2n+1}, a), d(ABu, STx_{2n+1}, a), d(ABu, Lu, a),$
$d(STx_{2n+1}, Mx_{2n+1}, a), d(ABu, Mx_{2n+1}, a), d(STx_{2n+1}, Lu, a)) \le 0$

which on letting $n \to \infty$ and using (4.1.12) then (4.1.13)reduces to
$F(d(Lu, z, a), d(z, z, a), d(z, Lu, a), d(z, z, a), d(z, z, a), d(z, Lu, a)) \le 0$
or$F(d(Lu, z, a), 0, d(z, Lu, a), 0, 0, d(z, Lu, a)) \le 0$
implying thereby $d(z, Lu, a) = 0$ for all $a \in X$ (due to (3.4)).
Hence z = Lu =ABu.
Since $L(X) \subseteq ST(X)$, there exists $v \in (ST)^{-1}(z)$ such that STv= z.
By (4.1.1) for x = u, y = vand (4.1.13) we have

(4.1.14) $F(d(Lu, Mv, a), d(ABu, STv, a), d(ABu, Lu, a), d(STv, Mv, a),$
$d(ABu, Mv, a), d(STv, Lu, a)) \le 0$

or$F(d(z, Mv, a), 0, 0, d(z, Mv, a), d(z, Mv, a), 0) \le 0$
yielding thereby $d(z, Mv, a) = 0$ for all $a \in X$ (due to (3.3)).
Therefore z = Mv.
Hence Lu = ABu= Mv= STv= z which establishes (4.1.3) and (4.1.4).
If one assumes that ST(X) is a complete subspace of X, then analogous arguments establish (4.1.3) and (4.1.4). The remaining two cases also pertain essentially to the previous cases. Indeed, if L(X) is complete, then $z \in L(X) \subseteq ST(X)$. Similarly if M(X) is complete,
then $z \in M(X) \subseteq S(X)$. Thus in all cases, (4.1.3) and (4.1.4) are completely established.
Since the pair (L, AB) is sub compatible and sub sequentially continuous, therefore, there exist two sequences $\{x_{2n}\}$ and $\{x_{2n+1}\}$ in X such that
(4.1.15) $\lim Lx_{2n} = \lim ABx_{2n} = u$ as $n\to \infty$ X and which

for some $u \in$ satisfy

$\lim d(L(AB)\, x_{2n}, (AB)L\, x_{2n}, a) = d(Lu, ABu, a) = 0$, for all $a \in X$,

which gives $Lu = ABu$.

i.e., u is the coincidence point of L and AB

Since the pair (M, ST) is sub compatible and sub sequentially continuous,

(4.1.16) $\lim Mx_{2n+1} = \lim STx_{2n+1} = v$ for some $v \in X$ and which satisfy

$\lim d(M(ST)\, x_{2n+1}, (ST)M\, x_{2n+1}, a) = d(Mv, STv, a) = 0$, for all $a \in X$,

Therefore $Mv = STv$ i.e., v is a coincidence point of M and ST.

Now using (4.1.1) for $x = x_{2n}$ and $y = x_{2n+1}$, we get

(4.1.17) $F(d(Lx_{2n}, Mx_{2n+1}, a), d(ABx_{2n}, STx_{2n+1}, a), d(ABx_{2n}, Lx_{2n}, a),$
$d(STx_{2n+1}, Mx_{2n+1}, a), d(ABx_{2n}, Mx_{2n+1}, a), d(STx_{2n+1}, Lx_{2n}, a)) \le 0$

Letting $n \to \infty$ and using (4.1.15) we have

$F(d(u, v, a), d(u,v,a), 0, 0, d(u,v,a), d(u,v,a)) \le 0$

Yielding thereby $d(u, v, a) = 0$, for all $a \in X$, (due to (3.4)),

Therefore $u = v$

Again using (4.1.1) for $x = u$, $y = x_{2n+1}$ we obtain

(4.1.18) $F(d(Lu, Mx_{2n+1}, a), d(ABu, STx_{2n+1}, a), d(ABu, Lu, a),$
$d(STx_{2n+1}, Mx_{2n+1}, a), d(ABu, Mx_{2n+1}, a), d(STx_{2n+1}, Lu, a)) \le 0$

Letting $n \to \infty$ and using (4.1.15) we have

$F(d(Lu, v, a), d(Lu,v,a), 0, 0, d(Lu,v,a), d(Lu,v,a)) \le 0$

Yielding thereby $d(Lu, v, a) = 0$, for all $a \in X$, (due to (3.4)),

Therefore $Lu = v = u$.

