INTEGRABILITY AND L^1-CONVERGENCE OF MIXED TRIGONOMETRIC SERIES

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Abstract: In this paper, we study mixed trigonometric series with special coefficient of sequences. We are mainly concerned with the following problems: (i) the series in the question is point wise convergent (ii) the sum of the series is integrable (iii) the series is the Fourier series of its sum (iv) the series converges in L^1-norm.

Keywords:-Dirichlet kernel, Fejer kernel, L^1-convergence, point wise convergence.

Introduction: we consider the double trigonometric series

$$f(x,y)=\sum_{j=1}^{\infty}\sum_{k=0}^{\infty}\lambda_k a_{jk}\sin jx\cos ky \qquad (1)$$

on the positive quadrant $Q=[0,\pi]\times[0,\pi]$ of two dimensional torus, where $\lambda_0=\frac{1}{2}$ and $\lambda_k=1$ for $k=1,2,3,\ldots,$ and $\{a_{jk}\}$ is a double sequence of real numbers.

We denote the rectangular partial sum of the series (1) by S_{mn} *i.e.*

$$S_{mn}(x,y)=\sum_{j=1}^{m}\sum_{k=0}^{n}\lambda_k a_{jk}\sin jx\cos ky \quad (m,.n\geq 1)$$

and let $f(x,y)=\lim_{m,n\to\infty}S_{mn}$.

In the literature so far available, many authors like F. Moricz ([1], [2]), K. Kaur, S.S.Bhatia and B. Ram [5], J. Kaur and S. S. Bhatia([3],[4]) have studied the point wise convergence of double trigonometric series under certain conditions on the coefficients.

In this concern Moricz [1], defined the following classes:

Definition 1.

We say that $\{a_{jk}\}\in BV_2^*$ *if*

$$a_{jk}\to 0 \quad as \quad j+k\to\infty, \qquad (2)$$

$$\sum_{j=1}^{\infty}\sum_{k=0}^{\infty} j\left|\Delta_{11}c_{jk}\right|<\infty,$$

where

$$c_{jk}=\frac{a_{jk}}{j} \quad (j=1,2,3,\ldots.;\ k=0,1,2,\ldots.)$$

Definition 2.

We say that $\{a_{jk}\}\in C_2^*$ *if condition (2) is satisfied and for every* $\varepsilon>0$ *there exists* $\delta>0$ *such that for all* $1\leq m\leq M;\ 0\leq n\leq N$ *we have*

$$C(m,M;n,N;\delta):=\iint_{D_\delta}\left|\sum_{j=m}^{M}\sum_{k=n}^{N}[D_j(x)]_x D_k(y)\Delta_{11}c_{jk}\right|dxdy\leq\varepsilon$$

where

$$D_\delta:=Q-(\delta,\pi]\times(\delta,\pi] =\{(x,y):0\leq x,y\leq\pi\ \&\ \min(x,y)\leq\delta\}$$

Here, $[.]_x$ means the partial derivative $\frac{\partial[.]}{\partial x}$.

Concerning the convergence of the double trigonometric series Moriz [1] proved the following result:

Theorem A. [1] *If* $\{a_{jk}\}\in BV_2^*\cap C_2^*$, *then sum* $f(x,y)$ *of the series (1) belongs to* $L^1(Q)$ *and (1) is the Fourier series of* $f(x,y)$.

The aim of this paper is to study the L^1-convergence of mixed series under a class $\left(S_d^2\right)^*$ of coefficient sequences defined as follows:

Definition 3.

A double null sequence $\{a_{jk}\}\in\left(S_d^2\right)^*$ *if there exists a null sequence* $\{A_{jk}\}$ *of positive numbers such that*

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} j(j+1)(k+1)\left|\Delta_{11}A_{jk}\right|<\infty, \qquad (3)$$

$$\left|\Delta_{11}c_{jk}\right|\leq A_{jk};\quad \forall\ (j,k=1,2,3,\ldots) \qquad (4)$$

Main Result :

The main result reads as follow:

Theorem.

Let $\{a_{jk}\}\in\left(S_d^2\right)^*$, *Then sum* $f(x,y)$ *of the series (1) belongs to* $L^1(Q)$ *and* $\left\|S_{mn}-f\right\|=o(1)$ *as* $m,n\to\infty$ *if and only if* $a_{mn}\ln(m+2)\ln(n+2)\to 0$ *as* $m,n\to\infty$.

Proof: First we shall show that the point wise limit f of

$$S_{mn}(x,y)=\sum_{j=1}^{m}\sum_{k=0}^{n}a_{jk}\sin jx\cos ky \quad (m,n\geq 1)$$

exists in $Q=[0,\pi]\times[0,\pi]$ and that f is a Fourier series i.e. $f\in L^1(Q)$.

