EXISTENCE AND UNIQUENESS OF SOLUTION FOR IMPULSIVE FRACTIONAL ORDER STOCHASTIC INTEGRO-DIFFERENTIAL EQUATION

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Abstract: This paper is concerned with the existence of solution for an impulsive neutral fractional order functional stochastic integro-differential equation. The existence and uniqueness result is shown by using the fixed point technique on a Hilbert space.

Keywords: Fractional stochastic differential equation, Functional differential equations, Impulsive conditions, Fixed point theorem.

Introduction: Fractional differential equations have gained considerable importance due to their applications in various fields, such as physics, fluid mechanics, viscoelasticity, chemistry, engineering, control and porous media etc. In recent years, the topic of impulsive fractional functional differential equations with infinite delay is hot in point of view of research, concerning the general motivations of the theory one can see the papers [1]-[12] and references therein.

The deterministic systems made by men or nature often fluctuate due to environmental noise which is random or at least appears to be so, these systems are modelled as stochastic differential equations. So these systems are very important for discussion. Stochastic differential equations with infinite delay play an important role in recent years as mathematical models of various phenomena in both physical and social sciences. Other than the environmental noise, sometimes we have to consider the impulsive effects which exist in many evolution processes because the impulsive effects may bring an abrupt change at certain moments of time. For the literatures on stochastic one can see [13]-[17] and references therein.

Recently, Feckan et al. [18] presented a counter example to show an essence error in the formula of solutions to the impulsive Cauchy problems:

$$ {}^{c}D_t^q u(t) = f(t, u(t)), $$

$$ t \in J' = J \setminus \{t_1, t_2, ..., t_m\}, J = [0,1] $$

$$ \Delta u(t_k) = I_k(u(t_k^-)), k = 1,2,...,m, u(0) = u_0 $$

and defined the new definitions for the solutions and then established the existence and uniqueness results using classical fixed point theorems.

In this article, we are concerned with the existence and uniqueness of solution for impulsive fractionalfunctional integro-differential equation of the form:

$$ {}^{c}D_t^\alpha Q(x_t) = J_t^{1-\alpha}[f(t, x_t, K(x))]\frac{dw(t)}{dt}, $$

$$ t \in J, t \neq t_k, \quad (1.1) $$

$$ \Delta x(t_k) = I_k(x(t_k^-)), k = 1,2,...,m \quad (1.2) $$

$$ x(t) = \phi(t) \in B_h, t \in (-\infty, 0] \quad (1.3) $$

where J = [0, T], 0 < T < ∞; and ${}^{c}D_t^\alpha$, α ε (0,1) is the Caputo fractional derivative. The functions $f : J \times B_h \times H \to H$ and $Q(\phi) = \phi(0) + g(0, \phi)$ are given and satisfy some assumptions, where B_h is a phase space. We assume that $x_t = (-\infty, 0] \to H, x_t(s) = x(t+s), s \leq 0,$ belong to an abstract phase space B_h defined later. The term $Kx(t)$ is given by $Kx(t) = \int_0^t S(t,s)x(s)ds$

where $S \in C(D, R^+)$,the set of all positive functions which are continuous on $D = \{(t,s) \in R^2 : 0 \leq s \leq t < T\}$. Here $0 \leq t_0 < t_1 < ... < t_m < t_{m+1} \leq T, I_k \in C(H, H),$ $(k = 1,2,..,m)$ are bounded functions, $\Delta x(t_k) = x(t_k^+) - x(t_k^-), x(t_k^+) = \lim_{h\to 0} x(t_k + h)$ and $x(t_k^-) = \lim_{h\to 0} x(t_k - h)$ represent the right and left-hand limits of x(t) at $t = t_k$ respectively, also we take $x(t_i^-) = x(t_i)$ .The initial data $\phi = \{\phi(t), t \in (-\infty, 0]\}$ is an F_0-measurable, B_h-valued random variable independent of $w(t)$ with finite second moments.

To the best of our knowledge, the existence and uniqueness of mild solution for the system (1.1)-(1.3) is an untreated topic yet in the literature and this fact is the motivation of the present work.

