COMMON FIXED POINT THEOREMS FOR S-WEAKLY COMMUTING, S-COMPATIBLE MAPPINGS OF COMPLETE S-FUZZY METRIC SPACES

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Abstract. In this paper we prove common fixed point theorem for S-Weakly commuting, S-Compatible maps in S-Fuzzy metric spaces.
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Introduction: Concept of S-fuzzy metric space is introduced by Singh, B. and Chauhan, M. S. [1]. They established the Banach Contraction Principle in this space. Sessa [7]introduced the weak commutativity for a pair of self maps. Further Jungck [3,4] extended these facts via concept of compatible maps. Pant, R.P. [5] introduced the notion of R-weakly commutativity of mappings in metric spaces and proved some common fixed points theorems. Later on Vasuki R. [6] defined R-weakly commuting maps in fuzzy metric spaces. In this paper we define S-Weakly commuting, S-Compatible maps in S-Fuzzy metric spaces and prove some results.

Preliminaries:

Definition 2.1 [1]. The 3- tuple (X, S, *) is said to be a S-fuzzy metric space if X is an arbitrary set, * a continuous t-norm and S is a fuzzy set on $X^3 \times (0, \infty)$, satisfying the following conditions

(i) $S(x, y, z, t) > 0$,

(ii) $S(x, y, z, t) = 1$ if and only if $x = y = z$ (coincidence),

(iii) $S(x, y, z, t) = S(y, z, x, t) = S(z, y, x, t) = \ldots$ (Symmetry),

(iv) $S(x, y, z, r + s + t) \geq S(x, y, w, r) * S(x, w, z, s) * S(w, y, z, t)$, (tetrahedral inequality)

(v) $S(x, y, z, .) : (0, \infty) \to [0,1]$ is continuous for all x, y, z, w ∈ X and r, s, t > 0.

Definition 2.2 [1]. A sequence {xn} in a S-fuzzy metric space (X, S, *) is a Cauchy sequence if and only if for each ε > 0, t > 0, there exists no ∈ N such that $S(x_n, x_m, x_p, t) > 1- \varepsilon$ for all n, m, p ≥ no.

Definition 2.3 [1]. A sequence{xn} in a fuzzy metric space ***(X, M, *)*** converges to x if and only if for each ε > 0, t > 0, there exists no ∈ N such that $M(x_n,x,t) > 1-\varepsilon$ for all n ≥ no.

Definition 2.4 [1]. A S-fuzzy metric space in which every Cauchy sequence is a convergent sequence, is called a complete S-fuzzy metric space.

Geometrically, S(x, y, z, t) represents the fuzzy perimeter of the triangle whose vertices are the points x, y and z with respect to t > 0.

Definition 2.5 [7]. Let A and B be mappings from a metric space (X, d) into itself.

Then A and B are said to be weakly commuting mappings on X if

$d(ABx, BAx) \leq d(Ax, Bx)$ for all x in X.

Definition 2.6 [3,4]. Let A and B be mappings from a metric space (X, d) into itself.Then A and B are said to be compatible mappings on X if

$\lim_{n \to \infty} d(ABx_n, BAx_n) = 0$ where {xn} is a sequence in X such that

$\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Bx_n = t$ for some point t in X.

Definition 2.7 [2] Two mappings f and g of a fuzzy metric space (X, M, *) into itself are said to be weakly commuting if $M(fgx, gfx, t) \geq M(fx, gx, t)$ for each x in X.

Definition 2.8 [2]. Self mappings F and G of a fuzzy metric space (X, M, *) are said to be compatible iff $M(FGx_n, GFx_n, t) \to 1$ for all t > 0, whenever {xn} is a sequence in X such that $Gx_n, Fx_n \to y$ for some y in X.

Now we define S-weakly commuting maps and S-compatible maps in S-fuzzymetric space (X, S, *)

Definition 2.9. Two self maps A and B of a S-fuzzy metric space (X, S, *) are said to be S-weakly commuting if

$S(ABx, BAx, y, t) \geq S(Ax, Bx, z, t)$ where y = ABx or BAx and z = Ax or Bx for all x ∈ X.

