ANALYSIS OF AN $M^{[X]}/G/1$ FEEDBACK RETRIAL QUEUE WITH MULTI STAGE SERVICE, SINGLE VACATION AND SERVER INTERRUPTION

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Abstract: We analyze a batch arrival retrial queue with k stages of service, Bernoulli feedback and vacation. An arriving batch of customers finds the server busy, breakdown or on vacation enters an orbit/retrial group. If the server is free, one of the customers from the batch enters into the first stage of service and the rest of them join to the orbit. After completion of the i^{th}, (i = 1,2,..k-1) stage of service, the customer may choose $(i+1)^{th}$ stage with probability θ_i, with probability p_i may join into orbit as feedback customer or may leave the system with probability 1- θ_i- p_i = q_i, if the service is successfully. Finally, after completion of k^{th} phase service, he may join into the orbit as a feedback customer with probability p_k or may leave the system with probability 1-p_k. After completion of each stage service, if the orbit is empty, the server takes a vacation. Busy server may get to breakdown and the service channel will fail for a short interval of time. The steady state probability generating function for system size is obtained by using the supplementary variable method. Some system performance measures are discussed.

Keywords: retrial queue, multi stage service, vacation, breakdown.

Introduction: The theory of retrial queues have been extensively applied in the study of communication and computer networks. The special characteristic of retrial queues is that, a customer who finds a busy server does not leave the system or joins a queue. He joins an orbit (retrial group) from where he makes repeated attempts to obtain service. Several survey articles and monographs have been published on retrial queues; see Artalejo [1]. Salehirad and Badamchizadeh [4] have analysed multi stage M/G/1 queueing system with feedback. In 2012, further Bagyam and Chandrika [2] further developed a model with the concept of retrial queues. Queueing systems with vacations and server breakdowns have been found to be useful in modeling the systems in which the server has additional tasks. Single server queueing systems with server breakdowns and Bernoulli vacation have been studied by many researchers including Choudhury and Deka [3], Saravanarajan and Chandrasekaran [5].

This model finds many practical examples such as production system, bank services, computer and communication networks and flexible manufacturing system etc.

In this paper, we investigate steady state analysis of a batch arrival retrial queueing system with multi stage of service, Bernoulli feedback and single vacation where the server is subject to breakdown and repair.

Model Description: In this section, we consider a model for batch arrival feedback retrial queue with multi stage service and single vacation where the server is subject to breakdowns and repair. The detailed description of the model is given as follows:

Arrival process: Customers arrive in batches according to a compound Poisson process with rateλ. Let X_k denote the number of customers belonging to the k^{th} arrival batch, where X_k , k = 1,2 ,3,...are with a common distribution $\Pr[X_k = n] = \chi_n$, n = 1,2,3... $X(z)$ denotes the probability generating function of X. The first and second moments are $E(X)$ and $E(X(X-1))$.

Retrial process: We assume that there is no waiting space and therefore if an arriving batch of customers finds the server free, the arrival beings his service one from the batch and rest of them join into pool of blocked customers called an orbit. If an arriving batch finds the server being busy, vacation or breakdown, it joins into an orbit. Inter-retrial times have an arbitrary distribution $R(x)$ with correspo-nding Laplace-Stieltijes transform (LST) $R^*(s)$

Service process and Feedback process: The server provides k stages of service in succession. The First Stage Service (FSS) is followed by i stages of service. The service time for all the stages has a general distribution. It is denoted by the random variable S_i with distribution function $S_i(x)$ having LST $S_i^*(s)$ and first and second moments are $E(S_i)$ and $E(S_i^2)$, i = 1,2,...k.

After completion of i^{th} stage of service the customer may go to $(i+1)^{th}$ stage with probability θ_i or may join into the orbit with probability p_i or leaves the system with probability 1-θ_i-p_i = q_i, for i = 1,2,...k-1. If the customer in the last k^{th} stage may join to the orbit with probability p_k or leaves the system with probability q_k = 1-p_k. From this model, the service time or the time required by the customer to complete the service cycle is a random variable S is given by $S = \sum_{i=1}^{k} \Theta_{i-1} S_i$ having the LST

$S^*(s)=\prod_{i=1}^{k}\Theta_{i-1}S_i^*(s)$ and the expected value is $E(S)=\sum_{i=1}^{k}\Theta_{i-1}E(S_i)$. where $\Theta_i=\theta_1\theta_2...\theta_i$ and $\Theta_0=1$.