Preliminaries: In this section, we introduce some basic definitions, properties and lemmas which are required for establishing our results. Let H,K be two separable Hilbert spaces and L(H,K) be the space of bounded linear operators from K into H. For convenience, we will use the same notation $\|.\|$ to denote the norms in H,K and L(K,H), and use (.,.) to denote the inner product of H and K without any confusion. Let $(\Omega, F, \{F_t\}_{t\geq 0}, P)$ be a complete filtered probability space satisfying that F_0 contains all P-null sets of F . $W=(W_t)_{t\geq 0}$ be a Q-Wiener process defined on $(\Omega, F, \{F_t\}_{t\geq 0}, P)$ with the covariance operator Q such that $TrQ<\infty$. We assume
that there exists a complete orthonormal system $\{e_k\}_{k\geq 1}$ in K, a bounded sequence of nonnegative real numbers λ_k such that $Qe_k=\lambda_k e_k, k=1,2,\dots$ and a sequence of independent Brownian motions $\{\beta_k\}_{k\geq 1}$ such that

$$(w(t),e)_K=\sum_{k=1}^{\infty}\sqrt{\lambda_k}(e_k,e)_K\,\beta_k(t), e\in K, t\geq 0.$$

Let $L_2^0=L_2(Q^{\frac{1}{2}}K,H)$ be the space of all Hilbert Schmidt operators from $Q^{\frac{1}{2}}K$ to H with the inner product $<\varphi,\psi>_{L_2^0}=Tr[\varphi Q\psi*]$.

Now, we introduce an abstract space phase B_h Assume that $h:(-\infty,0]\to(0,\infty)$ with $l=\int_{-\infty}^{0}h(t)d(t)<\infty$ a continuous function. The phase B_h defined by $B_h=\{\phi:(-\infty,0]\to H\}$, for any $a>0,(E|\phi(\theta)|^2)^{1/2}$ is bounded and measurable function on [-a,o] with $\phi(0)=0$ and $\int_{-\infty}^{0}h(s)\sup_{s\leq\theta\leq 0}(E|\phi(\theta)|^2)^{1/2}ds<\infty$.

If B_h is endowed with the norm

$$\|\phi\|_{B_h}=\int_{-\infty}^{0}h(s)\sup_{s\leq\theta\leq 0}(E|\phi(\theta)|^2)^{1/2}ds, \phi\in B_h$$

then ($(B_h,\|.\|_{B_h})$) is a Banach space [19], [20].
Now we consider the space

$B_h^{'}=\{x:(-\infty,T]\to H$ such that $x|_{J_k}\in C(J_k,H)$ and there exists $x(t_k^+)$ and $x(t_k^-), x_0=\phi\in B_h, k=1,2,\dots,m\}$, where $x|_{J_k}$ is the restriction of x to $J_k=(t_k,t_{k+1}], k=0,1,2,\dots,m$. The function $\|.\|_{B_h^{'}}$ to be a semi-norm in $B_h^{'}$ it is defined by

$$\|x\|_{B_h^{'}}=\|\phi\|_{B_h}+\sup_{s\in[0,T]}(E\|x(s)\|^2)^{1/2}, x\in B_h^{'}$$

Lemma 2.1. Assume that $x\in B_h^{'}$ then for $t\in J, x_t\in B_h$. Moreover,

$$l(E\|x(s)\|^2)^{1/2}\leq l\sup_{s\in[0,T]}(E\|x(s)\|^2)^{1/2}+\|x_0\|_{B_h}$$

,where $l=\int_{-\infty}^{0}h(s)ds<\infty.$

Definition 2.2. The Riemann-Liouville fractional integral operator for order $\alpha>0$, of a function $f:R^+\to R$ and $f\in L^1(R^+,X)$ is defined by

$$J_t^0 f(t)=f(t),$$

$$J_t^{\alpha} f(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s)ds, \alpha>0, t>0$$

where $\Gamma(.)$ is the Gamma function.

Definition 2.3. Caputo's derivative of order $\alpha>0$ for a function $f:[0,\infty)\to R$
is defined as

$$D_t^{\alpha}f(t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t(t-s)^{n-\alpha-1}f^{(n)}(s)ds$$

$$=J^{n-\alpha}f^{(n)}(t)$$

for $n-1\leq\alpha<n, n\in N$. If $0<\alpha\leq 1$ then

$$D_t^{\alpha}f(t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t(t-s)^{-\alpha}f^{(1)}(s)ds.$$

Obviously, Caputo's derivative of a constant is equal to zero.

