DOUBLE DOMINATION ON GENERALIZED PETERSEN GRAPHS

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Abstract: Let $G = (V(G), E(G))$ be a graph. A subset S of $V(G)$ is a double dominating set if every vertex of $V(G)$ is dominated at least twice by the vertices of S. The double domination number, denoted by $\gamma_{dd}(G)$ is the minimum cardinality among all double dominating sets of G. In this paper, we have given an algorithm to construct the minimal double dominating set of the generalized Petersen graphs $P(n, k)$ with $n \geq 2k + 1$ for $k = 1, 2$ and 3 and found their cardinality number $\gamma_{dd}(G)$.

Introduction: Domination number has its origin when de Jaenisch attempted to determine the minimum number of queens required to cover chess board. Rouse Bell[1] introduced covering, independent covering and independence problems in chess board analysis. Mathematical study of domination number in graphs emerged with the conjecture posed by Vizing[2] in 1963.

Vizing's Conjecture: "Is the chromatic number of the cartesian product of two graphs is at least theproduct of their domination number".

Harary and Haynes[3] and Mustapha Chellali[4] have discussed the double domination of graphs and we have taken their definition for double domination.

In this paper, we have given an algorithm to construct the minimal double dominating set of the generalized Petersen graphs $P(n, k)$ with $n \geq 2k + 1$ for $k = 1, 2$ and 3 and found their cardinality number $\gamma_{dd}(G)$.

Definition 1.1 The *open neighborhood* of a vertex $v \in V(G)$ is denoted by$N(v)$ and is defined as

$$N(v) = \{u \in V(G) \,/\, uv \in E(G)\}$$

The *closed neighborhood* of a vertex $v \in V(G)$ is denoted by $N[v]$ and is defined as$N[v] = N(v) \cup \{v\}$.

Definition 1.2 The set $S \subset V$ of vertices in a graph $G = (V, E)$ is a dominating set if every vertex $v \in V$ is an element of S or adjacent to an element of S.

Definition 1.3 A subset S of $V(G)$ is a double dominating set of G if v is in S and has at least one neighbor in S or v is in $V - S$ and has at least two neighbors in S that is for every vertex $v \in V(G), |N[V] \cap S| \geq 2$.

Definition 1.4 The *double domination number* is denoted by $\gamma_{dd}(G)$and is the minimum cardinality among all double dominating sets of G.

Double domination is defined for simple graphs without isolated vertices.

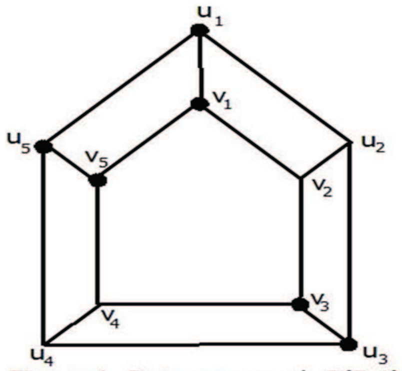

Figure 1. Petersen graph P(5,1)

Definition 1.5 Let n, k be positive integers such that $n \geq 3$ and $1 \leq k \leq \left[\frac{n}{2}\right]$. The generalized Petersen graph $P_{n,k}$is the graph whose vertex set is $\{a_i, b_i : 1 \leq i \leq n\}$ and whose edge set is $\{\{a_i, b_i\}, \{a_i, a_{i+1}\}, \{b_i, b_{i+k}\}: 1 \leq i \leq n\}$ where $a_{n+c} = a_c$ and $b_{n+c} = b_c$for every $c \geq 1$.

Throughout this paper, we take the outer vertices as $u_1, \ldots, u_n$ and the corresponding inner vertices as $v_1, \ldots, v_n$for $P(n, k)$.

Double domination of the generalized Petersen graph $P(n,1)$, $P(n,2)$ and $P(n,3)$.

Theorem 2.1 For all $n \geq 3$,the generalized Petersen graph $P(n, 1)$ has the minimal double dominating set $S = \{u_i, v_i: 1 \leq i \leq n \text{ and} i \text{ odd}\}$

Proof. The vertex u_i dominates the vertices $u_{i-1}, u_{i+1}$and v_i for $1 < i \leq n$ and i odd (modulo addition i); and the vertex $v_i$dominates the vertices $v_{i-1}, v_{i+1}$and $u_i$for $1 < i \leq n$ and i odd (modulo addition i). For $i = 1$, the vertex $u_1$dominates the vertices u_2, $u_n$and v_1; and the vertex $v_1$dominates the vertices v_2, $v_n$and u_1.

