AN EOQ MODEL FOR QUADRATIC DEMAND ITEMS WITH SHORTAGES

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Abstract: This paper presents an inventory model for deteriorating items with quadratic demand, instantaneous supply and shortages in inventory. A two-parameter Weibull distribution is taken to represent the time to deterioration. A theory for finding the optimal solution of the problem is developed. A numerical example is taken to illustrate the solution procedure.

Key-words: Weibull distribution, quadratic demand, deteriorating inventory, eoq model

Introduction: In inventory problems, deterioration is defined as damage, decay, spoilage, evaporation, obsolescence, loss of utility or loss of marginal value of goods that results in decrease the usefulness of the original one. Deterioration should not be neglected in inventory problems for the items like foodstuff, chemicals, pharmaceuticals, electronic goods, radioactive substances, etc. Emmons's (1968) models with two-parameter Weibull distribution deterioration were discussed by Covert and Philip (1973), Philip (1974), Giri et al.(2003), Ghosh and Chaudhuri (2004) etc. whereas Chakrabarty, Giri and Chaudhuri (1998) and other researchers used three-parameter Weibull distribution deterioration in their inventory models. Giri et al. (1999), Sana et al. (2004), Sana and Chaudhuri (2004a), etc., developed inventory models in this direction. Misra (1975) developed an EOQ model with a Weibull deterioration rate for perishable product where backordering is not allowed. These investigations were followed by several researchers like Deb and Chaudhuri (1986), Goswami and Chaudhuri (1991), Giri et al. (1996) etc. where a time-proportional deterioration rate is considered.

It has been empirically observed that the failure and life expectancy of many items can be expressed in terms of Weibull distribution. This empirical observation has encouraged researchers to represent the time of deterioration of a product by Weibull distribution. Ghare and Schrader's (1963) model was extended by Covert and Philip (1973) and obtained an EOQ model with a variable rate of deterioration by assuming a two-parameter Weibull distribution. Later, many researchers like Tadikamalla (1978), Chakrabarty et al. (1998), Mukhopadhyay et al. (2004, 2005) developed economic order quantity models. Therefore, the rate of deterioration is treated as time varying function in realistic models. Begum et al. (2010) develop an EOQ model for varying deteriorating items with Weibull distribution deterioration and price-dependent demand. They assume that the demand and deterioration rates are continuous and differentiable function of price and time.

Demand plays a key role in modeling of deteriorating inventory, researchers have recognized and studied the variations (or their combinations) of demand from the viewpoint of real life situations. Demand may be constant, time-varying, stock-dependent, price-dependent, etc. The constant demand is valid, only when the phase of the product life cycle is matured and also for finite periods of time. Wagner and Whitin (1958) discussed the discrete case of the dynamic version of EOQ. The classical no-shortage inventory policy for linear trend in demand was discussed by Donaldson (1977). EOQ models for deteriorating items with trended demand were considered by Bahari-Kashani (1989), Goswami and Chaudhuri (1991, 1992), Xu and Wang (1990), Kim (1995), Jalan et al. (1996), Jalan and Chaudhuri (1999), Lin et al. (2000), etc. Many research articles by Silver (1979), Henery (1979), McDonald (1979), Dave and Patel (1981), Sachan (1984), Deb and Chaudhuri (1986), Murdeshwar (1988), Hariga (1993), etc. analyzed linear time-varying demand. Later, Ghosh and Chaudhuri (2004,2006), Khanra and Chaudhuri (2003), etc. established their models with quadratic time-varying demand.

In this paper, we reconsider the model of Covert and Philip(1973) and extend it to include a time-quadratic demand rate and shortages in inventory. A two-parameter Weibull distribution is considered to represent the time to deterioration. The theory for finding the optimal solution of the problem is developed. A numerical example is taken to illustrate the solution procedure. Sensitivity of the optimal solution with respect to changes in different parameter values is also examined here.