Lemma 2.4. A measurable F_t adapted stochastic process $x:(-\infty,T]\to H$ is called a solution of the system (1.1)-(1.3) if $x_0=\phi\in B_h$ on $(-\infty,0]$, $\Delta x|_{t=t_k}=I_k(x(t_k^-)), k=1,2,\dots,m$ the restriction of x(.) to the interval $[0,T)\setminus t_1,t_2,\dots,t_m$ is continuous and satisfies the following fractional integral equation

$$x(t)=\begin{cases}\phi(t),\ t\in(-\infty,0]\\ \phi(0)+g(0,\phi)-g(t,x_t)+\\ \int_0^t f(s,x_s,Kx(s))dw(s),\ t\in[0,t_1]\\ \phi(0)+g(0,\phi)+g(t_1,x_{t_1}+I_1(x_{t_1^-}))\\ -g(t_1,x_{t_1})+I_1(x(t_1^-))-g(t,x_t)+\\ \int_0^t f(s,x_s,Kx(s))dw(s),\ t\in(t_1,t_2]\\ \ldots\ldots\\ \phi(0)+g(0,\phi)+\\ \sum_{i=1}^m[g(t_i,x_{t_i}+I_i(x_{t_i^-}))-g(t_i,x_{t_i})]\\ +\sum_{i=1}^m I_i(x(t_i^-))-g(t,x_t)+\\ \int_0^t f(s,x_s,Kx(s))dw(s),\ t\in(t_m,T]\end{cases}$$

Proof. If $t\in[0,t_1]$ then

$${}^cD_t^\alpha[x(t)+g(t,x_t)]=J_t^{1-\alpha}[f(t,x_t,K(x))]\frac{dw(t)}{dt}$$

now applying the Riemann-Liouville fractional integral operator on both sides, we get

$$x(t)+g(t,x_t)+c_1=\int_0^t f(s,x_s,Kx(s)dw(s).$$

Using initial condition, we get $c_1=-\phi(0)-g(0,\phi)$ .Then

$$x(t)=\phi(0)+g(0,\phi)-g(t,x_t)+\int_0^t f(s,x_s,Kx(s)dw(s). \quad (2.2)$$

If $t\in(t_1,t_2]$ then

$${}^cD_t^\alpha[x(t)+g(t,x_t)]=J_t^{1-\alpha}[f(t,x_t,K(x))]\frac{dw(t)}{dt},$$

$$x(t_1^+)=x(t_1^-)+I_1(x(t_1^-)).$$

Applying the Riemann-Liouville fractional integral operator on both sides, we get

$$x(t)+g(t,x_t)+c_2=\int_0^t f(s,x_s,Kx(s)dw(s).$$

Using initial condition, we get

$$x(t_1^+)+g(t_1,x_{t_1^+})+c_2=\int_0^{t_1} f(s,x_s,Kx(s)dw(s) \quad (2.3)$$

$$x(t_1)+I_1(x(t_1^-))+g(t_1,x_{t_1}+I_1(x_{t_1^-}))+c_2=\int_0^{t_1} f(s,x_s,Kx(s)dw(s) \quad (2.4)$$

thus

$$c_2=-\phi(0)-g(0,\phi)+g(t_1,x_{t_1})-g(t_1,x_{t_1}+I_1(x_{t_1^-}))-I_1(x(t_1^-))$$

then

$$x(t)=\phi(0)+g(0,\phi)+g(t_1,x_{t_1}+I_1(x_{t_1^-}))-g(t_1,x_{t_1})+I_1(x(t_1^-))-g(t,x_t)+\int_0^t f(s,x_sKx(s))dw(s). \quad (2.5)$$

Similarly, if $t\in(t_k,t_{k+1}]$ then we have

$$x(t)=\phi(0)+g(0,\phi)+\sum_{i=1}^k[g(t_i,x_{t_i}+I_i(x_{t_i^-}))-g(t_i,x_{t_i})]+\sum_{i=1}^k I_i(x(t_i^-))-g(t,x_t)+\int_0^t f(s,x_s,Kx(s))dw(s). \quad (2.6)$$

This is complete the proof of the lemma.