From $\{a_{jk}\}\in\left(S_d^2\right)^*$ it follows that $\{a_{jk}\}\in BV_2^*$. Indeed

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}(j+1)(k+1)\left|\Delta_{11}A_{jk}\right|$$

$$\leq\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}j(j+1)(k+1)\left|\Delta_{11}A_{jk}\right|<\infty.$$

On the other hand

$$m(m+1)(n+1)A_{mn}$$

$$\leq m(m+1)(n+1)\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\left|\Delta_{11}A_{jk}\right|$$

$$\leq\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}j(j+1)(k+1)\left|\Delta_{11}A_{jk}\right|=o(1). \quad (5)$$

Thus from

$$\sum_{j=1}^{m}\sum_{k=0}^{n}jA_{jk}=\sum_{j=1}^{m-1}\sum_{k=0}^{n-1}\frac{j(j+1)(k+1)}{2}\Delta_{11}A_{jk}-\sum_{j=1}^{m-1}\frac{j(j+1)(n+1)}{2}\Delta_{10}A_{jn}-\sum_{k=0}^{n-1}\frac{m(m+1)(k+1)}{2}\Delta_{01}A_{mk}+\frac{m(m+1)(n+1)}{2}A_{mn}$$

but we have

$$(n+1)\Delta_{10}A_{jn}=\sum_{k=n}^{\infty}k\Delta_{11}A_{jk}$$

and

$$m(m+1)\Delta_{01}A_{mk}=\sum_{j=m}^{\infty}j(j+1)\Delta_{11}A_{jk}$$

we get that the series $\sum_{j=1}^{\infty}\sum_{k=0}^{\infty}jkA_{jk}$ is convergent.

Since $\{a_{jk}\}\in\left(S_d^2\right)^*$ implies that $\left|\Delta_{11}c_{jk}\right|\leq A_{jk};\quad\forall\ (j,k=1,2,3,\ldots)$ it follows that $\{a_{jk}\}\in BV_2^*$.

Now from Theorem A we have that if $\{a_{jk}\}\in BV_2^*\cap C_2^*$, then $f\in L^1(Q)$. Thus it sufficient to prove that $\{a_{jk}\}\in\left(S_d^2\right)^*\Rightarrow\{a_{jk}\}\in C_2^*$.

By using double summation by parts, we get

$$\iint_{D_\delta}\left|\sum_{j=m}^{M}\sum_{k=n}^{N}[D_j(x)]_x D_k(y)\Delta_{11}c_{jk}\right|dxdy$$

$$\leq\int_0^\pi\int_0^\pi\left|\sum_{j=m}^{M}\sum_{k=n}^{N}[D_j(x)]_x D_k(y)\Delta_{11}c_{jk}\right|dxdy$$

$$\leq\int_0^\pi\int_0^\pi\left|\sum_{j=m}^{\infty}\sum_{k=n}^{\infty}[D_j(x)]_x D_k(y)A_{jk}\right|dxdy$$

$$\leq\int_0^\pi\int_0^\pi\left|\sum_{j=m}^{\infty}\sum_{k=n}^{\infty}\Delta_{11}A_{jk}j(j+1)F_j(x)(k+1)F_k(y)\right|dxdy$$

$$+\lim_{N\to\infty}\int_0^\pi\int_0^\pi\left|\sum_{j=m}^{\infty}\Delta_{10}A_{jN}j(j+1)F_j(x)(N+1)F_N(y)\right|dxdy$$

$$+\lim_{M\to\infty}\int_0^\pi\int_0^\pi\left|\sum_{k=n}^{\infty}\Delta_{01}A_{Mk}M(M+1)F_M(x)(k+1)F_k(y)\right|dxdy$$

$$+\lim_{M,N\to\infty}\int_0^\pi\int_0^\pi\left|M(M+1)NA_{MN}F_M(x)F_N(y)\right|dxdy \quad (6)$$

Where $D_n(x)$, $F_n(x)$ represent the Dirichlet and Fejer kernel respectively.

Since $\{a_{jk}\}\in\left(S_d^2\right)^*$ and $\int_0^\pi\left|F_n(x)\right|dx=\pi$, we have all the terms on the right hand side of inequality (6) is of o(1), as $M,N\to\infty$.

Hence, $f\in L^1(Q)$.

It remains to show that $\left\|S_{mn}-f\right\|=o(1)\ as\ m,n\to\infty$, for this we consider,

$$\left\|f-S_{mn}\right\|=\int_0^\pi\int_0^\pi\left|\sum_{j=m+1}^{\infty}\sum_{k=n+1}^{\infty}a_{jk}\sin jx\cos ky\right|dx\,dy$$

Applying double summation by parts, we have

$$\leq\int_0^\pi\int_0^\pi\left|\sum_{j=m}^{\infty}\sum_{k=n}^{\infty}\Delta_{11}c_{jk}\left[D_j(x)\right]_x D_k(y)\right|dxdy$$

$$+\lim_{N\to\infty}\int_0^\pi\int_0^\pi\left|\sum_{j=m}^{\infty}\Delta_{10}c_{jN}\left[D_j(x)\right]_x D_N(y)\right|dxdy$$

$$+\lim_{M\to\infty}\int_0^\pi\int_0^\pi\left|\sum_{k=n}^{\infty}\Delta_{01}c_{Mk}\left[D_M(x)\right]_x D_k(y)\right|dxdy$$

$$+\lim_{M,N\to\infty}\int_0^\pi\int_0^\pi\left|c_{MN}\left[D_M(x)\right]_x D_N(y)\right|dxdy \quad (7)$$

By the given hypothesis and $\int_0^\pi\left|\left[D_n(x)\right]_x\right|dx=o(n\log n)$, all the terms on the right hand side of inequality (7) are of $o(1)$ *as* $m, n \to \infty$.

The conclusion of the main result holds.

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