Notations and Assumptions : The model is developed using the following assumptions:

- The deterministic demand rate $D(t)$ varies in quadratic with time, i.e. $D = a + bt + ct^2$ where a, b and c are constants. Here $'a'$ is initial rate of demand, $'b'$ is the rate at which the demand rate increases.
- Lead time is zero.
- The replenishment is instantaneous.
- Shortages are allowed.
- The holding cost, ordering cost, shortage cost

and unit cost remain constant over time.

- The distribution of the time to deterioration follows a two parameter Weibull distribution and the deteriorated units are not replaced during a given cycle.

To develop the mathematical model of the inventory replenishment, the notations adopted in this paper is listed below:

- $K =$ a constant value $(0 < K < 1)$
- $c_1 =$ carrying cost per unit per unit time
- $c_2 =$ shortage cost per unit per unit time.
- $c_3 =$ ordering cost per order
- $c_4 =$ cost of a unit
- $T =$ length of the inventory cycle.
- $D(t) =$ demand rate at any instant 't'
- a, b, c are positive constants.
- $\theta(t)$ = instantaneous rate of deterioration of the inventory is followed by a two-parameter Weibull distribution. i.e. $\theta(t) = \alpha\beta\, t^{\beta-1}, \alpha, \beta, \gamma > 0$.
- α scale parameter, $\alpha > 0$.
- β shape parameter, $\beta > 0$.
- γ location parameter, $\gamma > 0$.

Mathematical formulation:

Let Q(t) be the instantaneous inventory level at any time $t \geq 0$. The instantaneous state of Q(t) at any time t is described by the differential equation

$$\frac{dQ(t)}{dt} + \theta(t)Q(t) = -(a + bt + ct^2), \qquad 0 \leq t \leq t_1 \qquad (1)$$

$$\frac{dQ(t)}{dt} = -(a + bt + ct^2), \qquad t_1 \leq t \leq T \qquad (2)$$

Taking $\theta(t) = \alpha\beta t^{\beta-1}, \quad \alpha, \beta, t > 0 \qquad (3)$

Equation (1) becomes

$$\frac{dQ(t)}{dt} + \alpha\beta t^{\beta-1} Q = -(a + bt + ct^2), \qquad 0 \leq t \leq t_1 \quad (4)$$

The solution of equation (4) yields

$$e^{\alpha t^\beta} Q(t) = q_0 - \int_0^t (a + bt + ct^2)\, e^{\alpha t^\beta} dt, \qquad 0 \leq t \leq t_1 \qquad (5)$$

Using the condition $Q(t_1) = 0$, in (5), we get

$$q_0 = \int_0^{t_1} (a + bt + ct^2)\, e^{\alpha t^\beta} dt \qquad (6)$$

The solution of equation (2) becomes

$$Q(t) = a(t_1 - t) + \frac{b}{2}({t_1}^2 - t^2) + \frac{c}{3}({t_1}^3 - t^3), \qquad t_1 \leq t \leq T \qquad (7)$$

Expanding equation (6) in infinite series and integrating term by term, we have

$$q_0 = a\sum_{n=0}^{\alpha} \frac{\alpha^n\, t_1^{\,n\beta+1}}{(n\beta+1)\, n!} + b\sum_{n=0}^{\alpha} \frac{\alpha^n\, t_1^{\,n\beta+2}}{(n\beta+2)\, n!} + c\sum_{n=0}^{\alpha} \frac{\alpha^n\, t_1^{\,n\beta+3}}{(n\beta+3)\, n!} \qquad (8)$$

Using equation (6) in equation(5), we have

$$Q(t) = \frac{\int_0^{t_1} (a + bt + ct^2).e^{\alpha t^\beta} dt - \int_0^{t} (a + bt + ct^2).e^{\alpha t^\beta} dt}{e^{\alpha t^\beta}}, \qquad 0 \leq t \leq t_1 \ (9)$$