Thus$S = \{u_i, v_i: 1 \leq i \leq n$ and i odd$\}$.

Example 2.2 Consider the Petersen graph $P(5,1)$. Let u_1,....,$u_5$be the outer vertices and v_1,....,$v_5$be the corresponding inner vertices.

By theorem 2.1, the minimal double dominating set Sin fig.1 is $\{u_1, u_3, u_5, v_1, v_3, v_5\}$.

Theorem 2.3 The generalized Petersen graph $P(n, 2)$with $n \geq 5$ has the minimal double dominating set as follows:

for n even, $S = \begin{cases} u_i, & 1 \le i \le n \text{ and } i \text{ odd} \\ v_i, & 1 \le i \le n-1 \end{cases}$

for n odd and $n > 5$,

$S = \begin{cases} u_i, & 1 \le i \le n \text{ and } i \text{ odd} \\ v_i, & 1 \le i \le n-2 \end{cases}$

for $n = 5$,

$S = \{u_1, u_3, u_5, v_1, v_2, v_3, v_4\}$

Proof. Case 1: Let n be even.

The vertex u_i dominates the vertices u_{i-1}, u_{i+1} and v_i for $1 < i \le n$ and i odd ; and the vertex v_i dominates the vertices v_{i+2}, v_{i+4} and u_i for $1 \le i \le n-1$ (modulo addition i). For $i = 1$, the vertex u_1 dominates the vertices u_2, u_n and v_1.

Thus $S = \begin{cases} u_i, & 1 \le i \le n \text{ and } i \text{ odd} \\ v_i, & 1 \le i \le n-1 \end{cases}$

Case 2: Let n be odd.

The vertex u_i dominates the vertices u_{i-1}, u_{i+1} and v_i for $1 < i \le n$ and i odd (modulo addition i); and the vertex v_i dominates the vertices v_{i+2}, v_{i+3} and u_i for $1 \le i \le n-2$ (modulo addition i). For $i = 1$, the vertex u_1 dominates the vertices u_2, u_n and v_1.

Thus $S = \begin{cases} u_i, & 1 \le i \le n \text{ and } i \text{ odd} \\ v_i, & 1 \le i \le n-2 \end{cases}$

Case 3: Let $n = 5$. The double dominating set S is $\{u_1, u_3, u_5, v_1, v_2, v_3, v_4\}$. Instead we can have either v_5 or u_2 for v_4 also for double domination.

Example 2.4 Let n be even. Consider the Petersen graph $P(6,2)$. Let $u_1, \ldots, u_6$ be the outer vertices and $v_1, \ldots, v_6$ be the corresponding inner vertices.

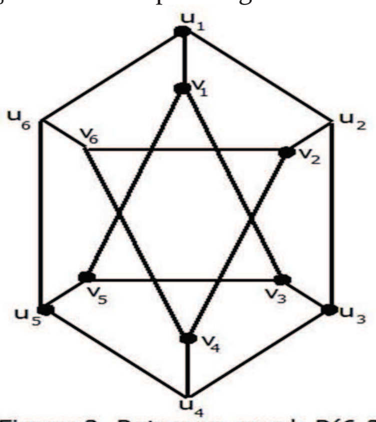

Figure 2. Petersen graph P(6,2)

By theorem 2.3, the minimal double dominating set S in fig.2 is $\{u_1, u_3, u_5, v_1, v_2, v_3, v_4, v_5\}$.

Example 2.5 Let n be odd. Consider the Petersen graph $P(7,2)$. Let $u_1, \ldots, u_7$ be the outer vertices and $v_1, \ldots, v_7$ be the corresponding inner vertices.

By theorem 2.3, the minimal double dominating set S in fig.3 is $\{u_1, u_3, u_5, u_7, v_1, v_2, v_3, v_4, v_5\}$.