Vacation process: After completion of each stage service, if the orbit is empty, the server takes a single vacation. At the end of a vacation, the server remains idle for the customer from the orbit or new arrival customers. The vacation time of the server is of random length *V* with distribution function *V*(*x*) and LST $V^*(s)$.

Breakdown process: While the server is working with any phase of service, it may breakdown at any time and the service channel will fail for a short interval of time i.e. server is down for a short interval of time. The breakdowns i.e. server's life times are generated by exogenous Poisson processes with rates α_i for i^{th} stage respectively for (i=1,2,...k).

Repair process: As soon as breakdown occurs the server is sent for repair, during that time it stops providing service to the primary customers till service channel is repaired. The customer who was just being served before server breakdown waits for the remaining service to complete. The repair time (denoted by G_i) distributions of the server for *i* stages are assumed to be arbitrarily distributed with d.f. $G_i(y)$ and LST $G_i(y)$ for (i=1,2,...k).

Various stochastic processes involved in the system are assumed to be independent of each other.

In the steady state, we assume that $R(0)=0$, $R(\infty)=1$, $S_i(0)=0$, $S_i(\infty)=1$, $V(0)=0$, $V(\infty)=1$ are continuous at x = 0 and $G_i(0)=0$, $G_i(\infty)=1$ are continuous at y = 0 ($1\le i\le k$). The state of the system at time t is $R^0(t)$, $S_i^0(t)$, $V^0(t)$ and $G_i^0(t)$ be the elapsed retrial times, the elapsed service times on i^{th} stage, the elapsed vacation time and the elapsed repair times on i^{th} stage.

Further, introduce the random variables

$$C(t)=\begin{cases}0, \text{ if the server is idle at time } t,\\ 1, \text{ if the server is busy on } i^{th} \text{ stage at time } t,\\ 2, \text{ if the server is on vacation time } t,\\ 3, \text{ if the server is repair on } i^{th} \text{ stage at time } t.\end{cases}$$

The state of system at time t can be described by the bivariate Markov process $\{C(t),\ N(t);\ t\ge0\}$ where $C(t)$ denotes the server state (0,1,2,3) depending if the server is idle, busy on i^{th} stage, vacation and repair on i^{th} stage respectively. $N(t)$ corresponding to the number of customers in orbit at time t. So that the functions $a(x)$, $\mu_i(x)$, $\gamma(x)$ and $\xi_i(y)$ are the conditional completion rates for retrial, service, vacation and repair respectively ($1\le i\le k$).

$$a(x)dx=\frac{dR(x)}{1-R(x)},\ \mu_i(x)dx=\frac{dS_i(x)}{1-S_i(x)},$$

$$\gamma(x)dx=\frac{dV(x)}{1-V(x)} \text{ and } \xi_i(y)dy=\frac{dG_i(y)}{1-G_i(y)}.$$

Then define $B_i^*=S_1^*S_2^*...S_i^*$ and $B_0^*=1$. The first moment M_{1i} and second moment M_{2i} of B_i^* are given by

$$M_{1i}=\lim_{z\to1}dB_i^*[A_i(z)]\Big/dz=\sum_{j=1}^{i}\lambda E(X)E(S_j)\left(1-\alpha_jE(G_j)\right)$$

$$M_{2i}=\lim_{z\to1}d^2B_i^*[A_i(z)]\Big/dz^2$$

$$=\sum_{j=1}^{i}\begin{pmatrix}\lambda E(X(X-1))E(S_j)\left(1-\alpha_jE(G_j)\right)+\alpha_j\left(\lambda E(X)\right)^2\\ E(S_j)E(G_j^2)+\left(\lambda E(X)\right)^2E(S_j^2)\left(1-\alpha_jE(G_j)\right)^2\end{pmatrix}$$

where $A_i(z)=b(z)+\alpha_i\left(1-G_i^*(b(z))\right)$ and $b(z)=\lambda\left(1-X(z)\right)$