And $= a(t_1 - t) + \frac{b}{2}\left(t_1^{\,2} - t^2\right) + \frac{c}{3}\left(t_1^{\,3} - t^3\right), \qquad t_1 \leq t \leq T \quad (10)$

The inventory level at the beginning of the cycle must be sufficient for meeting the total demand is

$$\int_0^{t_1} \left(a + bt + ct^2\right) dt = at_1 + \frac{b}{2}t_1^{\,2} + \frac{c}{3}t_1^{\,3}.$$

and the total deteriorated items is $q_0 - \int_0^{t_1} \left(a + bt + ct^2\right) dt = q_0 - at_1 - \frac{b}{2}t_1^{\,2} - \frac{c}{3}t_1^{\,3}.$

The average inventory holding cost in $(0, t_1)$ is $\frac{1}{2}\frac{c_1}{T} q_0 t_1$.

The average shortage cost in (t_1, T) is

$$\frac{c_2}{T}\int_{t_1}^{T} \left(a + bt + ct^2\right)(T - t)\, dt = \frac{c_2}{6T}\left[(T - t_1)^2\left\{3a + b(T + 2t_1) + \frac{c}{2}\left(T^2 + 3t_1^{\,2} + 2t_1T\right)\right\}\right]$$

Therefore, the total variable cost per unit time is

$$TVC(t_1,T)=\frac{c_4}{T}\left(q_0-at_1-\frac{b}{2}t_1^{\,2}-\frac{c}{3}t_1^{\,3}\right)+\frac{1}{2}\frac{c_1}{T}q_0\,t_1+\frac{c_3}{T}$$

$$+\frac{c_2(T-t_1)^2}{6T}\left\{3a+b(T+2t_1)+\frac{c}{2}\left(T^2+3t_1^{\,2}+2t_1T\right)\right\} \quad (11)$$

As the length of the shortage interval is a part of cycle time, therefore we may assume $t_1=KT,\ 0<K<1$; where K is a constant to be determined in an optimal manner. Using equation (7) in equation (11); we have,

$$TVC=\left(\frac{c_4a}{T}+\frac{1}{2}c_1Ka\right)\int_0^{KT}e^{\alpha t^{\beta}}dt+\left(\frac{c_4b}{T}+\frac{1}{2}c_1Kb\right)\int_0^{KT}t.e^{\alpha t^{\beta}}dt+\left(\frac{c_4c}{T}+\frac{1}{2}c_1Kc\right)\int_0^{KT}t^2e^{\alpha t^{\beta}}dt$$

$$-c_4aK-\frac{1}{2}c_4bK^2T-\frac{1}{3}c_4cK^3T^2+\frac{1}{2}c_2a(1-K)^2T+\frac{1}{6}c_2b(1-K)^2(1+2K)T^2$$

$$+\frac{1}{12}c_2c(1-K)^2\left(1+3K^2T+2KT\right)+\frac{c_3}{T}$$

$$TVC=\left(\frac{1}{2}c_1Ka+\frac{ac_4}{T}\right)\sum_{n=0}^{\alpha}\frac{\alpha^n(KT)^{n\beta+1}}{(n\beta+1)n!}+\left(\frac{1}{2}c_1Kb+\frac{bc_4}{T}\right)\sum_{n=0}^{\alpha}\frac{\alpha^n(KT)^{n\beta+2}}{(n\beta+2)n!}$$

$$+\left(\frac{1}{2}c_1Kc+\frac{cc_4}{T}\right)\sum_{n=0}^{\alpha}\frac{\alpha^n(KT)^{n\beta+3}}{(n\beta+3)n!}-aKc_4+\frac{1}{2}c_2aT(1-K)^2-\frac{1}{2}c_4bTK^2$$

$$+\frac{1}{6}c_2bT^2(1-K)^2(1+3TK^2+2KT)+\frac{c_3}{T} \quad (12)$$

Considering $\mathbf{K}$ as a decision variable, the necessary conditions for the minimization of average system cost $\mathrm{TVC(T,K)}$ are $\frac{\partial TVC}{\partial T}=0\ \ and\ \ \frac{\partial TVC}{\partial K}=0$ (13)