Therefore, $u = v$ is a common fixed point of L, M, AB and ST.

For uniqueness, let $w \neq u$ be another fixed point of L, M, AB and ST.

Then from (4.1.1), we have

(4.1.19) $F(d(Lu, Mw, a), d(ABu, STw, a), d(ABu, Lu, a), d(STw, Mw, a), d(ABu, Mw, a), d(STw, Lu, a)) \le 0$

$F(d(u, w, a), 0, 0, d(u, w, a), d(u, w, a)) \le 0$

which yielding thereby $u = w$, so AB, ST, L and M have a unique common fixed point in X.

Finally we need to show that u is a common fixed point of A,B,S,T,L and M.

Since (A,B), (A,L), (B,L) are commutative

$Au = A(ABu) = A(BAu) = (AB)(Au)$; $Au = ALu = LAu$

$Bu = B(ABu) = (BA)(Bu) = (AB)(Bu)$; $Bu = BLu = LBu$.

Which shows that Au, Bu are common fixed points of (AB, L) yielding there by $Au = u = Bu = Lu = ABu$ in the view of uniqueness of common fixed point of the pairs (AB, L).

Similarly using the, commutativity of (S, T), (S, M) and (T, M) it can be shown that

$Su = u = Tu = Mu = STu$. Now, we need to show that $Au = Su$ ($Bu = Tu$) also remains acommonfixed point of both the pairs (AB, L) and (ST, M).

From (4.1.1) we have

(4.1.20) $F(d(Lu, Mu, a), d(ABu, STu, a), d(Lu, ABu, a)\, d(Mu, STu, a),$
$d(ABu, Mu, a), d(STu, Lu, a)) \le 0$

$F(d(Lu,Mu,a), d(Lu,Mu,a), 0, 0, d(Lu,Mu,a), d(Lu,Mu,a)) \le 0$ Which yields $Lu = Mu$,

Hence $Au = Su$.

Similarly it can be shown that $Bu = Tu$.

Thus u is the unique common fixed point of A,B,S,T,Land M.

4.1.21 Corollary : The conclusions of Theorem 4.1 remain true if (for all $x, y, a \in X$) implicit relation (4.1.1) is replaced by any one of the following.

(4.1.22) $d(Lx, my, a) \le k \max\{ d(ABx, STy, a), d(ABx, Lx, a), d(STy, My, a), \frac{1}{2}(d(ABx, My, a) + d(STy, Lx, a))\}$, where $k \in (0,1)$.

(4.1.23) $d^2(Lx, My, a) \le d(Lx, My, a)[\alpha d(ABx, STy, a) + \beta d(ABx, Lx, a) + \gamma d(STy, My, a)] + \eta d(ABx, My, a).d(STy, Lx, a)$

where $\alpha > 0$; $\beta, \gamma, \eta \ge 0$, $\alpha + \beta + \gamma < 1$ and $\alpha + \eta < 1$.

(4.1.24) $d^3(Lx, My, a) \le \alpha d^2(Lx, My, a) d(ABx, STy, a) + \beta d(Lx, My, a) d(ABx, Lx, a) d(STy, My, a) + \gamma d^2(ABx, My, a) d(STy, Lx, a) + \eta d(ABx, My, a) d^2(STy, Lx, a)$

where $\alpha > 0$; $\beta, \gamma, \eta \ge 0$, $\alpha + \beta < 1$ and $\alpha + \eta + \gamma < 1$.

