# FUZZY CRITICAL PATH METHOD USING FUZZY TRIANGULAR NUMBERS

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**Abstract:** A new method is introduced for project scheduling in fuzzy environment. In this paper, we assume that the duration of activities are triangular fuzzy numbers. Here, we have to find the fuzzy earliest times and fuzzy project completion times by forward pass. We intend a new approach which we call fuzzy Modified Backward Pass (MBP) to compute the fuzzy latest times and fuzzy slack times. Due to these modification the intrinsic defects in the fuzzy environment is removed.

**Keywords:** Forward pass, Fuzzy slack times, Modified Backward Pass, Triangular Fuzzy Numbers.

**Introduction :** Scheduling is deemed to be one of the most fundamental and essential bases of the project management science.There are several methods for project scheduling such as CPM, PERT.
A fundamental approach to solve these problems is by applying fuzzy sets. Introducing the fuzzy set theory by Zadeh in 1965 opened promising new horizons to different scientific areas such as project scheduling. Fuzzy theory, with presuming imprecision in decision parameters and utilizing mental models of experts is an approach to adapt scheduling models into reality. To this end, several methods have been developed during the last three decades.
In this paper, a new method is introduced for project scheduling in fuzzy environment. This method is developed based on a number of assumptions and definitions in the fuzzy set and project scheduling. In the fuzzy project network considered in this paper, we assume that the duration of activities are triangular fuzzy numbers. The project characteristics such as fuzzy earliest times and fuzzy project completion time are calculated as triangular fuzzy number by forward pass. We propose a new approach which we call fuzzy Modified Backward Pass(MBP) to compute the fuzzy latest times and fuzzy slack times.
The advantage of MBP approach in that it does not use the fuzzy subtraction operator in its relations. Due to these modifications the inherent defects in the fuzzy environment is removed. Therefore, the obtained fuzzy latest times and fuzzy slack times in the MBP approach are correct and calculated, as well Finally , through a numerical example , calculation steps in this approach and results are illustrated.

**Preliminaries**

**Definition 2.1**

Let R be the space of real numbers. A fuzzy set $\widetilde{A}$ is a set of ordered pairs {x, $\mu_{\widetilde{A}}$ (x)|x$\in$R), where $\mu_{\widetilde{A}}$ (x):R→[0,1] and is upper semi continuous function $\mu_{\widetilde{A}}$(x) is called Membership function of fuzzy set.

**Definition 2.2**

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A convex fuzzy set $\widetilde{A}$ is a fuzzy set in which: $\mu_{\widetilde{A}}(\lambda x+(1-\lambda)y) \geq \min[\mu_{\widetilde{A}}(x), \mu_{\widetilde{A}}(y)]$ $\forall$ x,y $\in$ R, $\forall$ λ$\in$[0,1]

**Definition 2.3 :** A fuzzy set $\widetilde{A}$ is called positive if its membership function is such that $\mu_{\widetilde{A}}$(x)=0, $\forall$ x≤0

**Definition 2.4 :** A concave fuzzy set $\widetilde{A}$ is a fuzzy set in which $\forall$ x,y $\in$ R, $\forall$ λ$\in$[0,1]
$\mu_{\widetilde{A}}(\lambda x+(1-\lambda)y) \leq \max[\mu_{\widetilde{A}}(x), \mu_{\widetilde{A}}(y)]$.

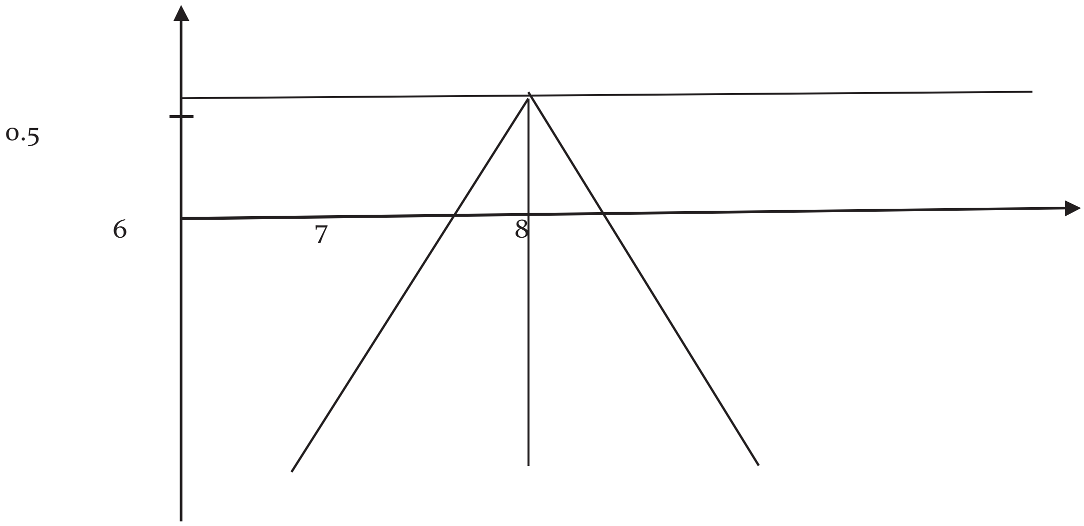
**Definition 2.5 :** The α-cut of $\mu_{\widetilde{A}}$ is a set consisting of those element x whose membership value exceed α. $\mu_{\widetilde{A}\alpha}$={x/ $\mu_{\widetilde{A}}$(x) ≥ α}. If α=1 the $\mu_{\widetilde{A}\alpha}$ determines a set of x totally belonging to $\mu_{\widetilde{A}}$ clearly lower the level of α, more the elements are admitted to corresponding to α-cut. (i.e.); if $\alpha_2 > \alpha_1$ then $\mu_{\widetilde{A}\alpha 2} < \mu_{\widetilde{A}\alpha 1}$
For a monotonically increasing sequence of α-cuts, we obtain a nested sequence of closed intervals corresponding to it is reverse order.