Then equation (13) becomes

$$\frac{1}{2}c_1a\sum_{n=0}^{\alpha}\frac{\alpha^n(KT)^{n\beta+2}}{n!}+\frac{1}{2}c_1b\sum_{n=0}^{\alpha}\frac{\alpha^n(KT)^{n\beta+3}}{n!}$$

$$+\frac{1}{2}c_1c\sum_{n=0}^{\alpha}\frac{\alpha^n(KT)^{n\beta+4}}{n!}+c_4a\sum_{n=0}^{\alpha}\left(\frac{n\beta}{n\beta+1}\right)\frac{\alpha^n(KT)^{n\beta+1}}{n!}+c_4b\sum_{n=0}^{\alpha}\left(\frac{n\beta+1}{n\beta+2}\right)\frac{\alpha^n(KT)^{n\beta+2}}{n!}$$

$$+c_4c\sum_{n=0}^{\alpha}\left(\frac{n\beta+2}{n\beta+3}\right)\frac{\alpha^n(KT)^{n\beta+3}}{n!}+\frac{1}{2}c_2aT^2(1-K)^2-\frac{1}{2}c_4bT^2K^2+\frac{1}{3}c_2bT^3(1-K)^2(1+2K)$$

$$+\frac{cc_2}{12}(1-K)^2(3K^2T^2+2KT^2)-\frac{2}{3}c_4c(KT)^3-c_3=0 \quad (14)$$

and

$$\frac{1}{2}c_1a\sum_{n=0}^{\alpha}\left(\frac{n\beta+2}{n\beta+1}\right)\frac{\alpha^n(KT)^{n\beta+1}}{n!}+\frac{1}{2}c_1b\sum_{n=0}^{\alpha}\left(\frac{n\beta+3}{n\beta+2}\right)\frac{\alpha^n(KT)^{n\beta+2}}{n!}+\frac{1}{2}c_1c\sum_{n=0}^{\alpha}\left(\frac{n\beta+4}{n\beta+3}\right)\frac{\alpha^n(KT)^{n\beta+3}}{n!}+c_4a\sum_{n=0}^{\alpha}\frac{\alpha^n(KT)^{n\beta}}{n!}+$$
$$c_4b\sum_{n=0}^{\alpha}\frac{\alpha^n(KT)^{n\beta+1}}{n!}+c_4c\sum_{n=0}^{\alpha}\frac{\alpha^n(KT)^{n\beta+2}}{n!}-c_2aT(1-K)-ac_4-bKTc_4-c_2bT^2K(1-K)-c_4cT^2K^2+$$
$$\frac{1}{6}cc_2(1-K)(T-3K^2T-1)=0 \quad (15)$$

The optimal values T^* of T and K^* of $\mathbf{K}$ are obtained by solving equation (14) and (15). The sufficient conditions that these values minimize $\mathrm{TVC(T,K)}$ are

$$\frac{\partial^2 TVC}{\partial T^{*2}}\cdot\frac{\partial^2 TVC}{\partial K^{*2}}-\left(\frac{\partial^2 TVC}{\partial T^*\partial K^*}\right)^2>0 \quad (16)$$ and
$$\frac{\partial^2 TVC}{\partial T^{*2}}>0\,,\quad \frac{\partial^2 TVC}{\partial K^{*2}}>0 \quad (17)$$

Equations (14) and (15) can only be solved with the help of a computer for a given set of parameter values by truncating the infinite series if $(KT)<1$.

