NUCLEUS IS NOT EQUAL TO THE CENTER IN (-1, 1) RINGS

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Abstract: If Nucleus is not equal to the center then (-1, 1) is associative under the following conditions: (i) the ring is a 2-torsion free, third power associative and has no zero divisors. (ii) the ring is a 2, 3- torsion free simple and not associative and (iii) the ring is a 2, 3- torsion free prime and third power associative.

Keywords :(-1, 1) ring, Nucleus, nucleus, torsion free ring.

Introduction: Hentzel [2] established that nil semi simple (-1, 1) rings are associative. Later Hentzel with Smith [3] has shown that a 2, 3- torsion free simple locally (-1, 1) nil ring must be associative. Paul [6] has proved that if R is semiprime and $([R, R], x, y) = 0$ then R is associative. Paul in [7] has proved that if R is a prime right alternative ring with commutators in the middle nucleus then either R is associative or the middle nucleus is in the center of R. In this paper using the results of [2,3,5,7] it is shown that a (-1, 1) ring can be associative if the ring is a 2-torsion free, third power associative semiprime, simple and prime.

A (-1, 1) ring is a nonassociative ring satisfying the condition

$(x, y, z) + (x, z, y) = 0$ ...(1)

and $(x, y, z) + (y, z, x) + (z, x, y) = 0$. ...(2)

for all $x, y, z \in R$.

In a nonassociative ring R we define an associator as $(x, y, z) = (xy)z - x(yz)$ and the commutator as $[x, y] = xy - yx$ for all $x, y, z \in R$. To make the notation more convenient we often use '·' to indicate multiplication as well as juxtaposition. In products, juxtaposition takes precedence, i, e,. $xy \cdot z \equiv (xy)z$. The nucleus of a ring R is defined as $N = \{n \in R / (n, R, R) = (R, n, R) = (R, R, n) = 0\}$and the commutative center C as $C = \{c \in R / [c, R] = 0\}$. A ring R is said to be a n – torsion free if there exist a positive integer n such that $nx = 0$ implies $x = 0$ for any $x \in R$. A ring R is prime if whenever A and B are ideals of R such that $AB = 0$, then either $A = 0$ or $B = 0$ and is said to be semiprime if for any ideal A of R, $A^2 = 0$ implies $A = 0$. These rings are also referred to as rings free from trivial ideals. And a ring is said to be simple if whenever A is an ideal of R, then either $A= R$ or $A = 0$.

Throughout this paper we denote R to be a 2, 3 – torsion free (-1, 1) ring. The associator ideal A is defined as all the finite sum of associators and the left multiple of associators i.e., $A = (R, R, R) + R(R, R, R)$.

Main section: The ring R satisfies the following identities [1] :

$(wx, y, z) - (w, xy, z) + (w, x, yz) = w(x, y, z) + (w, x, y)z$. ...(3)

and $[xy, z] = x[y, z] + [x, z]y + (x, y, z) - (x, z, y) + (z, x, y)$. ...(4)

Equation (3) is known as a Teichmuller identity and the equation (4) as a semi-jacobi identity and these two identities are valid in any arbitrary ring.

Also, it is known that a 3- torsion free (–1, 1) ring satisfies the following identity [1].

$(w, [x, y], z) = ([x, y], z, w) = (z, w, [x, y])$. ...(5)

From (2) and (5), we obtain $(w, [x, y], z) = ([x, y], z, w) = (z, w, [x, y]) = 0$. ...(6)

This implies that $[R, R] \subset N$.

Four applications of identity (3) leads to the equation

$(wx, y, z) - (w, xy, z) + (w, x, yz) = w(x, y, z) + (w, x, y)z$,

$-(xy, z, w) + (x, yz, w) - (x, y, zw) = -x(y, z, w) - (x, y, z)w$,

$(yz, w, x) - (y, zw, x) + (y, z, wx) = y(z, w, x) + (y, z, w)x$,

$-(zw, x, y) + (z, wx, y) - (z, w, xy) = -z(w, x, y) - (z, w, x)y$,

so that adding all the left side, we obtain after repeated applications of (2), the sum is zero. Adding the right hand sides we obtain

$[w, (x, y, z)] - [x, (y, z, w)] + [y, (z, w, x)] - [z, (w, x, y)]$.

