ON IDEALS OF A TERNARY SEMIGROUPS

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Abstract: In this paper the terms fully idempotent ternary semigroups, weekly prime ternary semigroups are introduced. It is proved that, if T is a duo ternary semigroup with an identity element. Then every ideal of T is prime if and only if it is fully idempotent and the set of ideals of S is totally ordered under inclusion. It is proved that, every prime ideal of a ternary semigroup is weekly prime ideal. Further it is proved that, if P is an ideal of a ternary semigroup T, then the following are equivalent. 1) P is weekly prime. 2) If X, Y, Z are three ideals of T such that
(P ∪ X) (P ∪ Y) (P ∪ Z) ⊆ P ⇒ X ⊆ P or Y ⊆ P or Z ⊆ P. 3) If X, Y, Z are three ideals of T such that X ⊇ P and XYZ ⊆ P ⇒ X = P or Y ⊆ P or Z ⊆ P. 4) If X, Y, Z are three ideals of T such that (P ∪ X)YZ ⊆ P ⇒ X ⊆ P or Y ⊆ P or Z ⊆ P. 5) If a, b, c are three elements of T such that (<a>∪ P) (<b>∪ P) (<c>∪ P) ⊆ P, implies that a ∈ P or b ∈ P or c ∈ P.

Key words : Fully idempotent ternary semigroup, weekly prime ideals.

Introduction : Ternary semigroups was introduced by Sen and Saha as a generalization of semigroup. ANJANEYULU.A[1], made a study on ideal theory of duo semigroups, SARALA.Y, ANJANEYULU.A, MADHUSUDHANA RAO [3] studied about the ideal theory of ternary semigroups,. In this paper we study about the fully prime ternary semigroups, weeklyprime ideals in duo ternary semigroups.

Preliminaries : Definition 2.1 : Let T be a non-empty set. Then T is said to be a ternary semigroup if there exist a mapping from T×T×T to T which maps (x_1, x_2, x_3) $\to [x_1x_2x_3]$ satisfying the condition : $[(x_1x_2x_3)x_4x_5] = [x_1(x_2x_3x_4)x_5] = [x_1x_2(x_3x_4x_5)]$ $\forall\ x_i \in$ T, $1 \le i \le 5$.

Definition 2.2 : A nonempty subset A of a ternary semigroup T is said to be left ternary idealor left idealof T if b, c∈ T, a∈ A implies bca∈ A.

Definition 2.3 : A nonempty subset of a ternary semigroup T is said to be a lateral ternary idealor simply lateral ideal of T if b, c∈ T , a∈ A implies bac ∈ A.

Definition 2.4 : A nonempty subset A of a ternary semigroup T is a right ternary idealor simply right ideal of T if b, c ∈ T , a∈ A implies abc∈ A

Definition 2.5 : A nonempty subset A of a ternary semigroup T is said to be ternary ideal or simply an ideal of T if b, c ∈ T , a ∈ A implies bca∈ A, bac∈ A, abc∈A.

Definition 2.6 : An ideal A of a ternary semigroup T is said to be a prime ideal of T provided X,Y,Z are ideals of T and XYZ ⊆ A ⇒ X ⊆A or Y ⊆A or Z ⊆A.

Definition 2.7 : An element a of a ternary semigroup T is said to be regular if there exist x, y∈ T such that axaya = a.

Definition 2.8 : A ternary semigroup T is said to be regular ternary semigroup provided every element is regular.

Definition 2.9 : An element a of a ternary semigroup T is said to be semisimple if a∈ $< a >^3$ i.e. $< a >^3$ = <a >.

Definition 2.10 : A ternary semigroup T is called semisimpleternary semigroup provided every element in T is semisimple.

Definition 2.11: A ternary semigroup T is said to be a left duo ternary semigroup if every left ideal of T is bothright and lateral ideal of T.

Definition 2.12 : A ternary semigroup T is said to be a right duo ternary semigroup if every right ideal of T is bothleft and lateral ideal of T.

Definition 2.13 : A ternary semigroup T is said to be a duo ternary semigroup if it is both a left duo ternary semigroup and a right duo ternary semigroup.

3. Fully idempotent ternary semigroups :

Definition 3.1 :A ternary ideal A of a ternary semigroup Tis said to be globallyidempotent, if A^3 = A.

Definition 3.2 :A ternary semigroup T is said to be fully idempotent,if every ideal is globally idempotent.

Theorem 3.3 : Let T be a fully idempotent ternary semigroup and A be an ideal of T. Then A^n = A for all odd natural number n.

Proof: Since A is globally ideal of T, we have A^3 = A. Therefore A = A^3= A^5 = ... for all oddnatural numbers. Hence A^n = A for all odd natural number n.

Theorem 3.4 : Let T be a fully idempotent ternary semigroup and A be an ideal of T. Then A^w = A.

Proof: By theorem 2.3, we haveA^n = A for all odd natural number n.Therefore A = $\bigcap_{\substack{i=1 \\ \text{i is odd}}}^{\infty} A^i = A^w$.