(4.1.25) $d^3(Lx, My, a) \le \alpha \frac{d^2(ABx,Lx,a) d^2(STy,My,a) + d^2(ABx,Ly,a) d^2(STy,Lx,a)}{1 + d(ABx,STy,a) + d(ABx,Lx,a) + d(STy,My,a)}$

Where $\alpha \in (0, 1)$.

(4.1.26) $d^2(Lx, My, a) \le \alpha d^2(ABx, STy, a) + \beta \frac{d(ABx,My,a) d(STy,Lx,a)}{1 + d^2(ABx,Lx,a) + d^2(STy,My,a)}$

where $\alpha > 0$, $\beta \ge 0$ and $\alpha + \beta < 1$.

(4.1.27) $d(Lx, My, a) \le a_1 \frac{d^2(ABx,Lx,a) + d^2(STy,My,a)}{d(ABx,Lx,a) + d(STy,My,a)} + a_2\, d(ABx, STy, a) + a_3(d(ABx, My, a) + d(STy, Lx, a))$

where $a_i \ge 0$ with at least one a_i non zero and $a_1 + a_2 + 2a_3 < 1$.

(4.1.28) $d(Lx, My, a) \le \alpha d(ABx, STy, a) + \frac{\beta d(ABx,Lx,a) d(STy,My,a) + \gamma d(ABx,My,a) d(STy,Lx,a)}{d(ABx,Lx,a) + d(STy,My,a)}$

where $\alpha, \beta, \gamma \ge 0$ such that $1 < 2\alpha + \beta < 2$.

(4.1.29) $d(Lx, My, a) \le a_1 d(ABx, STy, a) + a_2 d(ABx, Lx, a) + a_3 d(STy, My, a) + a_4 d(ABx, My, a) + a_5 d(STy, Lx, a)$, where $\sum_{i=1}^{5} a_i < 1$

(4.1.30) $d(Lx, My, a) \le \alpha[\, \beta \max\{d(ABx, STy, a), d(ABx, Lx, a), d(STy, My, a), \frac{1}{2}(d(ABx, My, a) + d(STy, Lx, a))\} + (1-\beta)$
$[\max d^2(ABx, STy, a), d(ABx, Lx, a) d(STy, My, a), d(ABx, My, a) d(STy, Lx, a), \frac{d(ABx,Lx,a) d(STy,Lx,a)}{2}, \frac{d(STy,My,a) d(ABx,My,a)}{2}\}]^{\frac{1}{2}}]$

where $\alpha \in (0, 1)$ and $0 \le \beta \le 1$.

(4.1.31) $d^2(Lx, My, a) \le \alpha \max\{d^2(ABx, STy, a),$

d^2(ABx, Lx,a), d^2(STy,My,a)}+βmax{ $\frac{d(ABx,Lx,a)d(ABx,My,a)}{2}, \frac{d(STy,My,a)d(STy,Lx,a)}{2}$}
+ γd(ABx,My, a)d(STy, Lx, a)
whereα, β, γ ≥ 0 and α + β + γ <1.

(4.1.32) d(Lx,My, a) ≤ ϕ (max{d(ABx, STy, a), d(ABx, Lx, a), d(STy,My, a),$\frac{1}{2}$[d(ABx,My, a) + d(STy,Lx,a)]})
whereϕ : $R^+ \rightarrow R^+$ is an upper semicontinuous and increasing function with ϕ (0) = 0 and ϕ (t) < t for each t >0.

(4.1.33) d(Lx,My, a) ≤ ϕ (d(ABx, STy, a), d(ABx, Lx, a), d(STy,My, a), d(ABx,My, a), d(STy,Lx, a))
whereϕ :$R_+^5 \rightarrow R^+$ is an upper semicontinuous and nondecreasing function in each coordinate variable such that ϕ (t, t, αt, βt, γt) < t for each t >0 and α, β, γ ≥ 0 withα + β + γ ≤ 3

(4.1.34) d^2(Lx,My, a) ≤ ϕ (d^2(ABx, STy, a), d(ABx, Lx, a)d(STy,My, a), d(ABx,My, a)d(STy,Lx, a),d(ABx, Lx, a)d(STy, Lx, a), d(STy, My,a)d(ABx,My, a))
whereϕ : $R_+^5 \rightarrow R^+$ is an upper semicontinuous and nondecreasing function in each coordinate variable such that ϕ (t, t, αt, βt, γt) < t for each t >0 and α, β, γ ≥ 0 withα + β + γ ≤ 3

Proof. The proof follows from Theorem 4.1 and Examples 3.6 to 3.18 the above.