**Definition 2.6** A fuzzy set is a function $\mu$ :R→[0,1] with the following properties:

1. $\mu_{\widetilde{A}}$ is normal
2. $\mu_{\widetilde{A}}(\lambda x+(1-\lambda)y) \geq \min[\mu_{\widetilde{A}}(x), \mu_{\widetilde{A}}(y)]$, $\forall$ x,y $\in$ R, $\forall$ λ$\in$[0,1]
3. $\mu_{\widetilde{A}}$ is upper semi continuous on R (ie); $\forall$ $x_0 \in$ R and $\forall$ $\in$>0 there exists a
   Neighbourhood V($x_0$) such that $\mu_{\widetilde{A}}(x) \leq \mu_{\widetilde{A}}(x_0)$ +$\in$, $\forall$ x$\in$V($x_0$)
4. The set $\overline{\sup(\mu_{\widetilde{A}})}$ is compact in R, where

$\overline{\sup(\mu_{\widetilde{A}})}$ ={ x $\in$ R; $\mu_{\widetilde{A}}(x)$>0}.

Let us denote $R_F$ by the set of all fuzzy numbers. Here our main concentration lies only about a triangular number




Which is one form of fuzzy number 7? It is fuzzy number around 7.

In general let a, b, c $\in$ R, a<b<c. the fuzzy number $\mu_{\tilde{A}}$:R→[0, 1] denoted by [a,b,c] is defined by $\mu_{\tilde{A}}(x)$=0, x≤a or x≥ c

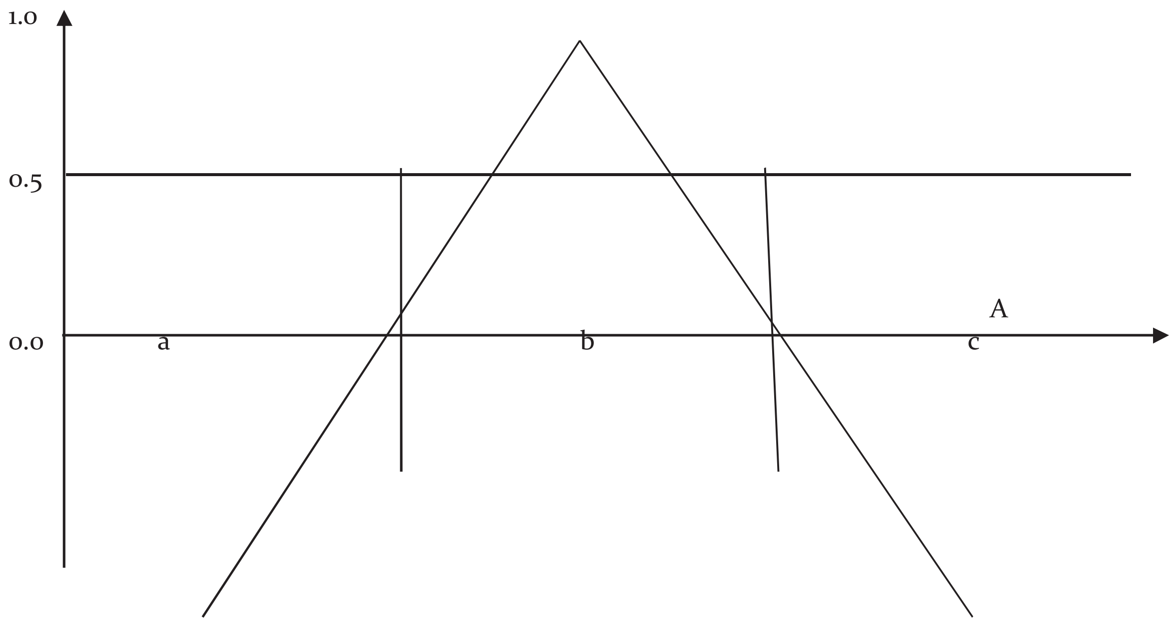
$\mu_{\tilde{A}}(x) = \dfrac{x-a}{b-a}$ if x$\in$ [a,b] and $\mu_{\tilde{A}}(x) = \dfrac{c-x}{c-b}$ if x$\in$ [b,c] is called a triangular fuzzy number.(ie); the membership function is defined as

$$\mu_{\tilde{A}}(x) = \begin{cases} 0 & \text{, for } x<a \\ \dfrac{x-a}{b-a} & \text{, for } a\le x\le b \\ \dfrac{c-x}{c-b} & \text{, for } b\le x\le c \\ 0 & \text{, for } x>c \end{cases} \quad \text{--------------------(1)}$$

The α-cuts of a triangular fuzzy numbers define a set of closed intervals.

