EXISTENCE AND STABILITY RESULTS OF THE IMPULSIVE FRACTIONAL FUNCTIONAL DIFFERENTIAL EQUATION

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Abstract: In this investigation, we study the existence and uniformly stability results of solutions for a class of abstract impulsive differential equations with state dependent delay. Existence results are proved by using fixed point theorem then we show the stability of the solution. One application is also presented to verifying the existence result.

Keywords: Fixed point theorem, Fractional order differential equation, Functional differential equation, Impulsive condition.

Introduction: We consider the following impulsive fractional functional integro-differential equation of the form:

$$ {}^cD_t^{\alpha} y(t) = J_t^{2-\alpha} f(t, y_{\rho(t,y_t)}, B(u)), \quad t \in J = [0,T], t \neq t_k, \qquad (1,1)$$

$$y(t) = \phi(t), \quad y'(0) = 0, t \in [-d,0]. \qquad (1.2)$$

$$\Delta y|_{t=t_k} = I_k(y(t_k^-)), k = 1,2,\dots,m$$
$$\Delta y|_{t=t_k} = Q_k(y(t_k^-)), k = 1,2,\dots,m, \qquad (1.3)$$

where $0 < T < \infty, D_t^{\alpha}$, Caputo's derivative of order $\alpha \in (1,2]$, y' denotes the derivative of y with respect to t and $f : J \times PC_0 \times X \to X$ is given continuous function. $\phi(t) \in PC_0$ and history function y_t the element of PC_0 defined by $y_t(\theta) = y(t+\theta), \theta \in [-d,0]$. Here $0 = t_1 < t_2 < \dots < t_m < t_{m+1} = T$, $Q_k, I_k \in C(X,X), (k = 1,2,\dots m)$, are bounded functions. We have $\Delta y(t_k) = y(t_k^+) - y(t_k^-)$ and $\Delta y'(t_k) = y'(t_k^+) - y'(t_k^-)$ represent the right and left hand limits of $y(t)$ at $t = t_k$ and $y(t_k^-) = y(t_k)$, $y'(t_k^-) = y'(t_k)$ respectively. The term $B(y) = \int_0^t K(t,s) y ds$, where $K \in C(D, R^+)$, is the set of all positive functions which are continuous on $D = \{(t,s) \in R^2 : 0 \le s \le t < T\}$ and $B^* = \sup_{t \in [0,T]} \int_0^t k(t,s) ds < \infty$.

Preliminaries: Let $(X, \|.\|_X)$ be a complex Banach space of functions with the norm $\|y\|_X = \sup_{t\in J}\{|y(t)| : y \in X\}$. PC_0 is set of continuous functions on $[-d,0]$. To treat the impulsive condition, defined the following space $PC_t = PC([-d,t] : X), 0 < t \le T < \infty$, be a Banach space of all such functions $y : [-d,t] \to X$, which are continuous everywhere except for a finite number of points $t_i, i = 1,2,\dots,m$ at which $y(t_i^+)$ and $y(t_i^-)$ exists with $y(t_i^-) = y(t_i)$ and endowed with the norm $\|y\|_{PC_t} = \sup_{t\in[-d,t]}\{\|y(t)\|_X, y \in PC_t\}$ and $PC^1{}_t = PC^1([-d,t] : X), 0 < t \le T < \infty$, be a Banach space of all such functions $y' : [-d,t] \to X$, which are continuously differentiable everywhere for finite number of points $t_i, i = 1,2,\dots,m$ at which $y'(t_i^+)$ and $y'(t_i^-)$ exists with $y'(t_i^-) = y'(t_i)$ and endowed with the norm

$$\|y\|_{PC^1_t} = \sup_{t\in[-d,t]}\{\|y(t)\|_{PC_t}, \|y'(t)\|_{PC_t}, y \in PC_t\}$$

Definition: A function $y : [-d,T] \to X$ such that $y \in PC_T^1$ is a solution of the system (1.1)-(1.3) if and only if it satisfied the following integral equation

$$y(t)=\begin{cases}\phi(t), t\in[-d,0]\\ \phi(0)+\int_0^t (t-s)f(s,y_{\rho(s,y_s)},B(y))ds, t\in[0,t_1]\\ \phi(0)+\sum_{0<t_k<t} I_k(y)+\sum_{0<t_k<t}(t-t_k)Q_k(y)\\ +\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)f(s,y_{\rho(s,y_s)},B(y))ds\\ +\sum_{0<t_k<t}(t-t_k)\int_{t_{k-1}}^{t_k}f(s,y_{\rho(s,y_s)},B(y))ds\\ \int_{t_k}^{t}(t-s)f(s,y_{\rho(s,y_s)},B(y))ds, t\in(t_k,t_{k+1}]\end{cases} \quad (2.1)$$

Existence Result: In this section, we study the existence result of the solution of the system (1.1)-(1.3) by applying the Schauder fixed point theorem. Let the function $\rho : J\times PC_0 \to [-d,T]$ is continuous and $\phi(0)\in PC_0$, and we shall need the following assumptions:

(H_1) f, I_k, Q_k are continuous functions and there exist positive constants M_1, M_2, M_3 and M_4 such that

$$\| f(t,\psi,y)\|_X \le M_1\|\psi\|_{PC_0}+M_2\|y\|_X;$$

$$\| I_k(y)\|_X \le M_3\|y\|_X; \|Q_k(y)\|_X \le M_4\|y\|_X$$

for all $\psi\in PC_0$ and $y\in X$.