Thus $[w, (x, y, z)] - [x, (y, z, w)] + [y, (z, w, x)] - [z, (w, x, y)] = 0$. ...(7)

From (1), (2) and (4) we obtain

$[xy, z] = x[y, z] + [x, z]y + 2(x, y, z) + (z, x, y)$. ...(8)

Let $n \in N$. Subsituting n for w in (7), we obtain

$[n, (x, y, z)] = 0$. ...(9) i.e., n commutes with all associators. From (3) and (9), we obtain

$[n, w(x, y, z)] = -[n, (w, x, y)z]$. ...(10)

If u and v are any two associators in R, then substituting $z = n$, $x = u$ and $y = v$ in (4) results in $[uv, n] = 0$. ...(11)

If $u = (a, b, c)$, then $[(a, b, c)v, n]= 0$ using (10) we obtain $-[a(b, v, c,), n] = 0$.

From this and (4) we have $-a[(b, c, v), n] - [a, n](b, c, v) = 0$.

Then from (9) we obtain

$[a, n](b, c, v) = 0$ ...(12)

Lemma 1 : Let $S = \sum [N, R] + R[N, R]$. Then S is an ideal of R.

Proof : Let $n \in N$ and $x, y \in R$, then $\sum [n, y] + x[n, y] \in S$.

Let $r \in R$ then $r\{\sum [n, y] + x[n, y]\} = \sum\{r[n, y] +$

$r(x[n, y]))\} = \sum\{r[n, y] - (r, x, [n, y]) + (rx)[n, y]\} = \sum\{r[n, y] + (rx)[n, y]\} \in S$. Hence $RS \subset S$.

Now substituting $z = n$ in (8) we obtain $[n, xy] = x[n, y] + [n, x]y$. ...(13)

Hence $\{\sum [n, y] + x[n, y]\}r = \sum\{[n, y]r + (x[n, y])r\} = \sum\{[n, y]r + [n, xy]r - ([n, x]y)r\} = \sum\{[n, y]r + [n, xy]r - ([n, x], y, r) - [n, x](yr)\} = \sum\{[n, y]r + [n, xy]r - [n, x](yr)\} \in S$.

Hence $SR \subset S$. Therefore S is an ideal of R. ■

Lemma 2 : Let $U = \{u \in R \,/\, [N, R]u = 0\}$. Then U is an ideal of R.

Proof : Let $u \in U$, then $0 = [n, x]u = ([n, x]u)r = ([n, x], u, r) + [n, x](ur) = [n, x](ur)$.

Hence $UR \subset U$. Also from (13) $[n, x](ru) = ([n, x]r)u = ([n, xr])u - (x[n, r])u = 0$. Hence $RU \subset U$. Thus U is an ideal of R. ■

Theorem 1 : Let R be a prime $(-1, 1)$ ring and nucleus is not equal to the center. Then R is associative.

Proof : Suppose R is not associative.

From (5), we have $[n, R] \subset N$. ...(14)

From Lemmas (1) and (2), we obtain $\sum\{[N, R]u + (R[N, R])u\} = \sum (R[N, R])u = 0$.

Hence $SU = 0$. Since R is a prime ring, then either $S = 0$ or $U = 0$.

Now taking $N \neq C$ from the hypothesis we have $S \neq 0$ and therefore $U = 0$.

Now we have $[n, x](y, y, z) = (nx)(y, y, z) - (xn)(y, y, z) = n(x(y, y, z)) - x(n(y, y, z)) = n(x(y, y, z)) - x((y, y, z)n) = n(x(y, y, z)) - (x(y, y, z))n = [n, x(y, y, z)]$. ...(15)

Using (3), we obtain $x(y, y, z) = (xy, y, z) - (x, yy, z) + (x, y, yz) - (x, y, y)z$.

Commuting this with n and using (9) and (2), we obtain $[n, x(y, y, z)] = 0$.

Hence we obtain $[n, x](y, y, z) = 0$. ...(16)

Using (13) and (5), we obtain $0 = [n, x](y, y, z) = (y[n, x], y, z) = ([n, xy], y, z) - ([n, y]x, y, z) = -([n, y]x, y, z) = -[n, y](x, y, z)$. Hence $[n, y](x, y, z) = 0$. ...(17)

In equation (3) using Lemma 2 we obtain $(y, y, z) \in U = 0$, so that R is left alternative. Hence in the light of (1) R must be alternative. Therefore from (2), we have

$0 = (R, R, R) + (R, R, R) + (R, R, R) = 3(R, R, R) = 0$. ...(18)

Linearization of (17) gives $[n, R](R, (R, R, R), R) = -[n, (R, R, R)](R, R, R) = 0$ using (9).