The intervals are, [(b-a) α+a, (b-c) α+c] , $\forall$ α$\in$ [0,1]

Suppose α=0.5 then the closed interval associated is $\left[\dfrac{b+a}{2}, \dfrac{b+c}{2}\right]$ which is represented as follows:



**2.7 Operations on Triangular Fuzzy Numbers**

The operations on fuzzy triangular numbers are defined below:

Let A= $[a_1,b_1,c_1]$ and B=$[a_2,b_2,c_2]$ be any two triangular fuzzy numbers then :

**(1) Addition:**

$[a_1,b_1,c_1]+[a_2,b_2,c_2]=[a_1+a_2,\ b_1+b_2,\ c_1+c_2]$ ------------------>(2)

**(2) Subtraction:**

$[a_1,b_1,c_1]-[a_2,b_2,c_2]=[a_1-c_2,\ b_1-b_2,\ c_1-a_2]$ ------------------> (3)

**(3) Multiplication:**

$[a_1,b_1,c_1].\ [a_2,b_2,c_2]=[a_1a_2,\ b_1b_2,\ c_1c_2]$ ------------------> (4)

**(4) Division:**

The inverse of $[a_1,b_1,c_1]$ is

$$\left.\begin{array}{l}\dfrac{1}{[a_1+b_1+c_1]}=[1/c_1,\ 1/b_1,1/a_1],\text{ and so}\\[2ex] \dfrac{[a_1+b_1+c_1]}{[a_2+b_2+c_2]}=\left[\dfrac{a_1}{c_2},\dfrac{b_1}{b_2},\dfrac{c_1}{a_2}\right]\end{array}\right\}$$ ------------------->(5)

**III. Fuzzy Project Network :** A Network N=<V, A, D>, being a fuzzy project model , is given V is a set of nodes(events) and $A\subset V\times V$ is a set of arcs (activities). The network N is a directed and acyclic graph in the fuzzy environment. The set V={1,2,...,n} is labled in such a way that the following condition holds: $(i,j)\in A \Rightarrow i<j$. In the fuzzy environment the duration of this activity (D) is a positive triangular fuzzy numbers:

$\widetilde{D}_{ij}=[d_{ij}^1,d_{ij}^2,d_{ij}^3]$.

Let us denote by $P(j)=\{i\in V|(i,j)\in A\}$the set of predecessors and by $S(i)=\{j\in V|(i,j)\in A\}$the set of successor event $i\in V$ respectively. Starting time of the fuzzy project model is a positive triangular numbers:

$\widetilde{T}_s=[ts^1,ts^2,ts^3]$.

**3.1 Fuzzy Project Scheduling :** Fuzzy Project scheduling consists of the forward pass and modified backward pass (MBP) calculations to obtain the substantial project characteristics. In this section , for the fuzzy Project network, these characteristics such as earliest times, latest times and slack times are obtained carrying out the calculation as follows:

**3.2 Fuzzy forwardpass Calculations :** The earliest times and the Project completion time in Project network can be detected by forward pass. In this case using the relations of CPM in the fuzzy environment , results in the following fuzzy forward Calculations:

$$\widetilde{E}_j=[e_j^1,e_j^2,e_j^3]=\begin{cases}\underset{i\in p(j)}{M\widetilde{A}X}\{\widetilde{E}_i+\widetilde{D}_{ij}\} & ,P(j)\neq\Phi \\ \widetilde{T}_s=[ts^1,ts^2,ts^3] & ,P(j)=\Phi\end{cases}$$ -------------(6)

$E\widetilde{S}_{ij}=[es_{ij}^1,es_{ij}^2,es_{ij}^3]=\widetilde{E}_i$ -----------------(7)

$E\widetilde{F}_{ij}=[e\,E\widetilde{S}_{ij}=f_{ij}^1,ef_{ij}^2,ef_{ij}^3]=E\widetilde{S}_{ij}+\widetilde{D}_{ij}$ -----------------(8)

$\widetilde{T}_F=[t_F^1,t_F^2,t_F^3]=\underset{i\in V}{M\widetilde{A}X}\,\widetilde{E}_i$ -----------------(9)

In the above, $\widetilde{E}_j$ is the fuzzy earliest time of event j; $\widetilde{T}_s$ is the fuzzy time of starting the project; $E\widetilde{S}_{ij}$ is the fuzzy earliest starting of activity (i,j); $E\widetilde{F}_{ij}$ is the fuzzy earliest finishing of activity (I,j) and $\widetilde{T}_F$ is the fuzzy time of project completion.

Based on the above equation $\widetilde{E}_j$ , $E\widetilde{S}_{ij}$ , $E\widetilde{F}_{ij}$ and $\widetilde{T}_F$ can be calculated as positive triangular fuzzy numbers.

**3.3 Fuzzy Modified Backward Pass (MBP) Calculations**

Backward Pass Calculations are employed to calculate the latest time in the project network. In this case, if the backward Pass Calculations of CPM are entirely done in the fuzzy environment, the fuzzy latest time of event j( $\widetilde{L}_i$ ) can be written as:

$$\tilde{L}_i = \begin{cases} \underset{i \in p(j)}{M\tilde{I}N}\{\tilde{L}_j - \tilde{D}_{ij}\} & , S(i) \neq \Phi \\ \tilde{T}_F & , S(i) = \Phi \end{cases} \quad \text{--------------------(10)}$$

As mentioned above, the fundamental manner of the backward pass is based on an inversion between addition and subtraction. In a crisp environment, the equation of A+B-B =A is always correct but the addition and subtraction are not always inverse in the fuzzy theory, It means that A and B do not satisfy the relation $\tilde{A} \oplus \tilde{B}\ \tilde{B} = \tilde{A}$ .Therefore, the fuzzy backward pass in the equation above faces serious problems.