Theorem 3.5 : Let T be a duoternary semigroup. Then T is fully idempotent if and only if T is semisimple ternary semigroup.

Proof: Suppose that T is fully idempotent. Let a∈ T, then <a> is an ideal of T. Therefore <a> is an idempotent ideal of T, implies that $<a>^3$ = <a> .

Hence $a \in \langle a \rangle = \langle a \rangle^3$. Therefore a is semisimple. Hence T is semisimple ternary semigroup.

Conversely, suppose that T is semisimple ternary semigroup. Let A be an ideal of T. Clearly $A^3 \subseteq A$. Let $a \in A$. Then $\langle a \rangle \subseteq A$. Since T is semisimple ternary semigroup, $a \in \langle a \rangle^3$ and hence $a \in \langle a \rangle^3 \subseteq A^3$. Hence $A \subseteq A^3$. Thus $A^3 = A$. Therefore T is fully idempotent.

Theorem 3.6 : Let T be a duoternary semigroup. Then T is fully idempotent if and only if T is regular ternary semigroup.

Proof: Suppose that T is fully idempotent and $a \in T$. By theorem 4.11, T is semisimple ternary semigroup. Therefore $a \in \langle a \rangle^3 \Rightarrow a = paqaras$ for some p, q, r, $s \in T$. Since T is duo, TaT = aTT = TTa. Hence $paq \in aTT$ and $ras \in TTa$. Therefore paq= alm and ras = jka for some j, k, l, $m \in T$. Therefore a = paqaras=almajka = axa where x = lmajk and hence axaxa = axa = a. Thus a is regular in T. Hence T is regular ternary semigroup.

Conversely, suppose that T is regular ternary semigroup and A be an ideal of T. Clearly $A^3 \subseteq A$. Let $a \in A$. Then $\langle a \rangle \subseteq A$. Since T is regular ternary semigroup, a = axaya for some x, $y \in T$ and hence $a \in \langle a \rangle^3 \subseteq A^3$. Hence $A \subseteq A^3$. Thus $A^3 = A$. Therefore T is fully idempotent.

Definition 3.7 :An element a of a ternary semigroup is said to be an E – invertiable element if there exist x in T such that axaxa = a and xaxax = x.

Definition 3.8 :A ternary semigroup T is said to be an E – inverse ternary semigroup,if every element of T is element an E – invertiable element.

Theorem 3.9 : Let T be a fully idempotent cancellative duoternary semigroup. Then T is an E- inverse ternary semigroup.

Proof: Let $a \in T$. Since T is fully idempotent, by theorem 3.6, a is regular in T. Therefore, there exists x in T such that axaxa = a. Now axaxaxa = axa and T is cancellative, implies that xaxax = x. Hence T is an E- inverse ternary semigroup.

Definition 3.10 :An element a of a ternary semigroup is said to be a g – regular element if there exist x in T such that xaxax = x.

Definition 3.11 :A ternary semigroup T is said to be g – regularternary semigroup,if every element of T is a g – regular element.

Corollary 3.12 : Let T be a fully idempotent cancellative duoternary semigroup. Then T is an E-inverse ternary semigroup.

Theorem 3.13 : Let T be a duoternary semigroup. Then T is fully idempotent ternary semigroup if and only if $I \cap J \cap K = IJK$ for any ideals I, J, K of ternary semigroup T.

Proof: Suppose that T is fully idempotent and I, J, K are three ideals of T. Clearly, $IJK \subseteq I \cap J \cap K$. Let $a \in I \cap J \cap K$. Since T is fully idempotent and $a \in T$, implies that by theorem 3.6, a is regular in T. Therefore there exists $x \in T$ such that axaxa = a. Since $a \in J$, $x \in T$ and J is an ideal of T, implies that $xax \in J$. Hence $axaxa \in IJK$. That is $a \in IJK$. Therefore $I \cap J \cap K \subseteq IJK$. Hence $I \cap J \cap K = IJK$.

Conversely, suppose that $I \cap J \cap K = IJK$ for any ideals I, J, K of ternary semigroup T. Hence $I^3 = III = I \cap I \cap I = I$. That is $I^3 = I$ for any ideals I of T. Thus I is globally idempotent and hence T is fully idempotent ternary semigroup.

Theorem 3.14 : If T is a duoternary semigroup with an identity element. Then every ideal of T is prime if and only if it is fully idempotent and the set of ideals of S is totally ordered under inclusion.

Proof: Suppose that every ideal of Tis prime. Let A be an ideal of T, then A is prime. Now $A^3 = AAA$ is prime ideal of T, implies that $A \subseteq A^3$ and hence $A^3 = A$. Thus A is idempotent. Hence T is fully idempotent. Now suppose thatX, Yare any two ideals of T, then $XYT \subseteq X \cap Y$. Since $X \cap Y$is prime,either $X \subseteq X \cap Y$or $Y \subseteq X \cap Y$. Hence $X \subseteq Y$or $Y \subseteq X$.