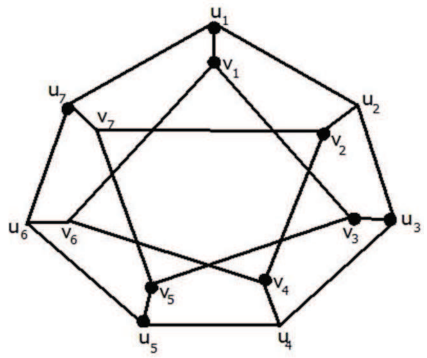

Figure 3. Petersen graph P(7,2)

Theorem 2.6 For all $n \ge 7$, the generalized Petersen graph $P(n, 3)$ has the minimal double dominating set as follows:

for n even, $S = \begin{cases} u_i, & 1 \le i \le n \text{ and } i \text{ odd} \\ v_i, & 1 \le i \le n \text{ and } i \text{ odd} \end{cases}$

for n odd, $S = \begin{cases} u_i, & 1 \le i \le n \text{ and } i \text{ odd} \\ v_j, & 1 \le i \le n \text{ and } i \text{ odd} \\ & \text{either } v_2 \text{ or } v_{n-1} \end{cases}$

Proof. Case 1: Let n be even.

The vertex u_i dominates the vertices u_{i-1}, u_{i+1} and v_i for $1 < i \le n$ and i odd (modulo addition i); and the vertex v_i dominates the vertices v_{i+3}, v_{i+5} and u_i for $1 \le i \le n$ and i odd (modulo addition i). For $i = 1$, the vertex u_1 dominates the vertices u_2, u_n and v_1.

Thus $S = \begin{cases} u_i, & 1 \le i \le n \text{ and } i \text{ odd} \\ v_i, & 1 \le i \le n \text{ and } i \text{ odd} \end{cases}$

Case 2: Let n be odd.

The vertex u_i dominates the vertices u_{i-1}, u_{i+1} and v_i for $1 < i \le n$ and i odd (modulo addition i); and the vertex v_i dominates the vertices v_{i+3}, v_{i+4} and u_i for $1 \le i \le n$ and i odd (modulo addition i). For $i = 1$, the vertex u_1 dominates the vertices u_2, u_n and v_1. In case of n odd, an additional vertex v_2 or v_{n-1} is added to S in order to satisfy the definition of double dominating set.

Thus $S = \begin{cases} u_i, & 1 \le i \le n \text{ and } i \text{ odd} \\ v_j, & 1 \le i \le n \text{ and } i \text{ odd} \\ & \text{either } v_2 \text{ or } v_{n-1} \end{cases}$

Example 2.7 Let n be odd. Consider the Petersen graph $P(7,3)$. Let $u_1, \ldots, u_7$ be the outer vertices and $v_1, \ldots, v_7$ be the corresponding inner vertices.

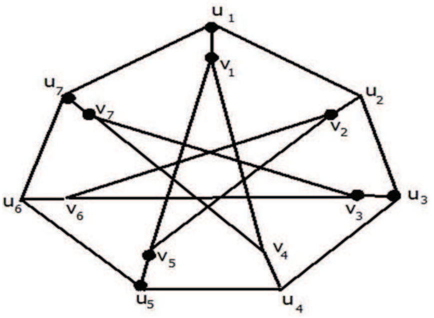

Figure 4. Petersen graph P(7,3)

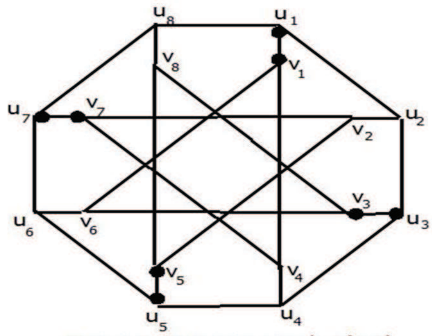

Figure 5. Petersen graph P(8,3)

By theorem 2.6, the minimal double dominating set S in fig.4 is $\{u_1, u_3, u_5, u_7, v_1, v_2, v_3, v_5, v_7\}$.

Example 2.8 Let n be odd. Consider the Petersen graph $P(8,3)$. Let u_1,....,$u_8$be the outer vertices and v_1,....,$v_8$be the corresponding inner vertices.

By theore.2.6, the minimal double dominating set Sin fig.5 is $\{u_1, u_3, u_5, u_7, v_1, v_3, v_5, v_7\}$.

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