For example, for a typical project data, in which S(2)={3}, $\tilde{L}_3$ =[10,20,22] and $\tilde{D}_{23}$ =[5,10,15], using (10), it is found that,

$\tilde{L}_2 = \tilde{L}_3 - \tilde{D}_{23}$ = [10,20,22]- [5,10,15] = [10-15, 20-10, 22-5] = [-5,10,17]

It is clear that $\tilde{L}_2$ is a triangular number with a negative part. It depicts that the latest time of event 2 may happen in a negative time. But the negative time is not feasible since it is not defined in the project scheduling. To avoid this problem, we propose a new approach which we call modified backward pass (MBP). Therefore, according to the concept of $\tilde{L}_i$ and using (10), the fuzzy latest time of event i( $\tilde{L}_i$ ) can be defined as:

$$\tilde{L}_i = \begin{cases} \underset{j \in S(i)}{M\tilde{I}N}\ \tilde{X}_j \mid \tilde{X}_j + \tilde{D}_{ij} = \tilde{L}_j & ,S(i) \neq \Phi \\ \tilde{T}_F & ,S(i) = \Phi \end{cases} \quad \text{---------------(11)}$$

Using this relation leads to a full positive [$l_i^1$, $l_i^2$, $l_i^3$]. Therefore, the problem due to the appearance of the negative time is removed. But in some cases, the calculated $\tilde{L}_i$ may not satisfy the definition of triangular numbers. For example, for a typical project data, in which S(2)=3, $\tilde{L}_3$ =[10,20,22]and $\tilde{D}_{23}$ =[5,10,15].

Using (11) is found that:

[$x_j^1$, $x_j^2$, $x_j^3$]+ [5,10,15]= [10,20,22] [$x_j^1$+5, $x_j^2$+10, $x_j^3$+15]= [10,20,22]

$x_j^1$=22-5 ; $x_j^2$ =20-10 ; $x_j^3$=10-15 $x_j^1$=17 ;$x_j^2$=10 ;$x_j^3$=-5

$\tilde{L}_2$ =[17,10,-5].It can be seen that, $\tilde{L}_2$ is not a triangular fuzzy number because it violates the convex conditions (17≤10≤-5). Therefore, the calculation should be done in such a way that $\tilde{L}_i$ becomes a Triangular fuzzy number. By adding the triangular condition to the above relations, the following relation is obtained:

$$\tilde{L}_i = \begin{cases} \underset{j \in S(i)}{M\tilde{I}N}\ M\tilde{A}X\ \tilde{X}_j \mid \tilde{X}_j + \tilde{D}_{ij} \leq \tilde{L}_j, \quad \tilde{X}_j \text{ is a positive Triangular number}, & S(i) \neq \Phi \quad \text{-------------(12)} \\ \tilde{T}_F, & S(i) = \Phi \end{cases}$$

In the above relation, we define the relation ≤ for any two triangular fuzzy numbers such as $\tilde{A}$ =[$a^1$,$a^2$,$a^3$] and $\tilde{B}$ =[$b^1$,$b^2$,$b^3$] as:

$\tilde{A} \leq \tilde{B}$ =$a^1 \leq b^1$, $a^2 \leq b^2$, $a^3 \leq b^3$, where S(i) ≠ Φ

Relation (12) results in the following fuzzy mathematical programming problem:

$$\left.\begin{array}{l} \tilde{L}_i = \underset{j \in S(i)}{M\tilde{I}N}\ M\tilde{A}X\ \tilde{X}_j = [x_j^1, x_j^2, x_j^3] \\ \text{Subject to} \quad \tilde{X}_j + \tilde{D}_{ij} \leq \tilde{L}_j, \ j \in S(i) \\ \quad 0 \leq x_j^1 \leq x_j^2 \leq x_j^3 \end{array}\right\} \text{------------------------ (13)}$$

This problem can be rearranged in a more convenient form as following:

$\widetilde{L}_i = \widetilde{MIN}\ \widetilde{Y}_i = [y_i^1, y_i^2, y_i^3]$

Subject to,

$x_j^1 \le y_i^1\ \forall j \in S(i)$

$x_j^2 \le y_i^2\ \forall j \in S(i)$

$x_j^3 \le y_i^3\ \forall j \in S(i)$

$x_j^1 \le l_j^1 - d_{ij}^1\ \forall j \in S(i)$

$x_j^1 \le l_j^2 - d_{ij}^2\ \forall j \in S(i)$

$x_j^1 \le l_j^3 - d_{ij}^3\ \forall j \in S(i)$

$x_j^2 \le x_j^3$

$x_j^1 \le x_j^2$

$0 \le x_j^1$

By replacing the objective function with $\widetilde{MIN}\ y_i^1, y_i^2, y_i^3$ the problem above converts to a linear programming problem. It can be easily observed that the optimal solution of this problem. $\widetilde{L}_i$ is obtained as a positive triangular fuzzy numbers using a simple recursive relation:

$$\left.\begin{aligned} &\widetilde{L}_i = [l_i^1, l_i^2, l_i^3] \\ &l_i^3 = \max(0, \min(l_j^3 - d_{ij}^1)) \\ &l_i^2 = \max(0, \min(l_i^3, \min_{j \in S(i)}(l_j^2 - d_{ij}^2))) \\ &l_i^1 = \max(0, \min(l_i^2, \min_{j \in S(i)}(l_j^1 - d_{ij}^3))) \end{aligned}\right\} \quad \text{-------------- (15)}$$

In the MBP, following the calculation of $\widetilde{L}_i$, the fuzzy latest finishing of activities ($\widetilde{LF}_{ij}$) is calculated as follows: $\widetilde{LF}_{ij} = [lf_{ij}^1, lf_{ij}^2, lf_{ij}^3] = \widetilde{L}_j$ ------------------ (16)

Based on the equation above, $\widetilde{LF}_{ij}$, can be calculated as a positive triangular fuzzy numbers.