3. Existence and uniqueness result

In this section, we establish the existence and uniqueness of solutions to the system (1.1)-(1.3). In order to establish the results, we impose the following conditions:

(H1) $f:t\times B_h\times H\to H$ is jointly continuous and there are positive constants L_{f1},L_{f2} such that

$$\|f(t,\gamma,x)-f(t,\psi,y)\|_H^2\le L_{f1}\|\gamma-\psi\|_{B_h}^2+L_{f2}E\|x-y\|_H^2.$$

(H2) $g:t\times B_h\to H$ is continuous and there is positive constant L_g such that

$$E\|g(s,z)-g(s,z^*)\|_H^2\le L_g\|z-z^*\|_H^2$$

(H3) $I_k:B_h\to H$ is continuous and there is positive constant L_I such that

$$E\|I_k(x)-I_k(y)\|_H^2 \le L_I E\|x-y\|_H^2$$

for all $x,y,z,z^* \in H; \gamma,\psi \in B_h, t\in[0,T]$ and each k = 1, 2, ..., m

Our result is based on Banach Contractions principle.

Theorem 3.1. Suppose that the assumptions (H1) - (H3) hold, and

$$\Delta = \{m(5L_g[L_I l + l] + 5L_g l + 5L_I) + 5L_g l + 5T^2[L_{f1} l + L_{f2}K^*]\} < 1$$

where $K^* = \sup_{t\in[0,T]} \int_0^t \|S(t,s)ds\|$ Then the system (1.1)-(1.3) has a unique solutions on J.

Proof. We prove the result by transforming the problem (1.1)-(1.3) in to fixed point problem. Consider the operator $P: B_h^{'} \to B_h^{'}$ defined by

$$P(x)(t) = \begin{cases} \phi(0) + g(0,\phi) - g(t,x_t) + \int_0^t f(s,x_s,Kx(s))dw(s), t\in[0,t_1] \\ \phi(0)+g(0,\phi)+g(t_1,x_{t_1}+I_1(x_{t_1^-})) - g(t_1,x_{t_1}) + I_1(x(t_1^-)) - g(t,x_t) + \int_0^t f(s,x_s,Kx(s))dw(s), t\in(t_1,t_2] \\ \ldots\ldots \\ \phi(0)+g(0,\phi)+ \sum_{i=1}^m [g(t_i,x_{t_i}+I_i(x_{t_i^-})) - g(t_i,x_{t_i})] + \sum_{i=1}^m I_i(x(t_i^-)) - g(t,x_t) + \int_0^t f(s,x_s,Kx(s))dw(s), t\in(t_m,T] \end{cases} \quad (3.1)$$

Let $y(.) : (-\infty, T] \to H$ be the function defined by

$$y(t) = \begin{cases} \phi(t), t\in(-\infty,0] \\ 0, t\in J \end{cases}$$

then $y_0 = \phi$ (3.2)

For each $z: J\to H$ with $z|_{t_k} \in C(I_k, H), k=1,2,\ldots,m$ and z(0) = 0, we denote by $\bar{z}$ the function defined by

$$\bar{z} = \begin{cases} 0, t\in(-\infty,0] \\ z(t), t\in J \end{cases} \quad (3.3).$$

If x(.) satisfies the system (2.1), then we can decompose x(.) as $x(t) = y(t) + \bar{z}(t)$, which implies $x_t = y_t + \bar{z}_t$ for $t\in J$ and the function z(.) satisfies