(H_2) f is jointly continuous functions and there exist positive constants L_{f1}, L_{f2} such that

$$\| f(t,\psi,y)-f(t,\xi,z)\|_X \le L_{f1}\|\psi-\xi\|_{PC_0} + L_{f2}\|y-z\|_X, \forall \psi,\xi\in PC_0, y,z\in X.$$

(H_3) I_k, Q_k are continuous functions and there exist positive constants L_I, L_Q such that

$$\| I_k(y)-I_k(z)\|_X \le L_1\|y-z\|_X;$$

$$\| Q_k(y)-Q_k(z)\|_X \le L_Q\|y-z\|_X, \forall y,z\in X.$$

Theorem3.1. Let the assumptions (H_1) is satisfied. Then the problem (1.1)-(1.3) has at least one solution on J.

Proof. We transform the problem (1.1)-(1.3) in to a fixed point problem. Let consider the space $B_r=\{y\in PC_T : \|y\|\le r\}$. It is obvious that B_r is closed bounded and convex subset of PC_T.

Consider the operator $P: B_r \to B_r$ defined by

$$Py(t)=\begin{cases}\phi(0)+\int_0^t (t-s)f(s,y_{\rho(s,y_s)},B(y))ds,\ t\in[0,t_1]\\ \phi(0)+\sum_{0<t_k<t} I_k(y)+\sum_{0<t_k<t}(t-t_k)Q_k(y)\\ +\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)f(s,y_{\rho(s,y_s)},B(y))ds\\ +\sum_{0<t_k<t}(t-t_k)\int_{t_{k-1}}^{t_k}f(s,y_{\rho(s,y_s)},B(y))ds\\ +\int_{t_k}^{t}(t-s)f(s,y_{\rho(s,y_s)},B(y))ds,\ t\in(t_k,t_{k+1}].\end{cases} \quad (3.1).$$

First, we shall show that P is continuous so consider a sequence $y^n \to y$ in B_r for the interval $(t_k,t_{k+1}]$, $k=1,2,\ldots,m.$ Since the function f, I_k, Q_k are continuous, hence $\| P(y^n)-P(y)\|_X \to 0, as\ n\to\infty.$ Which implies that the mapping P is continuous on PC_r. Let $y\in B_r$ then we have

$$\| P(y)\|_X \le \|\phi(0)\|_{PC_0} + mM_3 r + mTM_4 r + m\frac{T^2}{2}\left(M_1+M_2B^*\right)r$$

$$m\frac{T^2}{2}\left(M_1+M_2B^*\right)r+\frac{T^2}{2}\left(M_1+M_2B^*\right)r$$

It

$$\le \|\phi(0)\|_{PC_0} + m r\left(M_3+TM_4\right) + \frac{T^2}{2}\left(M_1+M_2B^*\right)r\left(1+2m\right)=C.$$

Itproves that P maps bounded set into bounded set in B_r for all subinterval $(t_k,t_{k+1}]$, $k=1,2,3,\ldots,m.$

Finally, we shall show that P maps bounded sets into equi-continuous sets in B_r. Let $l_1,l_2\in(t_k,t_{k+1}]$ with $l_1<l_2$, $0\le k\le m$, then we have

$$\|(Py)(l_2)-(Py)(l_1)\|_X \le m(l_2-l_1)M_4 r + m(l_2-l_1)T\left(M_1+M_2B^*\right)r+(l_2-l_1) T\left(M_1+M_2B^*\right)r+\frac{(l_2-l_1)^2}{2}\left(M_1+M_2B^*\right).$$

We have as $l_2 \to l_1$, then $\|P(y)(l_2) - P(y)(l_1)\|_X \to 0$. This implies that P is equi-continuous on all the subintervals $(t_k, t_{k+1}]$, $k = 1,2,3,\ldots,m$. Thus, by Arzela-Ascoli Theorem, it follows that P is completely continuous map. Therefore, by Schauder fixed point theorem, the operator P has a fixed point, which in turn implies that (1.1)-(1.3) has at least one solution on $[0,T]$. This completes the proof of the theorem.

