ALGORITHMIC ASPECTS OF CONTINUOUS THREE-STEP DECOMPOSITION OF SOME GRAPHS

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Abstract: Let G = (V, E) be a finite connected simple graph. In this paper we investigate the continuous three step star decompositions of complete bipartite graphs, complete tripartite graphs, Spider graphs and Lobster graphs. Also the continuous three step path decompositions of the tensor product of P_n and K_2 investigated. The corresponding algorithms are also developed.

Key Words: Complete Bipartite Graph, Complete Tripartite Graphs, Continuous Three Step Star Decomposition, Lobster Graphs ,Spider Graphs, Tensor product.

Introduction: An undirected simple graph with the property that there is a path between every pair of vertices is known as a connected graph. The degree of a vertex u of any graph is the number of edges incident with u and is denoted by d(u) and the distance between the two vertices u and v of G is the length of the shortest u-v path in G and is denoted by d(u,v)[3]. A decomposition of G into subgraphs {G_i / i = 1, 4, 7, 10…. .., 3n - 2}, where if each G_i is connected and $|E(G_i)|$ = 3i - 2, for all i = 1, 2, …... n and a, d ε Z, G is called a Continuous Three-Step Decomposition (CTSD) of the graph For graph terminology we refer to J.A.Bondy and U.S.R. Murty[1].

2. The Continuous Three-step decomposition of some Partite Graphs.

Theorem 2.1: The complete bipartite graph $K_{m,n}$ (m ≤ r) can be decomposed into stars $S_1, S_4, \ldots \ldots S_{3n-2}$ if $mr = \frac{n}{2}(3n-1)$ and r ≥ 3n - 2.

Proof: The theorem can be proved by the method of mathematical induction. Let P(n) be the decomposition of n subsets $S_1, S_4, \ldots, S_{3n-2}$, $mr = \frac{n}{2}(3n-1)$ and r ≥ 3n - 2. Let P(n) be the statement P(n): $S_1, S_4, \ldots S_{3n-2}$ if $mr = \frac{n}{2}(3n-1)$ and r ≥ 3n - 2. When n = 1, P(1): S1, $mr = \frac{1}{2} \times 2 = 1$, r ≥ 1. Therefore the theorem is true for n = 1. Assume that the theorem is true for n = k. $S_1, S_4, \ldots S_{3k-2}$ if $mr = \frac{k}{2}(3k-1)$ and r ≥ 3k - 2. To prove P(n) is true for n = k + 1.ie) To prove $S_1, S_4, \ldots S_{3k+1}$ if $mr = \frac{(k+1)}{2}(3k+2)$ and r ≥ 3k + 1. Now, P(k+1): $S_1, S_4, \ldots S_{3k-2}, S_{3k+1}$ = $\frac{k}{2}(3k-1) + (3k+1). \frac{k}{2}(3k+2-3) + (3k+2-1) = \frac{3k+2}{2}(k+1)$. Thus the theorem is proved.

Algorithm 3.2: To decompose the bipartite graph $K_{m,n}$

Step 1: Read m, r
Step 2: Calculate q = mr
Step 3: Calculate n = $(1+\sqrt{1+24q}) / 6$
Step 4: Calculate x = 3n - 2
Step 5: If(x < r) then "Not possible"
else doc = x
For(i = x to 1 step - 3
Decompose(x)
Step 5 : Stop

Output:

Table 2.1 The number of decompositions

Sl.No	m	R	q	x	n
1	1	12	12	3	7
2	2	11	22	4	10
3	3	17	51	6	16
4	3	39	117	9	25
5	3	70	210	12	34
6	4	23	92	8	22
7	4	44	176	11	31
8	5	29	145	10	28
9	5	66	330	15	43
10	7	41	287	14	40

Result 2.4[4]: A complete tripartite graph $K_{m1, m2, m3}$ admits CTSD if $q(K_{m1, m2, m3}) = m_1m_2 + m_2m_3 + m_1m_3$ is of the form $\frac{n}{2}(3n-1)$, where n is a positive integer and $\max(m_1, m_2, m_3) \geq 3n - 2$.

Algorithm 2.5: To decompose the complete tripartite graph

Step 1: Read a, b, c
Step 2: Calculate q = ab + bc + ca
Step 3: sum = 0
Step 4: For i = 1 to n step 3
sum = sum + i
if(q == sum)
print $K_{a,b,c}$
Step 5: Calculate y = 3n - 2
If max(a, b, c) > y
Decompose(q)
Step 6 : Stop

Example 3.6: $K_{1,2,11}$, $K_{1,3,8}$ $K_{1,2,11}$, $K_{1,3,12}$, $K_{2,3,9}$, $K_{2,4,14}$, $K_{3,4,15}$, $K_{3,4,19}$.

3. Continuous Three Step Decomposition of

Spider Graphs

Definition 3.1[10] A tree T with exactly one vertex of degree ≥ 3 is called a spider tree.