Simillary we get the results in three self maps and two self maps by taking L=M, T=B=I_X; A=S, T=B=I_X and L=M, A=S, T=B=I_X in corollary 4.1.21 respectively.

Put B = T = I_x in corollary 4.1.21, we get result in four self maps.

If we put B = T= I_x where I_x identity self map on X in theorem 4.1 then we get the following result.

4.1.35 Corollary:Let L,M,A and S be self mappings of a 2-metric space
(X, d) satisfying the conditions
F(d(Lx,My, a), d(Ax, Sy, a), d(Ax, Lx, a),d(Sy,My, a), d(Ax,My, a),
d(Sy, Lx, a)) ≤ 0
for all x,y∈ X and for all a ∈ X, where F enjoys the property (3.5) and
L(X) ⊆ S (X) and M(X) ⊆ A (X).
If one of L(X),M(X),A (X) or S (X) is a complete subspace of X, then

(4.1.36) the pair (L,A) has a point of coincidence,

(4.1.37) the pair (M, S) has a point of coincidence.

Moreover, L,M,A and Shave a unique common fixed point provided both the pairs (L, A) and (M, S) is sub compatible and sub sequentially continuous mappings. Taking L= M and T = B = I_X in theorem 4.1 we get the following corollary.

4.1.38 Corollary: Let L,A and S be self mappings of a 2-metric space (X, d) satisfying the conditions
F(d(Lx,Ly, a), d(Ax, Sy, a), d(Ax, Lx, a),d(Sy,Ly, a), d(Ax,Ly, a),
d(Sy, Lx, a)) ≤ 0 for all x,y∈ X and for all a ∈ X, where F enjoys the property (3.5) and L(X) ⊆ S (X) and L(X) ⊆ A (X). If one of L(X),A (X) or S (X) is a complete subspace of X, then

(4.1.39) the pair (L,A) has a point of coincidence,

(4.1.40) the pair (L, S) has a point of coincidence.

Moreover, L, A and Shave a unique common fixed point provided both the pairs (L, A) and
(L, S) is subs compatible and sub sequentially continuous mappings.

Taking A= S and T = B = I_X in theorem 4.1 we get the following corollary.

4.1.41 Corollary:Let L,M and S be self mappings of a 2-metric space (X, d) satisfying the conditions
F(d(Lx,My, a), d(Sx, Sy, a), d(Sx, Lx, a), d(Sy,My, a), d(Sx,My, a),
d(Sy, Lx, a)) ≤ 0
for all x,y∈ X and for all a ∈ X, where F enjoys the property (3.5) and L(X) ⊆ S (X) and M(X) ⊆ S (X).
If one of L(X),M(X) or S (X) is a complete subspace of X, then

(4.1.42) the pair (L,S) has a point of coincidence,

(4.1.43) the pair (M, S) has a point of coincidence.

Moreover, L,M and Shave a unique common fixed point provided either the pair (L, S) and (M,S) is sub compatible and sub sequentially continuous mappings.Taking L= M, A = S and T = B = I_X in theorem 4.1 we get the following corollary.

4.1.44 Corollary:Let L and A be self mappings of a 2-metric space (X, d) satisfying the conditions
F(d(Lx,Ly, a), d(Ax, Ay, a), d(Ax, Lx, a), d(Ay,Ly, a), d(Ax,Ly, a),
(Ay, Lx, a)) ≤ 0
for all x,y∈ X and for all a ∈ X, where F enjoys the property (3.5) and L(X) ⊆ A (X). If one of L(X) or A (X) is a complete subspace of X, then

(4.1.45) the pair (L,A) has a point of coincidence.

Moreover, L and Ahave a unique common fixed point provided the pair (L, A) is sub compatibleof sub sequentially continuous mappings.

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