SIMPLE RINGS WITH ASSOCIATORS IN THE LEFT NUCLEUS

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Abstract: In this paper, we show that a simple ring R of char. ≠2 with associators in the left nucleus is associative.

Keywords: Nonassociative ring, simple ring, characteristic, associator and nucleus.

Introduction: In [**2**] Yen proved the result of Kleinfeld [**1**] for the simple ring case under the weaker hypothesis $(R,R,R) \subseteq N_l \cap N_m$. Yen [**4**] also proved that if R is a simple ring of char. ≠2 with associators in the left nucleus, then the associators are also in the middle nucleus and hence R is associative. In this paper, we show that the associator is in the right nucleus if R is a simple ring with the associator is in the left nucleus. From Yen's result it follows that R is associative.

Preliminaries: Let R be a nonassociative ring. We shall denote the commutator and the associator by $(x,y) = xy\text{-}yx$ and $(x,y,z) = (xy)z\text{-}x(yz)$ for all x,y,z in R respectively. The nucleus N of a ring R is defined as $N = \{n \in R \ / \ (n,R,R) = (R,n,R) = (R,R,n) = 0\}$.The center C of R is defined as $C = \{c \in N \ / \ (c, R) = 0\}$.We know that a ring R is simple if $R^2 \neq 0$ and the only nonzero ideal of R is itself. Since R^2 is a non-zero ideal of R, we have $R^2 = R$. We know that a ring R is semiprime if the only ideal of R which squares to zero is the zero ideal.

Throughout this paper R represents a simple ring of char. ≠2 with all associators are in the left nucleus.

Main Results: If R is a ring of char. ≠ 2 with associators in the left nucleus, then

$(R,R,R) \subseteq N_l$. ...(1)

We have the identity valid in any ring, the so called Teichmuller identity:

$(wx,y,z) - (w,xy,z) + (w,x,yz) = w(x,y,z) + (w,x,y)z,$ ...(2)

for all w,x,y,z in R.

suppose that $n \in N_l$. Substituting $w = n$ in (2), we get

$(nx,y,z) = n(x,y,z),$

for all x,y,z in R and $n \in N_l$.

If N_l is a Lie ideal of R, then we have

$(nx,y,z) = n(x,y,z) = (xn,y,z),$ ...(3)

for all x,y,z in R and $n \in N_l$.

Let I be the associator ideal of a nonassociative ring R. By (2), I can be characterized as all finite sums of associators and right (or let) multiples of associators. Hence, we obtain

$I = (R,R,R) + (R,R,R)R.$

$= (R,R,R) + R(R,R,R).$ ...(4)

Now prove the following theorem.

Theorem 1: *If R is a simple ring of char. ≠2 with associators in the left nucleus then R is associative.*

Proof: Since R is simple, we have either $I = 0$, in which case R is associative, or $I = R$.

Assume that R is not associative. So $I = R$.

Using (4) and (1), we have

$R = R^2$

$= IR$

$= \{(R,R,R) + (R,R,R)R\}R$

$= (R,R,R)R + (R,R,R)\,R^2$

$= (R,R,R)R.$ ...(5)

Using (2) and (1), we get

$w(x,y,z) + (w,x,y)z \in N_l$ ...(6)

for all w,x,y,z in R.

By replacing n with $(p,q,(r,s,t))$ in (3), we get

$(p,q,(r,s,t))(x,y,z) = ((p,q,(r,s,t))x,y,z),$

where $p,q,r,s,t \in R$ and $(p,q,(r,s,t)) \in N_l$.

Now (2) gives

$(pq,(r,s,t),x) - (p,q(r,s,t),x) + (p,q,(r,s,t)x)$

$= p(q,(r,s,t),x) + (p,q,(r,s,t))x.$

By forming the associator with y,z and using (1), and (3), we obtain

$((p,q,(r,s,t))x,y,z) = -(p(q,(r,s,t),x),y,z)$

$= -((q,(r,s,t),x)p,y,z)$

$= (q((r,s,t),x,p),y,z)$

$= 0.$

Thus $(p,q,(r,s,t))x \in N_l$.

i.e., $(R,R,(R,R,R))R \subseteq N_l$. ...(7)

But $((p,q,(r,s,t))x,y,z) = 0$ implies

$(p,q,(r,s,t))\,(x,y,z) = 0$, because of (3).

i.e, $(R,R(R,R,R))I = 0$. Since we are considering the case where $I = R$, this gives

$(R,R(R,R,R))R = 0.$ ...(8)

Assuming that $x \in (R,R,(R,R,R))$ and $w,y,z,t \in R$.