4. Stability Result: We use the concept of stability from [2] see Theorem 3.1

Theorem 4.1 Assume that conditions *(H₂)* and *(H₃)* hold and

$$\delta = mL_1 + mTL_Q + (1+2m)\frac{T^2}{2}\left(L_{f1} + L_{f2}B^*\right) < 1.$$

Then the solution of the system (1.1)-(1.3) is uniformly stable.

Proof. We have $y(t)$ be a solution of (2.1), and let $z(t)$ be a solution of (2.1) satisfying the nonlocal initial value $z(t) = \psi$, where $\psi \in PC_T$. Then for $t \in [0, t_1]$ we have

$$\|(y)(t) - (z)(t)\|_X \le \|\phi(0) - \psi(0)\|_X$$

$$\le \|\phi(0) - \psi(0)\|_X + \frac{T^2}{2}\left(L_{f1} + L_{f2}B^*\right)\|y - z\|_X.$$

For $t \in (t_k, t_{k+1}]$, we have

$$\|(y)(t) - (z)(t)\|_X \le$$

$$\le \|\phi(0) - \psi(0)\|_X + mL_1\|y - z\|_X + mTL_Q\|y - z\|_X$$

$$+ m\frac{T^2}{2}\left(L_{f1} + L_{f2}B^*\right)\|y - z\|_X$$

$$+ m\frac{T^2}{2}\left(L_{f1} + L_{f2}B^*\right)\|y - z\|_X$$

$$+ \frac{T^2}{2}\left(L_{f1} + L_{f2}B^*\right)\|y - z\|_X$$

$$\le \|\phi(0) - \psi(0)\|_X + [mL_1 + mTL_Q$$

$$+ (1+2m)\frac{T^2}{2}\left(L_{f1} + L_{f2}B^*\right)\|y - z\|_X]$$

Therefore, if $\|\phi - \psi\|_X < \delta(\varepsilon)$, then $\|y(t) - z(t)\|_X < \varepsilon$, which implies that the solution of system (1.1)-(1.3) is uniformly stable. This completes the proof of the theorem.

5. Example: Consider the following example for verifying our results.

$$D_t^\alpha x(t) = \frac{1}{\Gamma(2-\alpha)}\int_0^t (t-s)^{1-\alpha}$$

$$\Big[\frac{x(s - \sigma(x(s)))}{25 + x^2(s - \sigma(x(s)))}$$

$$+ \int_0^s \frac{\cos(s-\xi)x(\xi - \sigma(x(\xi)))}{4 + x(\xi - \sigma(x(\xi)))}d\xi\Big]ds, \qquad (5.1)$$

$t \in [0,T], t \ne t_i$,

$$\Delta x(t_i) = \int_{-d}^{t_i} \frac{\gamma_i(t_i - s)x(s)}{25}ds,$$

$$\Delta x'(t_i) = \int_{-d}^{t_i} \frac{\gamma_i(t_i - s)x(s)}{9}ds,$$

$$x(t) = \phi(t), \quad t \in [-d, 0], \quad x'(0) = 0,$$

where

$$\gamma_i \in C([0,\infty), X), \sigma \in C(X, [0,\infty)),$$

$$0 < t_1 < t_2 < t_3 < T.$$

Set $\gamma > 0$, and choose PC_T to be defined by, $PC_T = \{\phi \in PC([-d,T], X) : \lim_{t \to -d}\phi(t)\ exist\}$ with the norm $\|\phi\|_{PC_0} = \sup_{t\in[-d,T]}|\phi(t)|, \phi \in PC_0$. Set

$$\rho(t,\varphi) = t - \sigma(\varphi(0)), \varphi \in J \times PC_T,$$

$$f(t,\varphi,Bx) = \frac{\varphi}{25 + (\varphi)^2}$$

$$+ \int_0^t \cos(t-s)\frac{x}{(4+x)}ds; \quad \varphi \in PC_T,$$

$$Q_k(x(t_k)) = \int_{-d}^{t_i} \frac{\gamma_i(t_i - s)x(s)}{25}ds,$$

$$I_k(x(t_k)) = \int_{-d}^{t_i} \frac{\gamma_i(t_i - s)x(s)}{9}ds.$$

Under the above condition we can represent the system (5.1) in the abstract form (1.1)-(1.3). We see that all the assumption of the theorem 3.1 are satisfied with

$$\| f(t,\varphi,By)\| \le \frac{\|\varphi\|}{25} + B^*\frac{\|x\|}{4},$$

$$\| Q_k(x(t_k))\| \le \gamma * \frac{1}{25}\|x\|,$$

$$\| I_k(x(t_k))\| \le \gamma * \frac{1}{9}\|x\|.$$

Here

$$M_1 = \frac{1}{25}, M_2 = \frac{1}{4}, M_3 = \frac{\gamma *}{9}, M_4 = \frac{\gamma *}{25}, B^* = 1,$$

where $\gamma^* = \int_{-d}^{t} \gamma_i (t_i - s) ds < 0.$ Which implies that system (5.1) has a solution on [0,1].

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