$$z(t) = \begin{cases} \phi(0)+g(0,\phi) - g(t,y_t+\bar{z}_t) + \int_0^t f(s,y_s+\bar{z}_s, K(y(s)+\bar{z}(s)))\times dw(s), t\in[0,t_1] \\ \phi(0)+g(0,\phi) + g(t_1, y_{t_1}+\bar{z}_{t_1} + I_1(y_{t_1^-}+\bar{z}_{t_1^-})) - g(t_1,y_{t_1}+\bar{z}_{t_1}) + I_1(y(t_1^-)+\bar{z}(t_1^-)) - g(t,y_t+\bar{z}_t) + \int_0^t f(s,y_s+\bar{z}_s, K(y(s)+\bar{z}(s)))dw(s), t\in(t_1,t_2] \\ \ldots \\ \phi(0)+g(0,\phi) + \sum_{i=1}^m [g(t_i,y_{t_i}+\bar{z}_{t_i}+I_i(y_{t_i^-}+\bar{z}_{t_i^-})) - g(t_i,y_{t_i}+\bar{z}_{t_i})] + \sum_{i=1}^m I_i(y(t_i^-)+\bar{z}(t_i^-)) - g(t,y_t+\bar{z}_t) + \int_0^t f(s,y_s+\bar{z}_s, K(y(s)+\bar{z}(s)))dw(s), t\in(t_m,T] \end{cases} \quad (3.4)$$

Set $B_h^{''}$ such that $z_0 = 0$ and for any $z\in B_h^{''}$ we have

$$\|z\|_{B_h^{''}} = \|z_0\|_{B_h} + \sup_{t\in J}(E\|z(t)\|^2)^{1/2} = \sup_{t\in J}(E\|z(t)\|^2)^{1/2}$$

.

Thus is a Banach space. Define operator $N: B_h^{''} \to B_h^{''}$

$$(Nz)(t) = \begin{cases} \phi(0) + g(0,\phi) - g(t, y_t + \bar{z}_t) + \\ \int_0^t f(s, y_s + \bar{z}_s, K(y(s) + \bar{z}(s))) \times \\ dw(s), t \in [0, t_1] \\ \phi(0) + g(0,\phi) \\ + g(t_1, y_{t_1} + \bar{z}_{t_1} + I_1(y_{t_1^-} + \bar{z}_{t_1^-})) \\ - g(t_1, y_{t_1} + \bar{z}_{t_1}) + I_1(y(t_1^-) + \bar{z}(t_1^-)) \\ - g(t, y_t + \bar{z}_t) + \int_0^t f(s, y_s + \bar{z}_s, \\ K(y(s) + \bar{z}(s)))dw(s), t \in (t_1, t_2] \\ \ldots \\ \phi(0) + g(0,\phi) + \qquad (3.5) \\ \sum_{i=1}^{m} [g(t_i, y_{t_i} + \bar{z}_{t_i} + I_i(y_{t_i^-} + \bar{z}_{t_i^-})) - \\ g(t_i, y_{t_i} + \bar{z}_{t_i})] + \sum_{i=1}^{m} I_i(y(t_i^-) + \bar{z}(t_i^-)) \\ - g(t, y_t + \bar{z}_t) + \int_0^t f(s, y_s + \bar{z}_s, \\ K(y(s) + \bar{z}(s)))dw(s), t \in (t_m, T]. \end{cases}$$

In order to prove existence result, it is enough to show that N has a unique fixed point. Let $z, z^* \in B_h''$, then for $t \in [0, t_1]$, we have

$$E\left\|(Nz)(t) - (Nz^*)(t)\right\|_H^2$$
$$\leq 2E\left\|g(t, y_t + \bar{z}_t) - g(t, y_t + \bar{z}_t^*)\right\|_H^2$$
$$+ 2E \,\| \int_0^t [f(s, y_s + \bar{z}_s, K(y(s) + \bar{z}(s))) -$$
$$f(s, y_s + \bar{z}_s^*, K(y(s) + \bar{z}^*(s)))]dw(s) \,\|_H^2$$

$$\leq [2L_g l + 2T^2 (L_{f1} l + L_{f2} K^*)]\left\|z - z^*\right\|_{B_h''}^2.$$

