ON PRIME DERIVATIONS WITH d^{3n+1} CONTAINED IN THE CENTER

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Abstract : Let C be a center of a prime nonassociative ring R with derivation 'd' such that $d^n(R) \subseteq C$ where n is a fixed positive integer. Then it is shown that either R is commutative or $d^{3n+1} = 0$.
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Introduction: There are numbers of conditions which imply the commutativity of a certain derivation rings. In his note on derivations, Herstein [3] showed that if a ring R with 2-divisibility admits a nonzero derivation d such that $[d(x), d(y)] = 0$, for all $x, y \in R$, then R is commutative. Suh [4] proved that it R is a prime ring with a derivation d such that $d(R) \subseteq N$ then either R is associative or $d^3 = 0$ In this paper we extend Suh's result for center and show that if R is a prime ring with a derivation d such that $d^{(n)}(R) \subseteq C$ where n is a fixed positive integer then either R is commutative or $d^{3n+1} = 0$.

Let R be a non associative ring. A usual notation for associator $(x, y, z) = (xy)z - x(yz)$, for all $x, y, z \in R$ and for commutator $[x, y] = xy - yx$ for all $x, y \in R$ is adopted. Denote the nucleus of R by N consisting of all element n in R such that $(n, R, R) = (R, n, R) = (R, R, n) = 0$ and the center by C where C consists of all elements c in N such that $[c, R] = 0$. An additive mapping d from a ring R to R is called a derivation if $d(xy) = d(x)y + xd(y), \forall x, y \in R$. A ring R is said to be prime if $xay = 0$ implies $x = 0$ or $y = 0$ $\forall x, a, y \in R$.

Throughout we assume that $d^{(n)}(R) \subseteq C$ ...(1)

where R is a prime ring with derivation d and n is a fixed positive integer.

In any arbitrary ring the following two identities are well known to hold.

$[xy, z] + [yz, x] + [zx, y] = [x, y, z] + [y, z, x] + [z, x, y]$ ... (2) and

$[xy, z] - x[y, z] - [x, z]y = (x, y, z) + (z, x, y) - (x, z, y)$ ...(3) for all $x, y, z \in R$.

By our assumption (1) and using the Leibnotz's formula we obtain $d^{3n+1}(R) \subseteq C$.

Now suppose that $c \in C \subseteq N$. Then with $x = c$ in (3) we have $[cy, z] = c[y, z]$ ... (4)

and replacing $y = c$ in (3) we have $[xc, z] = [x, z]c$, for all $c \in C \subseteq N$. ...(5)

As a consequence of (3), (4) and (5) we have that C is a commutative sub ring of R.

From (1) we note that $d^i(R) \subseteq d^n(R) \subseteq C$ for all integers $i > n$

Replace x by $d^{3n+1}(R)$ in (2) we obtain $[d^{3n+1}(R) \cdot R, R] + [R^2, d^{3n+1}(R)] + [R \cdot d^{3n+1}(R), R] = (d^{3n+1}(R), R, R) + (R, R, d^{3n+1}(R)) + (R, d^{3n+1}(R), R)$.

By the definition of center we obtain $[R^2, d^{3n+1}(R)] = 0$ and the right hand side all sum to zero. Thus we have $[d^{3n+1}(R) \cdot R, R] + [R \cdot d^{3n+1}(R), R] = 0$ which gives $[d^{3n+1}(R) \cdot R, R] + d^{3n+1}(R) \cdot R, R] - [(d^{3n+1}(R), R], R] = 0$ implying $2[d^{3n+1}(R) \cdot R, R] = 0$ and by 2-divisibility we obtain $[d^{3n+1}(R) R, R] = 0$ which in turn shows that $d^{3n+1}(R) \cdot R \subseteq C$. ...(6)

lly it is easy to see that $R - d^{3n+1}(R) \subseteq C$ ... (7)

The commutator ideal I of R is the smallest ideal which contains all commutators in R. The commutator ideal is zero if and only if R is commutative. Thus I can be characterized as all the finite sum of commutators and left (or right) multiple of commutators.

Hence now substituting $z = d^{3n+1}(R)$ in (3) gives

$0 = [R^2, d^{3n+1}(R)] - R[R, d^{3n+1}(R)] - [R, d^{3n+1}((R)] R$
$= [[R, d^{3n+1}(R)],R] - [R, d^{3n+1}(R)] R - [R,d^{3n+1}(R)] R$
$= -2 [R, d^{3n+1}(R)] R$
$= 2 [d^{3n+1}(R),R] R$
$= [d^{3n+1}(R), R] R$
$= [[d^{3n+1}(R),R] + R [d^{3n+1}(R),R]$
$= R [d^{3n+1}(R),R$ ...(8)

Again substituting $y = d^{3n+1}(R)$ in (3) we obtain $[R \cdot d^{3n+1}(R), R] - R[d^{3n+1}(R), R] - [R, R]d^{3n+1}(R) = 0$.

Using (8) and (7) we obtain $[R, R] d^{3n+1}(R) = 0$.

Applying $I = [R,R] + R[R,R] = [R,R] + [R,R]R$ these two inequalities imply

$I \cdot d^{3n-1}(R) = 0$. ...(9)

Similarly by $d^{3n+1}(R) \cdot R \subseteq C$ we obtain $d^{3n+1}(R) \cdot I = 0$.

Lemma 1: Let F be generated by $d^{3n+1}(R)$. That is $F = d^{3n+1}(R) + d^{3n+1}(R)R + Rd^{3n+1}(R) + R \cdot d^{3n+1}(R)R$ then F is an ideal of R.

Proof: It is obvious that F is the additive subgroup of (R, +) and by equations (1) $d^{3n+1}(R) \subseteq C$, since $3n+1 \geq n$.

Also from equations (6) and (7) we have $d^{3n+1}(R) \cdot R + R \cdot d^{3n+1}(R) \subseteq C$. Thus by [4 lemma 1] F, is an ideal of R.

Theorem: If R is a prime ring with a derivation d such that $d^{(n)}(R) \subseteq C$ where n is a fixed positive integer then either R is commutative or $d^{3n+1} = 0$.

Proof: Since $3n+1 \geq n$ we have $d^{3n+1}(R) \subseteq C$. Using equations (6), (7) and (8) we see that $\{d^{3n+1}(R) + d^{3n+1}(R)R + Rd^{3n+1}(R) + R \cdot d^{3n+1}(R)R\} \cdot I = 0$. Hence $FI = 0$. By the primeness of R either $I = 0$ or $F = 0$. Thus either R is commutative or $d^{3n+1} = 0$.

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