Numerical Analysis:

Equations (14) and (15) are solved with the help of a computer based technique using the following parameter values: $c_1=\mathrm{Rs.}\,100.00$ per unit per day, $c_2=\mathrm{Rs.}\,10.00$ per unit per day, $c_3=\mathrm{Rs.}\,20.00$ per order, $c_4=\mathrm{Rs.}\,4.00$ per unit, $\alpha=0.002, \beta=1.5, a=20.0, b=2.0$ and $c=5.0$ Then the optimal cycle time are $T^*=0.443189$, optimal value $K^*=0.9977$, economic order quantity $q_0{}^*=$

1.50449 units, the value of $t_1^* = 0.0749091$ days, and total average cost $TVC^* = Rs. 63.3948$ per day. It is checked that this solution satisfies the sufficient conditions given in equation (16) and (17).

Sensitivity Analysis:

We now study the effects of changes in the value of system parameters $a, b, c, \alpha, \beta, c_1, c_2, c_3, c_4$ on the optimal cycle time T^*, the optimal length of inventory q_0^* and the minimum total relevant cost per unit time TVC^*. The sensitivity analysis is performed by changing each of the parameter by 50%, 20%, –20% and - 50%, taking one parameter at a time and keeping the remaining parameters unchanged. The results are shown in Table 1.

On the basis of the results of Table-1, the following observations can be made:

1) T^*, q_0^* and TVC^* are all insensitive towards changes in the parameter b, c.

2) α and β are infeasible to the changes in T^*, q_0^* and TVC^*.

3) With increase in c_2, c_3, c_4; q_0^* and TVC^* increases, but T^* decreases with increase in c_2 and c_4 and increases with increase in c_3.

c_1 is infeasible towards the solution. With increase in c_1; T^*, q_0^* and TVC^* are insensitive to changes in the parameter c_1

Table-1 Sensitivity Analysis

Changing Parameter	% change in the system parameter	% change in T^*	% change in q_0^*	% change in TVC^*
A	-50	39.51	-39.43	-27.19
	-20	11.80	-15.03	-10.67
	20	-8.98	14.41	10.46
	50	-19.19	35.13	25.90
B	-50	0.79	0.15	-1.05
	-20	0.31	0.06	-0.41
	20	-0.30	-0.05	0.41
	50	-0.76	-0.14	1.025
C	-50	0.13	1.30	-2.29
	-20	0.05	0.52	-0.92
	20	-0.05	-0.52	0.92
	50	-0.12	-1.30	2.31
c_1	-50	0.60	85.14	11.65
	-20	0.14	22.53	3.36
	20	-0.09	-15.33	-2.44
	50	-0.18	-31.49	-9.78
c_2	-50	37.22	-11.36	-30.78
	-20	10.53	-3.87	-11.13
	20	-7.71	3.30	10.04
	50	-16.17	7.48	23.68
c_3	-50	-31.46	-17.48	-21.32
	-20	-11.19	-6.23	-7.94
	20	10.01	5.59	7.34
	50	23.47	13.11	17.48
c_4	-50	2.82	-22.90	-9.68
	-20	1.37	-9.03	-4.33
	20	-1.70	8.86	5.01
	50	-4.94	21.81	14.03
α, β	-50	--	--	--
	-20	--	--	--
	20	--	--	--
	50	--	--	--

(--) represents infeasible solution

Concluding : An inventory replenishment policy is developed for deteriorating items with time-quadratic demand. The rate of deterioration is time-proportional and the time to deterioration is followed by a two-parameter Weibull distribution. We reconsider the model of Covert and Philip (1973) and extended it to a time-dependent demand rate and shortages in inventory using Weibull distribution. A numerical example is taken to illustrate the theory. The sensitivity of the optimal solution to changes in the parameter values is examined. From the above analysis, it is seen that α and β are the critical parameters in the sense that any error in the estimation of α and β resulting errors in the optimal results. Therefore, proper care must be taken to estimateβ. Again the above analysis shows that great care should be taken to estimate the value of the parameter c, c_3 and c_4.

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