For $t \in (t_1, t_2]$, we get

$$E\left\|(Nz)(t) - (Nz^*)(t)\right\|_H^2$$
$$\leq 5E \,\| [g(t_1, y_{t_1} + \bar{z}_{t_1} + I_1(y_{t_1^-} + \bar{z}_{t_1^-}))$$
$$- g(t_1, y_{t_1} + \bar{z}_{t_1}^* + I_1(y_{t_1^-} + \bar{z}_{t_1^-}^*))] \,\|_H^2$$
$$+ 5E\left\|[g(t_1, y_{t_1} + \bar{z}_{t_1}) - g(t_1, y_{t_1} + \bar{z}_{t_1}^*)]\right\|_H^2$$
$$+ 5E\left\|[I_1(y(t_1^-) + \bar{z}(t_1^-)) - I_1(y(t_1^-) + \bar{z}^*(t_1^-))]\right\|_H^2$$
$$+ 5E\left\|[g(t, y_t + \bar{z}_t) - g(t, y_t + \bar{z}_t^*)]\right\|_H^2$$
$$+ 5E \,\| \int_0^t [f(s, y_s + \bar{z}_s, K(y(s) + \bar{z}(s))) -$$
$$f(s, y_s + \bar{z}_s^*, K(y(s) + \bar{z}^*(s)))]dw(s) \,\|_H^2$$
$$\leq \{(5L_g [L_I l + l] + 5L_g l + 5L_I) + 5L_g l$$
$$+ 5T^2 [L_{f1} l + L_{f2} K^*]\}\left\|z - z^*\right\|_{B_h''}^2.$$

:

Similarly, when $t \in (t_i, t_{i+1}], i = 2,3,..m$ we obtained

$$E\left\|(Nz)(t) - (Nz^*)(t)\right\|_H^2 \leq \{k(5L_g [L_I l + l]$$
$$+ 5L_g l + 5L_I) + 5L_g l +$$
$$5T^2 [L_{f1} l + L_{f2} K^*]\}\left\|z - z^*\right\|_{B_h''}^2$$

Thus for all $t \in [0, T]$, we can summarize as

$$E\left\|(Nz)(t) - (Nz^*)(t)\right\|_H^2 \leq \{m(5L_g [L_I l + l]$$
$$+ 5L_g l + 5L_I) + 5L_g l +$$
$$5T^2 [L_{f1} l + L_{f2} K^*]\}\left\|z - z^*\right\|_{B_h''}^2$$
$$\leq \Delta \left\|z - z^*\right\|_{B_h''}^2$$

Since $\Delta < 1$, hence N is a contraction map and therefore it has a unique fixed point $z \in B_h''$ which is the solution of the system (1.1)-(1.3) on $(-\infty, T]$. This complete the proof of the theorem.

4.**Application**: To illustrate the application of theory we consider the following partial integro-differential equation with fractional derivative of the form

$$\frac{\partial^q Q(u_t)(x)}{\partial t} = \frac{1}{\Gamma(1-q)}\int_0^t (t-s)^{-q}$$

$$[\int_{-\infty}^{s} G(s,x,\xi-s)Q_2(u(\xi,s))d\xi$$

$$+\int_0^s k(\xi,s)e^{-u(\xi,x)}ds]\frac{d\beta(t)}{dt}, \quad (4.1)$$

$$x \in [0,\pi], t \in [0,b], t \neq t_k. \quad (4.2)$$

$$u(t,0) = 0 = u(t,\pi), t \geq 0. \quad (4.3)$$

$$u(t,x) = \phi(t,x), t \in (-\infty,0], x \in [0,\pi]. \quad (4.4)$$

$$\Delta u(t_i)(x)$$

$$= \int_{-\infty}^{t_i} q_i(t_i - s)u(s,x)ds, x \in [o,\pi], \quad (4.5)$$

where $\frac{\partial^q}{\partial t}$ is the Caputo's fractional derivatives of order $q \in (0,1), 0 < t_1 < t_2 < \ldots < t_n < T$ are prefixed numbers, $\phi \in B_h$ and

$$Q(u_t)(x) = u(t,x) + \int_{-\infty}^{t} a(t,x,s-t)Q_1(u(s,x))ds.$$

Consider $G = L^2[0,\pi]$ and $h(s) = e^{2s}, s < 0$ then $l = \int_{-\infty}^{0} h(s)ds = \frac{1}{2}$. We define

$$\| \phi \|_{B_h} = \int_{-\infty}^{0} h(s)\sup_{\theta\in[s,0]} \| \phi(\theta) \|_{L^2} \, ds.$$