Another important characteristics of the backward pass is the fuzzy latest starting of activities ($\widetilde{LS}_{ij}$).; In order to calculate $\widetilde{LS}_{ij}$ ,when the backward pass calculations of CPM are applied directly to fuzzy environment, the following relation is obtained: $\widetilde{LS}_{ij} = \widetilde{LF}_{ij} - \widetilde{D}_{ij}$ ------------------------ (17)

In the relation above, the use of fuzzy subtraction is required. Due to the presence of fuzzy subtraction similar to that used in calculation of $\widetilde{L}_i$, in some cases, the calculating $\widetilde{LS}_{ij}$ may face some obstacles. Then, by adding positive and triangular conditions and also using a linear programming problem similar to (14) and (15), the following relations would be obtained:

$\widetilde{LS}_{ij} = [ls_{ij}^1, ls_{ij}^2, ls_{ij}^3]$ $\quad ls_{ij}^3 = \max(0, (lf_{ij}^3 - d_{ij}^1))$

$ls_{ij}^2 = \max(0, \min(lf_{ij}^3, (lf_{ij}^2 - d_{ij}^2)))$

$ls_{ij}^1 = \max(0, \min(lf_{ij}^2, (lf_{ij}^1 - d_{ij}^3)))$

**3.4 Fuzzy Slack Times :** One of the main characteristics in project control and planning is the slack time. There are three types of slacks for any activity,(i.e.); fuzzy total slack ($\widetilde{TF}_{ij}$), fuzzy free slack($\widetilde{FF}_{ij}$), and fuzzy independent slack ($\widetilde{IF}_{ij}$). If classical relations of CPM are applied for the calculation of these characteristics in the fuzzy environment, we can write the following relations:

$\widetilde{TF}_{ij} = \widetilde{LF}_{ij} - \widetilde{EF}_{ij}$ --------------- (19)

$\widetilde{FF}_{ij} = \widetilde{E}_j - \widetilde{EF}_{ij}$ ---------------(20)

$\widetilde{IF}_{ij} = \widetilde{E}_j - \widetilde{L}_i - \widetilde{D}_{ij}$ ---------------(21)

By using the relation s above, the slack times may be out of positive triangular fuzzy numbers definition. Therefore, similar to the calculation of $\widetilde{L}_i$ in modified backward pass, the following relations are proposed for slack times: $\widetilde{TF}_{ij} = [tf_{ij}^1, tf_{ij}^2, tf_{ij}^3]$;

$tf_{ij}^3 = \max(0, (lf_{ij}^3 - ef_{ij}^1))$ ---------------(22)

$tf_{ij}^2 = \max(0, \min(tf_{ij}^3, (lf_{ij}^2 - ef_{ij}^2)))$

$tf_{ij}^1=\max(0,\min(tf_{ij}^2, (lf_{ij}^1-ef_{ij}^3)))$

$\widetilde{FF}_{ij}=[ff_{ij}^1, ff_{ij}^2, ff_{ij}^3];$

$ff_{ij}^3=\max(0, (e_j^3-ef_{ij}^1))$ ---------------(23)

$ff_{ij}^2=\max(0,\min(ff_{ij}^3, (e_j^2-ef_{ij}^2)))$

$ff_{ij}^1=\max(0,\min(ff_{ij}^2, (e_j^1-ef_{ij}^3)))$

$\widetilde{IF}_{ij}=[if_{ij}^1, if_{ij}^2, if_{ij}^3];$

$if_{ij}^3=\max(0, (e_j^3-l_i^3-d_{ij}^1))$ ---------------(24)

$if_{ij}^2=\max(0,\min(if_{ij}^3, (e_j^2-l_i^2-d_{ij}^2)))$

$if_{ij}^1=\max(0,\min(if_{ij}^2, (e_j^1-l_i^1-d_{ij}^3)))$

**Numerical Example:**

The network representing a structure of project is given in the following figure:

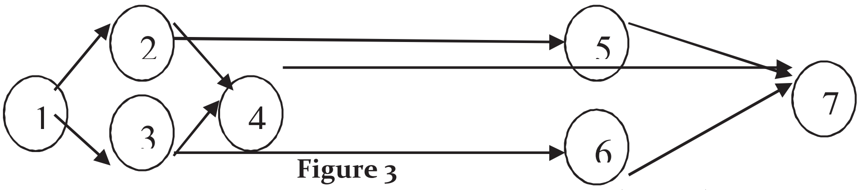



**Figure 3**

The duration of activities are positive triangular fuzzy numbers (Table 1).The fuzzy start time of this example is [0,0,0].

Table 1 $\widetilde{D}_{ij}$ of the numerical example

| Activity (i,j) | Duration ( $\widetilde{D}_{ij}$ ) |
|---|---|
| (1,2) | [25,28,32] |
| (1,3) | [40,55,65] |
| (2,4) | [32,37,43] |
| (3,4) | [20,25,35] |
| (2,5) | [35,38,42] |
| (3,6) | [42,45,55] |
| (4,7) | [60,65,75] |
| (5,7) | [65,75,85] |
| (6,7) | [15,18,22] |

Using the previously described relations, the main fuzzy characteristics for the numerical example are obtained. These values are positive triangular fuzzy numbers

$P(1) = \Phi \Rightarrow \widetilde{E}_1=[e_1^1,e_1^2,e_1^3] = \widetilde{T}_s=[0,0,0]$

$P(2)=\{1\} \Rightarrow \widetilde{E}_2=\widetilde{E}_1+\widetilde{D}_{12}=[0,0,0]+[25,28,32]= [25,28,32]$

$P(3)=\{1\} \Rightarrow \widetilde{E}_3=\widetilde{E}_1+\widetilde{D}_{13}=[0,0,0]+[40,55,65]= [40,55,65]$

$P(4)=\{2,3\} \Rightarrow \widetilde{E}_4=\text{Max}\{\widetilde{E}_2+\widetilde{D}_{24}, \widetilde{E}_3+\widetilde{D}_{34}\}$