Definition 2.10. Two self mappings A and B of a S-fuzzy metric space (X, S, *) are said to be S- compatible if

$\lim_{n \to \infty} S(ABx_n, BAx_n, z, t) = 1$ where $z = ABx_n$ or BAx_n, whenever {xn} is a sequence in X such that $\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Bx_n = y$, for some y in X.

Clearly, commutativity implies S-weak commutativity and S-weak commutativity implies S-compatibility, but neither implication is reversible always.

Common fixed point theorems for S-weakly commuting maps andS- compatible maps in complete S-fuzzy metric spaces.

We prove the following theorem for S-weakly

commuting maps.

Theorem 3.1. Let A, B, P and T be self maps of a complete S-fuzzy metric space $(X,S,*)$ with t-norm $*$ defined by $a * b = \min\{a, b\}$, $a, b \in [0, 1]$ satisfying theconditions

(i) $A(X) \subseteq T(X)$, $B(X) \subseteq P(X)$,

(ii) One of A, B, P or T is continuous,

(iii) (A, P) and (B, T) are S-weakly commuting pairs of maps,

(iv) for all $x, y, z \in X$, $0 < k < 1$, $t > 0$

$S(Ax, By, z, kt) \geq \min\{S(Px, Ty, z, t), S(Ax, Ty, z, t), S(By, Px, z, t), S(Ax, Px, z, t)\}$ and

(v) $S(x, y, z, t) \to 1$ as $t \to \infty$

Then A, B, P and T have a unique common fixed point in X.

Proof : Let $x_0 \in X$ be arbitrary, construct a sequence $\{y_n\}$ in X such that

$y_{2n+1} = Tx_{2n+1} = Ax_{2n}$ and $y_{2n} = Px_{2n} = Bx_{2n-1}$; $n = 0, 1, 2, \ldots$

using (iv), we have

$S(y_1, y_2, y_m, kt) = S(Ax_0, Bx_1, y_m, kt)$

$\geq \min\{S(Px_0, Tx_1, y_m, t), S(Ax_0, Tx_1, y_m, t), S(Bx_1, Px_0, y_m, t), S(Ax_0, Px_0, y_m, t),\}$

$= \min\{S(y_0, y_1, y_m, t), S(y_1, y_1, y_m, t), S(y_2, y_0, y_m, t), S(y_1, y_0, y_m, t),\}$

$\geq \min\{S(y_0, y_1, y_m, t), S(y_1, y_2, y_m, t), S(y_0, y_2, y_m, t)\}$.

This implies that

$S(y_1, y_2, y_m, kt) \geq S(y_0, y_1, y_m, t)$ or $S(y_0, y_2, y_m, t)$.

Further using (iv), we have

$S(y_2, y_3, y_m, kt) = S(Bx_1, Ax_2, y_m, kt) = S(Ax_2, Bx_1, y_m, kt)$

$\geq \min\{S(Px_2, Tx_1, y_m, t), S(Ax_2, Tx_1, y_m, t), S(Bx_1, Px_2, y_m, t), S(Ax_2, Px_2, y_m, t),\}$

$= \min\{S(y_2, y_1, y_m, t), S(y_3, y_1, y_m, t), S(y_2, y_2, y_m, t), S(y_3, y_2, y_m, t),\}$

$\geq \min\{S(y_1, y_2, y_m, t), S(y_1, y_3, y_m, t), S(y_2, y_3, y_m, t)\}$

which implies that

$S(y_2, y_3, y_m, kt) \geq S(y_1, y_2, y_m, t)$ or $S(y_1, y_3, y_m, t)$.

Proceeding in the same way, we get

$S(y_n, y_{n+1}, y_m, kt) \geq S(y_{n-1}, y_n, y_m, t)$ or $S(y_{n-1}, y_{n+1}, y_m, t)$

$\geq S(y_{n-2}, y_{n-1}, y_m, t/k)$ or $S(y_{n-2}, y_{n+1}, y_m, t/k)$

$\geq \ldots \ldots$

$\ldots \ldots$

$\geq S(y_0, y_1, y_m, t/k^{n-1})$ or $S(y_0, y_{n+1}, y_m, t/k^{n-1})$

i.e. $S(y_n, y_{n+1}, y_m, t) \geq S(y_0, y_1, y_m, t/k^n)$ or $S(y_0, y_{n+1}, y_m, t/k^n)$