Hence for $(t,\phi) \in [0,T] \times B_h$, where $\phi(\theta)(x) = \phi(\theta,x), (\theta,x) \in (-\infty,0]\times[0,\pi]$. Set $u(t)(x) = u(t,x)$,

$$g(t,\phi)(x) = \int_{-\infty}^{0} a(t,x,\theta)Q_1(\phi(\theta)(x))d\theta,$$

$$f(t,\phi,Ku(t))(x) = \int_{-\infty}^{0} G(t,x,\theta)Q_2(\phi(\theta)(x))d\theta + Ku(t)(x),$$

$$I_i(\phi)(x) = \int_{-\infty}^{0} q_i(-\theta)\phi(\theta)(x)d\theta,$$

where $Ku(t)(x) = \int_0^t k(s,t)e^{-u(s,x)}ds$. Then with these settings the equations (4.1)-(4.5) can be written in the abstract form of the equations (1.1)-(1.3). Suppose further that:

(i)The functions $Q_i, i = 1,2$ are continuous and $u(\theta,x), v(\theta,x)$ are continuous in $(-\infty,0]\times[0,\pi]$,

$$0 \leq Q_i(u(\theta)(x)) - Q_i(v(\theta)(x)) \leq \int_{-\infty}^{0} e^{2s} \| u(s,\cdot) - v(s,\cdot) \|_{L^2} \, ds.$$

(ii) The function $G(t,x,\theta)$, continuous in $[0,T]\times[0,\pi]\times(-\infty,0)$ and satisfying $\int_{-\infty}^{0} H^2(t,x,\theta)d\theta < C, C > 0$.

(iii) The functions $q_i \in C(R,R)$ and $d_i = (\int_{-\infty}^{0} \frac{q_i^2(-\theta)}{h(\theta)}d\theta) < \infty$ for $i = 1,2,\ldots,m$. Now we can see that: For $(t,\phi,Bu(t)) \in [0,T]\times B_h \times X$ we have

$$\|f(t,\phi,Bu(t)) - f(t,\psi,Bv(t))\|_{L^2} =$$

$$[\int_0^\pi \{\int_{-\infty}^{0} G(t,x,\theta)(Q_2(\phi(\theta)(x)) - Q_2(\psi(\theta)(x)))d\theta + Bu(t)(x) - Bv(t)(x)\}^2 dx]^{1/2}$$

$$\leq [\int_0^\pi \{\int_{-\infty}^{0} G(t,x,\theta)(Q_2(\phi(\theta)(x)) - Q_2(\psi(\theta)(x)))d\theta\}^2 dx]^{1/2}$$

$$+ [\int_0^\pi \{Bu(t)(x) - Bv(t)(x)\}^2 dx]^{1/2}$$

$$\leq [\int_0^\pi \{\int_{-\infty}^{0} G(t,x,\theta) \times (\int_{-\infty}^{0} e^{2s}\|\phi(s)(.) - \psi(s)(.)\|_{L^2} ds) \times d\theta\}^2 dx]^{1/2} + \|Bu(t) - Bv(t)\|_{L^2}$$

$$\le [\int_0^{\pi} \{ \int_{-\infty}^{0} G(t,x,\theta) \times$$
$$(\int_{-\infty}^{0} e^{2s} \sup_{s\in[\theta,0]} \|\phi(s)-\psi(s)\|_{L^2} ds) \times$$
$$d\theta \}^2 dx]^{1/2} + \|Bu(t)-Bv(t)\|_{L^2}$$
$$\le [\int_0^{\pi} \{ \int_{-\infty}^{0} G(t,x,\theta) d\theta \}^2 dx]^{1/2} \|\phi-\psi\|_{B_h}$$
$$+ \|Bu(t)-Bv(t)\|_{L^2}$$
$$\le C\sqrt{\pi} \|\phi-\psi\|_{B_h} + \|Bu(t)-Bv(t)\|_{L^2}.$$

Hence function f satisfies (H1) and in a similar way we can show that g and I_i may satisfy (H2)-(H3), respectively. All the conditions of Theorem 3.1 have been fulfilled so we deduced that the system (4.1)-(4.5) has a mild solution on $(-\infty, T]$.

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