Thus $(R, (R, R, R), R) \subset U = 0$. Hence from (1) and (2), we have $(R, R, (R, R, R)) = 0$ and $((R, R, R), R, R) = 0$. So $(R, R, R) \subseteq N$. ...(19) Now multipling (18) with an associator we obtain

$3(R, R, R)(R, R, R) = 0$. ...(20)

In [4] equation (19) implies that $2(R, R, R)(R, R, R) = 0$. ...(21)

From (20) and (21), we obtain $(R, R, R)(R, R, R) = 0$. ...(22)

From [4] the associator ideal which squares to zero must be associative. Hence R is associative. This completes the proof of the Theorem. ■

Lemma 3: Let $T = \{t \in R/ [t, N] = 0 = [tR, N] = [Rt, N]\}$. Then T is an ideal of R.

Proof: let $t \in T$, $n \in N$ and $x, y, z \in R$ then from (9) and by the definition of T we obtain $[tx \cdot y, n] = [t \cdot xy, n] = 0$ and from (4) $[y \cdot tx, n] - y[tx, n] - [y, n] \cdot tx - (y, tx, n) - (n, y, tx) + (y, n, tx) = 0$ we have $[y \cdot tx, n] = [y, n] \cdot tx$. ...(23)

Also from (4) $0 = [yt, n] - y[t, n] - [y, n]t - (y, t, n) - (n, y, t) + (y, n, t)$ we obtain $[y, n]t = 0$. Also with $y = n \in N$, $w = x$, $x = y$, and $z = t \in T$ we have

$([y, n], t, x) = 0$. ...(24)

But $([y, n], t, x) = ([y, n]t)x - [y, n] \cdot tx = -[y, n] \cdot tx = -[y \cdot tx, n]$ from (4).

Thus from (24) we have $[y \cdot tx, n] = 0$. Thus T is a right ideal of R. By going to the anti-isomorphic ring it can easily be proved that T is a left ideal of R. Therefore T is an ideal of R. ■

Theorem 2: If $N \neq C$ then a 2, 3-torsion free simple $(-1, 1)$ ring is associative.

Proof: Since R is a simple ring then either $T = R$ or $T = 0$. If $T = R$ then $N = C$ but from the hypothesis $N \neq C$. Thus $T = 0$. From (11) with $u = (x, y, z)$, where $x, y, z \in R$ we obtain $[(x, y, z)v, n] = 0$ using the identity $(x, y, z)w = -(w, x, y)z$. ....(25)

from [8]. Now using (17) and (1) we obtain $0 = [(x, y, z)v, n] = -[(v, x, y)c, n] = [(v, y, x)z, n] = -[v(y, x. z), n] = [v(y, z, x), n]$. Thus we have $(x, y, z) \in T$ and $(y, z, x) \in T$ and we have $(x, y, z) = 0 = (y, z, x)$. But from (2) we also see that $(z, x, y) = 0$. Replacing x by an associator (R, R, R) we are in the situation where all associators are in the nucleus. That is $(R, R, R) \subseteq N$. Now we shall use the result of [4] to conclude that R must be associative. Thus we complete the proof of the theorem. ■

Theorem 3: If $N \neq C$ then a 2-torsion free semiprime third power associative $(-1, 1)$ ring is associative.

Proof: Since $N \neq C$ there exist $n \in N$ and $x \in R$ such that $[x, n] \neq 0$. Hence from (12) we have $(y, z, v) = 0$ for all associators v, and $y, z \in R$. We can write this as $(R, R, (R, R, R)) = 0$. Substituting $v = (q, r, s)$ in (11) we see that $[u(q, r, s), n] = 0$. Using (10) we see that $-[(u, q, r)s, n] = 0$. From this and (4) we obtain $(u, q, r)[s, n] = 0$. But $N \neq C$ and so we have $(u, q, r) = 0$, for all associators u, and $q, r \in R$. We can write this as $((R, R, R), R, R) = 0$. Using $(R, R, (R, R, R)) = 0 = ((R, R, R), R, R)$ identities gives $(R, (R, R, R), R) = 0$. Thus $(R, R, R) \subseteq N$. Since R is semiprime we use the result of [4] to conclude that R must be associative. ■

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