Case I

When $S(y_n, y_{n+1}, y_m, t) \geq S(y_0, y_1, y_m, t/k^n)$. Then for $p, q \in N$ and $t > 0$, we have

$S(y_n, y_{n+p}, y_{n+p+q}, 3t) \geq S(y_n,y_{n+1},y_{n+p+q},t)*S(y_n,y_{n+1},y_{n+p},t)*S(y_{n+1},y_{n+p},y_{n+p+q}, t)$

$\geq S(y_0, y_1, y_{n+p+q}, t/k^n) * S(y_0, y_1, y_{n+p}, t/k^n) * S(y_{n+1}, y_{n+2}, y_{n+p+q}, t/3)$

$* S(y_{n+1}, y_{n+2}, y_{n+p}, t/3) * S(y_{n+2}, y_{n+p}, y_{n+p+q}, t/3)$

$\geq S(y_0, y_1, y_{n+p+q}, t/k^n) * S(y_0, y_1, y_{n+p}, t/k^n) * S(y_0, y_1, y_{n+p+q}, t/3k^{n+1})$

$* S(y_0, y_1, y_{n+p}, t/3k^{n+1}) * S(y_{n+2}, y_{n+p}, y_{n+p+q}, t/3)$

$\ldots \ldots$

$\ldots \ldots$

$\geq S(y_0,y_1, y_{n+p+q}, t/k^n) * S(y_0, y_1, y_{n+p}, t/k^n) * S(y_0, y_1, y_{n+p+q}, t/3k^{n+1})$

$*S(y_0, y_1, y_{n+p},t/3k^{n+1})*\ldots* S(y_0, y_1, y_{n+p+q},t/k^{n+p-2} 3^{p-2}) * S(y_0,y_1,y_{n+p},t/k^{n+p-2} 3^{p-2})*S(y_{n+p-1}, y_{n+p}, y_{n+p+q}, t/3^{p-2})$

$\geq S(y_0, y_1, y_{n+p+q}, t/k^n) * S(y_0, y_1, y_{n+p}, t/k^n) *S(y_0, y_1, y_{n+p+q}, t/3k^{n+1}) * S(y_0, y_1, y_{n+p}, t/3k^{n+1})$

$* \ldots * S(y_0, y_1, y_{n+p+q}, t/k^{n+p-2} 3^{p-2}) * S(y_0, y_1,y_{n+p}, t/k^{n+p-2} 3^{p-2}) *S(y_0,y_1,y_{n+p+q}, t/k^{n+p-1} 3^{p-2})$

taking limit as $n \to \infty$, we have

$\lim_{n\to\infty} S(y_n, y_{n+p}, y_{n+p+q}, 3t) \geq 1 * 1 * 1 * \ldots * 1$ ($2p-1$ times),

which implies that $S(y_n, y_{n+p}, y_{n+p+q}, 3t) \to 1$ as $n\to\infty$.

Case II

When $S(y_n, y_{n+1}, y_m, t) \geq S(y_0, y_{n+1}, y_m, t/k^n)$. Then on the lines of case I, we have

$S(y_n, y_{n+p}, y_{n+p+q}, 3t) \geq S(y_0, y_{n+1}, y_{n+p+q}, t/k^n) * S(y_0, y_{n+1}, y_{n+p}, t/k^n)$

$* S(y_0, y_{n+2}, y_{n+p+q}, t/3k^{n+1}) * S(y_0, y_{n+2}, y_{n+p}, t/3k^{n+1}) * \ldots * S(y_0, y_{n+p-2}, y_{n+p+q}, t/k^{n+p-2} 3^{p-2})$

$* S(y_0, y_{n+p-2}, y_{n+p}, t/k^{n+p-2} 3^{p-2}) * S(y_0, y_{n+p}, y_{n+p+q}, t/k^{n+p-1} 3^{p-2})$.

Taking limit as $n \to \infty$, we have $\lim_{n\to\infty} S(y_n, y_{n+p}, y_{n+p+q}, 3t) \geq 1 * 1 * 1 * \ldots * 1$ (2p-1 times), which implies that $S(y_n, y_{n+p}, y_{n+p+q}, 3t) \to 1$ as $n\to\infty$.