$= \text{Max}\{[25,28,32]+[32,37,43], [40,55,65]+[20,25,35]\}$

$=\text{Max}\{[57,65,75],[60,80,100]\}$

$\widetilde{E}_4=[60,80,100]$

$P(5)=\{2\} \Rightarrow \widetilde{E}_5=\widetilde{E}_2+\widetilde{D}_{25}=[25,28,32]+[35,38,42] = [60,66,74]$

$P(6)=\{3\} \Rightarrow \widetilde{E}_6=\widetilde{E}_3+\widetilde{D}_{36}=[40,55,65]+[42,45,55] = [82,100,120]$

$P(7)=\{4,5,6\} \Rightarrow \widetilde{E}_7=\text{Max}\{\widetilde{E}_4+\widetilde{D}_{47}, \widetilde{E}_5+\widetilde{D}_{57}, \widetilde{E}_6+\widetilde{D}_{67}\}$

$= \text{Max}\{[60,80,100]+[60,65,75],[60,66,74]+[65,75,85],$
$[82,100,120]+[15,18,22]\}$

$\widetilde{E}_7=[125,145,175]$

$\widetilde{L}_i$ =[ $l_i^1, l_i^2, l_i^3$] $\widetilde{L}_7$ =[ $l_7^1, l_7^1, l_6^3$]=[ 125,145,175]

S(6)={7} => $\widetilde{L}_6$ =[ $l_6^1, l_6^2, l_6^3$] $l_6^3$=max(0,min($l_6^3$-$d_{67}^1$)) =max(0,(175-15))=max(0,160))=160

$l_6^2$=max(0,min($l_6^3$,min($l_7^2$-$d_{67}^2$)) =max(0,min(153,(145-18)))=max(0,127))=127

$l_6^1$=max(0,min($l_6^2$ ,min($l_7^1$-$d_{67}^3$)) =max(0,min(127,(125-22)))=max(0,103))=103

$\widetilde{L}_6$ =[103,127,160]

Similarly S(5)={7} => $\widetilde{L}_5$ =[ $l_5^1, l_5^2, l_5^3$]=[40,70,110]

S(4)={7} => $\widetilde{L}_4$ =[ $l_4^1, l_4^2, l_4^3$]=[50,80,115]

S(3)={4,6} => $\widetilde{L}_3$ =[ $l_3^1, l_3^2, l_3^3$]=[15,55,95]

S(2)={4,5} => $\widetilde{L}_2$ =[ $l_2^1, l_2^2, l_2^3$]=[0,32,75]

S(1)={2,3} => $\widetilde{L}_1$ =[ $l_1^1, l_1^2, l_1^3$]=[0,0,30]

Table 2 represents fuzzy earliest and latest times of events by using (6) and (15).

Table 2 Calculated values of $\widetilde{E}_i$ and $\widetilde{L}_i$ for the numerical example.

| Event (i) | $\widetilde{E}_i$ | $\widetilde{L}_i$ |
|---|---|---|
| 1 | [0,0,0] | [0,0,30] |
| 2 | [25,28,32] | [0,32,75] |
| 3 | [40,55,65] | [15,55,95] |
| 4 | [60,80,100] | [50,80,115] |
| 5 | [60,66,74] | [40,70,110] |
| 6 | [82,100,120] | [103,127,160] |
| 7 | [125,145,175] | [125,145,175] |

The fuzzy time of Project completion, $\widetilde{T}_F$ is calculated using (9) as:[125,145,175]

$E\widetilde{S}_{ij}$ , $E\widetilde{F}_{ij}$ , $L\widetilde{S}_{ij}$ and $L\widetilde{F}_{ij}$ are obtained using (7),(8),(16) and (18) respectively. The results have been presented in Table 3.

$E\widetilde{S}_{12}$ =[$es_{12}^1$, $es_{12}^2$, $es_{12}^3$]= $\widetilde{E}_1$ =[0,0,0]

$E\widetilde{S}_{13}$ =[$es_{13}^1$, $es_{13}^2$, $es_{32}^3$]= $\widetilde{E}_1$ =[0,0,0]

$E\widetilde{S}_{24}$ =[$es_{24}^1$, $es_{24}^2$, $es_{24}^3$]= $\widetilde{E}_2$ =[25,28,32]

$E\widetilde{S}_{34}$ =[$es_{34}^1$, $es_{34}^2$, $es_{34}^3$]= $\widetilde{E}_3$ =[40,55,65]

$E\widetilde{S}_{25}$ =[$es_{25}^1$, $es_{25}^2$, $es_{25}^3$]= $\widetilde{E}_2$ =[25,28,32]

$E\widetilde{S}_{36}$ =[$es_{36}^1$, $es_{36}^2$, $es_{36}^3$]= $\widetilde{E}_3$ =[40,55,65]

$E\widetilde{S}_{47}$ =[$es_{47}^1$, $es_{47}^2$, $es_{47}^3$]= $\widetilde{E}_4$ =[60,80,100]

$E\widetilde{S}_{57}$ =[$es_{57}^1$, $es_{57}^2$, $es_{57}^3$]= $\widetilde{E}_5$ =[60,66,74]

$E\widetilde{S}_{67}$ =[$es_{67}^1$, $es_{67}^2$, $es_{67}^3$]= $\widetilde{E}_6$ =[82,100,120]

$E\widetilde{F}_{ij}$ = [$ef_{ij}^1$, $ef_{ij}^2$, $ef_{ij}^3$]= $E\widetilde{S}_{ij} + \widetilde{D}_{ij}$