Notation: Let W be the set of all pendant vertices of a spider tree T and u be the vertex of degree ≥ 3 in T.

Theorem 3.2 A spider tree T which is not a star admits CTSSD ($S_1, S_4, \ldots S_{(3n-2)}$) if and only if T - W is a path of length one.

Proof:

Assume that T admits CTSSD ($S_1, S_4, \ldots S_{(3n-2)}$).

To prove T is a path of length one. Suppose not, then is a path of length atleast two. Let T – W = P_2.

We take S_1 = vw. Then T - S_1 is a spider tree with $\frac{1}{2}(3n^2 - n - 2)$ edges. From these $\frac{1}{2}(3n^2 - n - 2)$ edges, there exits S_2 such that the two edges of S_2 are st and tu. But the existence of S_2 contradicts CTSSD. Hence T - W is a path of length one.

Conversely assume that T - W is a path of length one such that T is not a star. Let S_1 = vw , then the remaining edges can be decomposed into $S_4, \ldots S_{(3n-2)}$. Hence the theorem.

Theorem 3.3: If a spider tree T with diam(T) = 2 with d(u) = $\frac{n}{2}(3n - 1)$, then T admits CTSSD.

Proof: Since diam(T) = 2, T is a star. Also, since d(u) = n / (2) (3n -1) is $K_{1, \frac{n}{2}(3n-1)}$ Therefore q(T) = $\frac{n}{2}(3n - 1)$. Hence T admits CTSSD.

Theorem 3.4: If a spider tree with diam(T) = 3 with d(u) = $\frac{1}{2}(3n^2 - n - 2)$, then T admits CTSSD.

Proof: Since diam(T) = 3 with d(u) = $\frac{1}{2}(3n^2 - n - 2)$ there are $\frac{1}{2}(3n^2 - n - 4)$ pendant edges incident with u. Also there is a path of length two in T with origin is a pendant vertex (say v) and terminus is v.

Take S_1 = e with the terminus of S_1 is v. Then T - e is a star. Then we can easily decompose T - e into $S_4, \ldots S_{(3n-2)}$. Hence T admits CTSSD.

Algorithm 3.5 : To decompose the given graph into CTSSD

Step 1: Read q // the number of edges
Step 2: Initialize sum = 0
Step 3: For i = 1 to q step 3
    sum = sum + i
    if(q = sum) goto step 5
Step 4: "Not Possible"
Step 5 : Calculate y = 3n - 2
    For i = 4 to y step 3
    Decompose(i)
Step 6 : Stop

4. Continuous Three Step Decomposition of Lobster Graphs

Definition 4.1 Caterpillar is a tree in which the removal of pendant vertices results in a path.

Definition 4.2[9] Lobster is a tree in which the removal of pendant vertices results in a caterpillar.

Definition 4.3 The underlying path P_l of a Lobster L is a path obtained by removal of pendant vertices two times successively.

Remark 4.4 In this section, the Lobster L with q = $\frac{n}{2}(3n - 1)$

Theorem 4.5 Let L be a Lobster with underlying path P_l of length l. If L admits CTSPD($P_1, P_4, P_7, \ldots, P_{3n-2}$), then $1 + \sqrt{97 + 24l} \leq n \leq 9 + \sqrt{9 + 24l}$.

Proof: Let l be a Lobster with underlying path P_l of length l. Suppose L admits CTSPD ($P_1, P_4, P_7, \ldots, P_{3n-2}$), then diam(L) = l + 4.

Case i : The path P_1 alone can be obtained from L without taking any edge from P_l. For P_7 we must have 3 edges from P_l and etc for $P_{(3n-2)}$ we must have [(3n - 2) - 4] edges from P_l.

Therefore $l = 3 + 6 + \ldots + 3n - 6$
$= 3(1 + 2 + \ldots + n - 2)$
$= \frac{3}{2}(n - 1)(n - 2)$

$3n^2 - 9n - 2(l - 4) = 0$, Therefore $n = \frac{1}{6}(9 + \sqrt{24l + 9})$.

To get the lower bound,

Case ii The paths P_1 alone can be obtained from L without taking any edge from P_l. For P_4 we must have only 1 edge from Pl, P_7 we must have 3 edges from P_l and etc for $P_{(3n-2)}$ we must have [(3n - 2) - 4] edges from P_l.

Therefore $l = 1 + 3 + 6 + \ldots + 3n - 6$
$= 1 + 3(1 + 2 + \ldots + n - 2)$
$= 1 + \frac{3}{2}(n - 1)(n - 2)$

$3n^2 - 9n - 2(l - 4) = 0$, Therefore $n = \frac{1}{6}(9 + \sqrt{24l - 15})$.