Thus in both cases, $\{y_n\}$ is a Cauchy sequence. By the completeness of X, sequence $\{y_n\}$ and its subsequences $\{Ax_{2n}\}$, $\{Bx_{2n-1}\}$, $\{Px_{2n}\}$ and $\{Tx_{2n+1}\}$ converge to some u in X. Now if we suppose that P is continuous then $PAx_{2n}, PPx_{2n} \to Pu$.

Since (A,P) are S-weakly commuting, therefore

$S(APx_{2n}, PAx_{2n}, APx_{2n}, t) \geq S(Ax_{2n}, Px_{2n}, Ax_{2n}, t)$.

On letting $n\to\infty$, we have

$S(\lim_{n\to\infty} APx_{2n}, Pu, \lim_{n\to\infty} APx_{2n}, t) \geq S(u, u, u, t) = 1$

which implies that $APx_{2n} \to Pu$. Now using (iv), we have

$S(APx_{2n},Bx_{2n+1},u,kt) \geq \min \{S(PPx_{2n}, Tx_{2n+1}, u, t),S(APx_{2n}, Tx_{2n+1}, u, t),$
$S(Bx_{2n+1}, PPx_{2n}, u, t)\}$.

On letting $n\to\infty$, we have

$S(Pu, u, u, kt) \geq \min\{ S(Pu, u, u, t), S(Pu, u, u, t), S(u, Pu, u, t) ,S (Pu, Pu, u, t)\}$

or $S(Pu, u, u, kt) \geq S(Pu, u, u, t)$

which implies that $Pu = u$.

Further using (iv), we have

$S(Au, Bx_{2n+1}, u, kt) \geq \min\{S(Pu, Tx_{2n+1}, u, t), S(Au, Tx_{2n+1}, u, t),$
$S (Bx_{2n+1}, Pu, u, t), S(Au, Pu, u, t)\}$

on letting $n\to\infty$, we have

$S(Au, u, u, kt) \geq \min\{ S (u, u, u, t), S (Au, u, u, t), S(u, u, u, t), S(Au, u, u, t),\}$

or $S(Au, u, u, kt) \geq S (Au, u, u, t)$ which implies that $Au = u$.

Since $A(X) \subseteq T(X)$, there exists $v \in X$ such that $u = Tv = Pu$.

Using (iv), we have

$S(u, Bv, u, kt) = S(Au, Bv, u, kt)$
$\geq \min\{ S(Pu, Tv, u, t), S(Au, Tv, u, t), S(Bv, Pu, u, t), S(Au, Pu, u, t),\}$
$= \min\{ S(u, u, u, t), S(u, u, u, t), S(Bv, u, u, t), S(u, u, u, t)\}$

or $S(u, Bv, u, kt) \geq S (u, Bv, u, t)$

which implies that $Bv = u$. Thus $u = Bv = Tv$. Since (T, B) are S-weakly commuting, therefore

$S(TBv, BTv, TBv, t) \geq S (Tv, Bv, Tv, t) = 1$

which implies that $TBv = BTv$ and so $Tu = Bu$.

Using (iv), we have

$S (u, Tu, u, kt) = S (Au, Bu, u, kt)$
$\geq \min \{ S(Pu, Tu, u, t), S(Au, Tu, u, t), S(Bu, Pu, u, t), S (Au, Pu, u, t),\}$
$= \min \{ S(u, Tu, u, t), S(u, Tu, u, t), S(Tu, u, u, t), S (u, u, u, t),\}$
$S(u, Tu, u, kt) \geq S(u, Tu, u, t)$

which implies that $u = Tu = Bu$. Hence $u = Tu = Bu = Au = Pu$.

Shows u is a common fixed point of A, B, P and T.

Now to prove uniqueness of u, let w be another common fixed point of A, B, P and T.

Then from (iv), we have $S(u, w, u, kt) = S(Au, Bw, u, kt)$
$\geq \min\{S(Pu, Tw, u, t), S(Au, Tw, u, t), S(Bw, Pu, u, t), S (Au, Pu, u, t),\}$
$= \min\{S(u, w, u, t), S(u, w, u, t), S(w, u, u, t), S (u, u, u, t),\}$
$= S(u, w, u, t)$

or $S(u, w, u, kt) \geq S(u, w, u, t)$ which implies that $u = w$. Hence u is a unique common fixed point of A, B, P and T.

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