Conversely, suppose that P is an ideal of Tand let $XYZ \subseteq P$ for some ideals X, Y and Zof T.

By hypothesis, ideals of S is totally ordered under inclusion. we have either $X \subseteq Y \subseteq Z$ or $X \subseteq Z \subseteq Y$ or$Y \subseteq X \subseteq Z$ or $Y \subseteq Z \subseteq X$ or $Z \subseteq Y \subseteq X$ or $Z \subseteq X \subseteq Y$. If $X \subseteq Y \subseteq Z$, then $X = X^3 \subseteq XYZ \subseteq P$, the other casesare similar, hence P is a prime ideal.

4. Weekly Prime Left Ideals

Definition 4.1 : An left ideal P of a ternary semigroup T is said to be weekly prime ideal , if $XYZ \subseteq P$ for some ternary left ideals X, Y, Z of T containing P, then X = P or Y = P or Z = P.

Theorem 4.2 : Every left prime ideal of a ternary semigroup is weekly prime ideal.

Proof: Suppose that Pis a left ternary prime ideal of T. Let X, Y, Z be three left ideals of T containing P such that $XYZ \subseteq P$. Now $XYZ \subseteq P$ and P is prime $\Rightarrow X \subseteq P$ or $Y \subseteq P$ or $Z \subseteq P$ and hence X = P or Y = P or Z = P. Hence P is weekly prime.

Note 4.3 : Every weekly left prime ideal need not be a prime ideal.

Example 4.4 : Let T ={o, a, b, c} define a ternary operation. In T as show in the following table. Then T is a ternary semigroup. Here B = {o, b} and C = {o, c} are left ideal of T and here $BBB \subseteq C$ and $B \nsubseteq C$. Hence C is not left prime but C is left weekly prime.

.	o	a	b	c
o	o	o	o	o
a	o	a	b	o
b	o	o	o	o
c	o	a	b	c

Theorem 4.5 : Let P be a left ideal of a ternary semigroup T. Then the following are equivalent.

1. P is left weekly prime.

2. If X, Y, Z are three left ideals of T such that (P ∪ X)(P ∪ Y)(P ∪ Z) ⊆ P ⇒X ⊆ Por Y ⊆ P or Z ⊆ P.

3. If X, Y, Z are three left ideals of T such that X ⊇ P and XYZ ⊆ P ⇒X = Por Y ⊆ P or Z ⊆ P.

4. If X, Y, Z are three left ideals of T such that (P ∪ X)YZ ⊆ P ⇒X ⊆ Por Y ⊆ P or Z ⊆ P.

5. If a, b, c are three elements of T such that (<a>∪ P)(<b>∪ P)(<c>∪ P) ⊆ P, implies that a∈P or b∈P or c∈P.

Proof : (1) ⇒ (2) : Suppose that P is a weekly left prime ternary ideal of T. Let X, Y, Z be three left ternary ideals of T such that (P ∪ X)(P∪Y) (P∪Z) ⊆ P. Since P, X, Y, Z are left ideals of T, P ∪ X, P∪Y, P∪Z are left ideals of T containing P. Since P is weekly prime,

P ∪ X = P or P∪Y = P or P∪Z = P. Hence X ⊆ P or Y ⊆ P or Z ⊆ P.

(2) ⇒ (3) : Let X, Y, Z are three left ideals of T such that X ⊇ P and XYZ ⊆ P. Therefore (P ∪ X)(P∪Y) (P∪Z) ⊆ P. Hence by (2), X ⊆ P or Y ⊆ P or Z ⊆ P.

Thus X = P or Y ⊆ P or Z ⊆ P.

(3) ⇒ (4) : Let X, Y, Z are three left ideals of T such that(P ∪ X)YZ ⊆ P. From (3), we have P ∪X = P or Y ⊆ P or Z ⊆ P. Hence X ⊆ P or Y ⊆ P or Z ⊆ P.

(4) ⇒ (5): Let a, b, cbe three elements of T suchthat (a∪ P)($\langle b\rangle_l$∪ P)($\langle c\rangle_l$∪ P) ⊆ P.

Now (<a>∪ P)(<b>∪ P)(<c>∪ P) =<a>(<b>∪ P)(<c>∪ P)⊆ P⇒a∈ P or <b> ⊆ P or <c>⊆ P⇒a∈ P or b∈ P or c∈ P.

(5) ⇒ (1): Let X, Y, Z be three left ideals of T containing P such that XYZ ⊆ P.

Suppose that P ≠ X and P ≠ Y. Therefore there exists x∈ X, y∈ Y such that x, y∉ P.

Therefore <x >∪ P⊈ P and <y>∪ P⊈ P. Let z ∈ Z, then <z>∪ P⊆ Z.

Therefore (<x >∪P)(<y>∪ P)(<z>∪ P) ⊆XYZ ⊆P. By condition (5), x ∈ P or y ∈ P or z ∈ P. But x, y∉ P, implies that z ∈ P. Hence Z ⊆ P, implies that Z = P. Hence P is weekly prime ideal of T

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