$E\widetilde{F}_{12}$ = [$ef_{12}^1$, $ef_{12}^2$, $ef_{12}^3$]= $E\widetilde{S}_{12} + \widetilde{D}_{12}$ =[0,0,0]+[25,28,32]= [25,28,32]

$E\widetilde{F}_{13} = E\widetilde{S}_{13} + \widetilde{D}_{13}$ =[0,0,0]+[40,55,65]=[40,55,65]

$E\widetilde{F}_{24} = E\widetilde{S}_{24} + \widetilde{D}_{24}$ = [25,28,32]+[32,37,43]=[57,65,75]

$E\widetilde{F}_{34} = E\widetilde{S}_{34} + \widetilde{D}_{34}$ =[40,55,65]+[20,25,35]=[60,80,100]

$E\widetilde{F}_{25} = E\widetilde{S}_{25} + \widetilde{D}_{25}$ =[25,28,32]+[35,38,40]=[60,66,72]

$E\widetilde{F}_{36} = E\widetilde{S}_{36} + \widetilde{D}_{36}$ =[40,55,65]+[42,45,55]=[82,100,120]

$\widetilde{EF}_{47} = \widetilde{ES}_{47} + \widetilde{D}_{47}$ =[60,80,100]+[60,65,75]=[120,145,175]

$\widetilde{EF}_{57} = \widetilde{ES}_{57} + \widetilde{D}_{57}$ =[60,66,74]+[65,75,85]=[125,141,159]

$\widetilde{EF}_{67} = \widetilde{ES}_{67} + \widetilde{D}_{67}$ =[82,100,120]+[15,18,22]=[97,118,142]

$\widetilde{LF}_{ij}$ =[$lf_{ij}^1$, $lf_{ij}^2$, $lf_{ij}^3$]= $\widetilde{L}_j$

$\widetilde{LF}_{12} = \widetilde{L}_2$ =[0,32,75]

$\widetilde{LF}_{13} = \widetilde{L}_3$ =[15,55,95]

$\widetilde{LF}_{24} = \widetilde{L}_4$ =[50,80,115]

$\widetilde{LF}_{34} = \widetilde{L}_4$ =[50,80,115]

$\widetilde{LF}_{25} = \widetilde{L}_5$ =[40,70,110] $\widetilde{LF}_{36} = \widetilde{L}_6$ =[103,127,160]

$\widetilde{LF}_{47} = \widetilde{L}_7$ =[125,145,175]

$\widetilde{LF}_{57} = \widetilde{L}_7$ =[125,145,175]

$\widetilde{LF}_{67} = \widetilde{L}_7$ =[125,145,175]

$\widetilde{LS}_{ij}$ = [$ls_{ij}^1$, $ls_{ij}^2$, $ls_{ij}^3$]

$\widetilde{LS}_{12}$ = [$ls_{12}^1$, $ls_{12}^2$, $ls_{12}^3$]

$ls_{12}^3$= max(0,( $lf_{12}^3$- $d_{12}^1$))=max(0,(75-25))=50

$ls_{12}^2$= max(0,min($lf_{12}^3$,( $lf_{12}^2$-$d_{12}^2$)))= max(0,min(75,(32-28)))=4

$ls_{12}^1$= max(0,min($lf_{12}^2$,( $lf_{12}^1$-$d_{12}^3$))) = max(0,min(32,(0-32)))=0

$\widetilde{LS}_{12}$ =[0,4.50]

In the similar way, We obtain

$\widetilde{LS}_{13}$ =[0,0,55]

$\widetilde{LS}_{24}$ =[7,43,83]

$\widetilde{LS}_{34}$ =[15,55,95]

$\widetilde{LS}_{25}$ =[0,32,75]

$\widetilde{LS}_{36}$ =[48,82,118]

$\widetilde{LS}_{47}$ =[50,80,115]

$\widetilde{LS}_{57}$ =[40,70,110]

$\widetilde{LS}_{67}$ =[103,127,160]

Table 3 : Calculated values of $\widetilde{ES}_{ij}$ , $\widetilde{EF}_{ij}$ , $\widetilde{LS}_{ij}$ and $\widetilde{LF}_{ij}$ for the numerical example.

| (i,j) | $\widetilde{ES}_{ij}$ | $\widetilde{EF}_{ij}$ | $\widetilde{LS}_{ij}$ | $\widetilde{LF}_{ij}$ |
|---|---|---|---|---|
| (1,2) | [0,0,0] | [25,28,32] | [0,4,50] | [0,32,75] |
| (1,3) | [0,0,0] | [40,55.65] | [0,0,55] | [15,55,95] |
| (2,4) | [25,28,32] | [57,65,75] | [0,43,83] | [50,80,115] |
| (3,4) | [40,55.65] | [60,80,100] | [15,55,95] | [50,80,115] |
| (2,5) | [25,28,32] | [60,66,72] | [0,32,75] | [40,70,110] |
| (3,6) | [40,55.65] | [82,100,120] | [48,82,118] | [103,127,160] |
| (4,7) | [60,80,100] | [120,145.175] | [50,80,115] | [125,145.175] |
| (5,7) | [60,66,72] | [125,141,151] | [40,70,110] | [125,145.175] |
| (6,7) | [82,100,120] | [97,118,142] | [103,127,160] | [125,145.175] |

Using (22)-(24) and the result presented in tables 2 and 3, the fuzzy slack times are calculated (Table 4). These values can be used for calculation of the critically of the activities as well as determination of the critical path.