Case iii : All the paths P_{3i-2} must have 1, 2, 3, 6, … .. [(3n - 2) - 4] edges from P_l. Therefore $l = 1 + 2 + 3 + 6 + \ldots + 3n - 6 = 3 + 3(1 + 2 + \ldots + n - 2) = 3 + \frac{3}{2}(n - 1)(n - 2)$

$3n^2 - 9n - 2(l - 6) = 0$, Therefore $n = \frac{1}{6}(9 + \sqrt{24l - 63})$.

Case iv: P_{3i-2} and P_{3j-1} are two paths in the decomposition with origin $u^{||}_0$ and $u^{||}$ respectively.

Then we have $l = 2 + 6 + 10 + \ldots + 3n - 2$
$= 8 + \frac{(n-3)}{2}(8 + 3n)$

$3n^2 - n - 2(l - 4) = 0$, Therefore $n = \frac{1}{6}(9 + \sqrt{24l + 97})$.

Hence $1 + \sqrt{97 + 24l} \leq n \leq 9 + \sqrt{9 + 24l}$.

5. Continuous Three Step Decomposition of product graphs

Lemma 5.1[6]**:** Let $m \equiv 0 (\mathrm{mod}\ 2)$. The set {1, 4 , 7,……,3m - 2} can be partitioned into two sets S_1, and S_2 such that $\sum_{x \in S1} x = \sum_{y \in S2} y = n - 1$. Here $\frac{m}{2}(3m - 1) = 2n - 2$.

Proof: Let m = 4k, k ≥ 1, k ε Z. This lemma can be proved by induction on k.

When k = 1, m = 4. Now $n = \frac{m}{4}(3m - 1) + 1$. If S_1 = {1, 10} and S_2 = {4, 7} then $\sum_{x \in S1} x = 1 + 10 = 11 = 12 - 1$ *and* $\sum_{y \in S2} y = 4 + 7 = 11 = 12 - 1$. Hence the result is true for k = 1. Assume the result is true for k - 1. Hence the set {1, 4, 7, … … ,3(4k - 1) - 2} can be partitioned into two sets S_1, and S_2 such that $\sum_{x \varepsilon S1} x = \sum_{y \varepsilon S1} y$ = n – 1 = (k - 1)(12k - 13). To prove the result is proved for k. The set {1, 4, 7, … …,3(4k) - 2} can be partitioned into two sets $S^{|}_1$, and $S^{|}_2$ where $S^{|}_1 = S_1 \cup$ {12k - 2, 12k - 11} and $S^{|}_2 = S_2 \cup$ {12k - 5, 12k - 8}.

Now $\sum_{x \in S'1} x = \sum_{x \in S1} x + 12k - 2 + 12k - 11$

$= (k - 1)(12k - 13) + 24k - 13$

$= 12k^2 - k = k(12 - 1) = n - 1.$

$\sum_{y \in S'2} y = \sum_{y \in S2} y + 12k - 5 + 12k - 8$

$= (k - 1)(12k - 13) + 24k - 13 = 12k^2 - k = k(12 - 1) = n - 1.$

Hence by induction, the lemma is true for all k.

Theorem 5.2: For any integer n, $P_n \wedge K_2$ has continuous three step decomposition {H_1, H_4, … … … . ., H_{3m-2}} if and only if there exists an integer m satisfying the following properties.

(i) m = 4k or m = $\frac{1}{3}$(4k - 1),(k ≥ 1 , k ε Z).

(ii) $\frac{m}{2}$(3m - 1) = 2n - 2

Algorithm 5.3 : to compute the number of decompositions

(ii) $\frac{m}{2}$(3m - 1) = 2n - 2

Step 1 : Initialize n, m, doc, n1

Step 2 : Read the value of n1

Step 3 : for m = 4 to n step 4 do

(i) Calculate n = m (3n - 4) / 4 + 1

(ii) Calculate doc = 3m - 2

(iii) Print m, n and doc

Step 5: Goto step 3 until m > n1

Step 6: Stop.

Output: n = 10,

Table : 5.1 The values of m and n for different values of k

Sl.No	M	N	CTSD
1	4	12	H_1,……..,H_{10}
2	8	47	H_1,……..,H_{22}
3	12	106	H_1,……..,H_{34}
4	16	189	H_1,……..,H_{46}
5	20	296	H_1,……..,H_{58}
6	24	427	H_1,……..,H_{70}
7	28	582	H_1,……..,H_{82}
8	32	761	H_1,……..,H_{94}
9	36	964	H_1,……..,H_{106}
10	40	1191	H_1,……..,H_{118}

Corollary 5.4 For any integer n, $P_n \wedge P_s$ has s - 1 copies of continuous three step decomposition {H_1, H_4, ……….., H_{3m-2}} if and only if there exists an integer m satisfying the following properties.

(i) m = 4k or m = $\frac{17.}{3}$(4k - 1),(k ≥ 1, k ε Z).

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