Then by (8), (2) and (1), we get

$(wx,y,z) + (w,x,yz) = (wx,y,z) - (w,xy,z) + (w,x,yz)$

$= w(x,y,z) + (w,x,y)z$

$= 0.$

So $(wx,y,z)t = -(w,x,yz)t = 0$. This implies

$(R(R,R,(R,R,R)),R,R)R = 0.$ ...(9)

Then with $x \in (R,R,R)$ in (6) and applying (9), we get

$R((R,R,R),R,R) + (R,(R,R,R),R)R \subseteq N_l$

or $(R,(R,R,R),R)R \subseteq N_l$.

Therefore $((R,(R,R,R),R)R,R,R)R = 0.$ ...(10)

Using (5), (1), (3) and (10), we get

$(R,(R,R,R),R)R = (R,(R,R,R),R)\cdot(R,R,R)R$

$= (R,(R,R,R),R)\,(R,R,R)\cdot R$

$= ((R,(R,R,R),R)R,R,R)\cdot R$

$= 0.$ ...(11)

Using (7) and N_l is a Lie ideal of R, we get $R(R,R(R,R,R)) \subseteq N_l$. ...(12)

For all $x \in (R,R,(R,R,R))$ and $w,y,z \in R$, using the previous computation and by (12), we obtain $(w,x,yz) = -(wx,y,z) = 0$.

Since $R^2 = R$, the above equation implies

$(R,R,(R,R,R)) \subseteq N_m$. ...(13)

Let $T = (R,R,(R,R,R))$.

Now we define V_n inductively by

$V_0 = T$, $V_1 = RT$ and $V_{n+1} = RV_n$, $n = 1,2,3,\ldots\ldots$

Assume that $B = \sum_{n=0}^{\infty} V_n$. ...(14)

By using (8), (13) and (8), we have

$V_0R = TR = 0$,

$V_1R = RT\cdot R = R\cdot TR = 0$,

and $V_2R = R(RT)\cdot R$

$\subseteq ((R,R,T) + R^2T)\cdot R$

$= (R,R,T)R + R^2T\cdot R$

$= 0.$

Suppose that $V_iR = 0$, $i = 0.1,2,\ldots\ldots,m$ and $V_{m+1}R = 0$. Then using these and (8), we get

$V_{m+2}R = (RV_{m+1})\cdot R$

$= R(RV_m)\cdot R$

$\subseteq ((R,R,V_m) + R^2V_m)\cdot R$

$= (R,R,V_m)\cdot R + R^2V_m\cdot R$

$= (R,R,V_m)R + V_{m+1}R$

$= (R,R,V_m)R$

$= (R,R,R\cdot V_{m-1})R$

$= (R,R,R(RV_{m-2}))R$

$\subseteq (R,R,(R,R,V_{m-2}) + R^2V_{m-2})R$

$= (R,R,(R,R,V_{m-2}))R + (R,R,V_{m-1})R$

$= (R,R,V_{m-1})R.$

Continuing in this manner, we get

$V_{m+2}R \subseteq (R,R,V_m)R \subseteq (R,R,V_{m-1})R \subseteq \ldots\ldots\ldots\ldots\ldots\subseteq (R,R,V_2)R \subseteq (R,R,V_1)R = (R,R,RT)R.$

By (2) and (11), we get

$RT= R(R,R,(R,R,R)) \subseteq (R,R,R)+(R,(R,R,R),R)R= (R,R,R).$

Thus, applying the above equation and (8), we have

$V_{m+2}R \subseteq (R,R,RT)R \subseteq (R,R,(R,R,R))R = 0.$

Hence by induction, we obtain $B\cdot R = 0$. ...(15)

By (15), B is just the ideal of R generated by

$T = (R,R,(R,R,R)).$

By the simplicity of R and (15), we get $B = 0$.

Thus $(R,R,(R,R,R)) = 0$. So $(R,R,R) \subseteq N_l \cap N_r$.

Hence, by Theorem of [3], R is associative. This contradiction proves the theorem. This completes the proof of the theorem.

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