$\widetilde{TF}_{ij}$ =[$tf_{ij}^1$, $tf_{ij}^2$, $tf_{ij}^3$]

$\widetilde{TF}_{12}$ =[$tf_{12}^1$, $tf_{12}^2$, $tf_{12}^3$]

$tf_{ij}^3$=max(0,($lf_{ij}^3$-$ef_{ij}^1$))

$tf_{12}^3$= max(0,( $lf_{12}^3$-$ef_{12}^1$))=max(0,(75-25))=50

$tf_{12}^2$= max(0,min($tf_{12}^3$,($lf_{12}^2$-$ef_{12}^2$)))

=max(0,min(50,(32-28)))=4

$tf_{12}^1$= max(0,min($tf_{12}^2$,($lf_{12}^1$-$ef_{12}^3$)))

=max(0,min(4,(0-32)))=0

$\widetilde{TF}_{12}$ = [0,4,50]

In the similar way, we obtain

$\widetilde{TF}_{13}$ =[0,0,55]

$\widetilde{TF}_{24}$ =[0,15,58]

$\widetilde{TF}_{34}$ =[0,0,55]

$\widetilde{TF}_{25}$ =[0, 4,50]

$\widetilde{TF}_{36}$ =[0,27,78]

$\widetilde{TF}_{47}$ =[0,0,55]

$\widetilde{TF}_{57}$ =[0,4,50]

$\widetilde{TF}_{67}$ =[0,27,78]

$\widetilde{FF}_{ij}$ =[$ff_{ij}^1$,$ff_{ij}^2$, $ff_{ij}^3$]

$\widetilde{FF}_{12}$ =[$ff_{12}^1$, $ff_{12}^2$, $ff_{12}^3$]

$ff_{12}^3$=max(0,($e_2^3$-$ef_{12}^1$))=max(0,(32-25))=7

$ff_{12}^2$= max(0,min($ff_{12}^3$,( $e_2^2$-$ef_{12}^2$)))

=max(0,min(7,(28-28)))=0

$ff_{12}^1$= max(0,min($ff_{12}^1$,( $e_2^1$-$ef_{12}^3$)))

=max(0,min(0,(25-32)))=0

$\widetilde{FF}_{12}$ =[0,0,7]

$\widetilde{FF}_{13}$ =[0,0,25]

$\widetilde{FF}_{24}$ =[0,15,43]

$\widetilde{FF}_{34}$ =[0,0,60]

$\widetilde{FF}_{25}$ =[0,0,14]

$\widetilde{FF}_{36}$ =[0,0,38]

$\widetilde{FF}_{47}$ =[0,0,55]

$\widetilde{FF}_{57}$ =[0,4,50]

$\widetilde{FF}_{67}$ =[0,27,78]

$\widetilde{IF}_{ij}$ =[$if_{ij}^1$,$if_{ij}^2$, $if_{ij}^3$]

$\widetilde{FF}_{12}$ =[$if_{12}^1$, $if_{12}^2$, $if_{12}^3$]

$if_{12}^3$=max(0, ($e_2^3$-$l_1^3$-$d_{12}^1$))=0

$if_{12}^2$=max(0,min( $if_{12}^3$, ($e_2^2$-$l_1^2$-$d_{12}^2$)=0

$if_{12}^1$=max(0,min( $if_{12}^2$, ($e_2^1$-$l_1^1$-$d_{12}^3$)))=0

$\widetilde{IF}_{12}$ =[0,0,0]

$\widetilde{IF}_{13}$ =[8,10,10]

$\widetilde{IF}_{24}$ =[0,0,0]

$\widetilde{IF}_{34}$ =[0,0,0]

$\widetilde{IF}_{25}$ =[0,0,0]

$\widetilde{IF}_{36}$ =[0,0,0]

$\widetilde{IF}_{47}$ =[0,0,0]

$\widetilde{IF}_{57}$ =[0,0,0]

$\widetilde{IF}_{67}$ =[0,0,0]

Table 4 : Calculated values of $\widetilde{TF}_{ij}$ , $\widetilde{FF}_{ij}$ and $\widetilde{IF}_{ij}$ for the numerical example:

| (i,j) | $\widetilde{TF}_{ij}$ | $\widetilde{FF}_{ij}$ | $\widetilde{IF}_{ij}$ |
|---|---|---|---|
| (1,2) | [0,4,50] | [0,0,7] | [0,0,0] |
| (1,3) | [0,0,55] | [0,0,25] | [8,10,10] |
| (2,4) | [0,15,58] | [0,15,43] | [0,0,0] |
| (3,4) | [0,0,55] | [0,0,60] | [0,0,0] |
| (2,5) | [0,4,50] | [0,0,14] | [0,0,0] |
| (3,6) | [0,27,78] | [0,0,38] | [0,0,0] |
| (4,7) | [0,0,55] | [0,0,55] | [0,0,0] |
| (5,7) | [0,4,50] | [0,4,50] | [0,0,0] |
| (6,7) | [0,27,78] | [0,27,78] | [0,0,0] |

(1,3),(3,4) and (4,7) are the critical activities because their total floats are equal.

The critical path is

(1) — (3) — (4) — (7)

**Conclusion :** Previous work on network scheduling using fuzzy set theory provides methods for scheduling projects. These methods, however, do not support the backward pass calculations in direct manner similar to that used in the forward pass. In this project new method based on the fuzzy theory has been developed to solve the project scheduling problem under the fuzzy environment.

A major advantage of this method is to employ direct arithmetic fuzzy operations in obtaining meaningful computable results. In this method, we introduced a new approach which we called Modified Backward Pass (MBP). This approach based on a Linear Programming (LP) problem, removes negative and infeasible solutions which can be generated by other methods in the backward pass calculation. We drove the general form of the optimal solution of the LP problem which enables practitioners to obtain the optimal solution by simple recursive relation without solving only LP problem. Through a numerical example, calculations involved in this method have been illustrated.

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