Let $\{t_n;\ n = 1,2,...\}$ be the sequence of epochs at which either a service period completion occurs or a vacation time ends. The sequence of random vectors $Z_n=\{C(t_n+),\ N(t_n+)\}$ forms a Markov chain which is embedded in the retrial queueing system. The embedded Markov chain $\{Z_n;\ n\in N\}$ is ergodic if and only if, $\rho<1$. Where $\rho=E(X)(1-R^*(\lambda))+\omega$,

$$\text{where } \omega=\sum_{i=1}^{k}\Theta_{i-1}M_{1i}+\sum_{i=1}^{k}p_i\Theta_{i-1}-\sum_{i=1}^{k-1}\Theta_{i-1}M_{1i}.$$

Steady state distribution: In this section, we first develop the steady state difference-differential equations for the retrial system by treating the elapsed retrial time, the elapsed service time, the elapsed vacation time and the elapsed repair time as supplementary variables.

For the process $\{N(t),\ t\ge0\}$, we define the probabilities $P_0(t)=P\{C(t)=0,\ N(t)=0\}$ and the probability densities, for $t\ge0, x\ge0, n\ge0$ and ($1\le i\le k$)

$$P_n(x,t)dx=P\left\{C(t)=0, N(t)=n,\ x\le R^0(t)<x+dx\right\},$$

$$\prod_{i,n}(x,t)dx=P\left\{C(t)=1,\ N(t)=n,\ x\le S_i^0(t)<x+dx\right\},$$

$$Q_n(x,t)dx=P\left\{C(t)=2,\ N(t)=n,\ x\le V^0(t)<x+dx\right\},$$

$$R_{i,n}(x,y,t)dy=P\begin{Bmatrix}C(t)=3,\ N(t)=n,\\ y\le G_i^0(t)<y+dy\ \big/S_i^0(t)=x\end{Bmatrix}$$

We assume that the stability condition is fulfilled in the sequel and we can get

$$P_0=\lim_{t\to\infty}P_0(t),\ P_n(x)=\lim_{t\to\infty}P_n(x,t),\prod_{i,n}(x)=\lim_{t\to\infty}\prod_{i,n}(x,t),$$

$$Q_n(x)=\lim_{t\to\infty}Q_n(x,t),\ R_{i,n}(x,y)=\lim_{t\to\infty}R_{i,n}(x,y,t).$$

By the method of supplementary variable technique, we obtain the following system of equations that govern the dynamics of the system behavior for (i = 1,2,...k).

$$\lambda P_0 = \int_0^\infty Q_0(x)\gamma(x)dx \tag{1}$$

$$\frac{dP_n(x)}{dx} + [\lambda + a(x)]P_n(x) = 0,\ n \geq 1 \tag{2}$$

$$\frac{d\prod_{i,n}(x)}{dx} + [\lambda + \alpha_i + \mu_i(x)]\prod_{i,n}(x) = \lambda \sum_{k=1}^{n} \chi_k \prod_{i,n-k}(x) + \int_0^\infty \xi_i(y) R_{i,n}(x,y)dy, n \geq 1, \tag{3}$$

$$\frac{dQ_0(x)}{dx} + [\lambda + \gamma(x)]Q_0(x) = 0,\ n = 0 \tag{4}$$

$$\frac{dQ_n(x)}{dx} + [\lambda + \gamma(x)]Q_n(x) = \lambda \sum_{k=1}^{n} \chi_k Q_{i,n-k}(x),\ n \geq 1 \tag{5}$$

$$\frac{dR_{i,n}(x,y)}{dy} + [\lambda + \xi_i(y)]R_{i,n}(x,y) = \lambda \sum_{k=1}^{n} \chi_k R_{i,n-k}(x,y), \tag{6}$$

The steady state boundary conditions are

$$P_n(0) = \sum_{i=1}^{k-1} q_i \int_0^\infty \prod_{i,n}(x)\mu_i(x)dx + (1-p_k)\int_0^\infty \prod_{k,n}(x)\mu_k(x)dx + \int_0^\infty Q_n(x)\gamma(x)dx + \sum_{i=1}^{k} p_i \int_0^\infty \prod_{i,n-1}(x)\mu_i(x)dx,\ n \geq 1 \tag{7}$$

$$\prod_{1,n}(0) = \int_0^\infty P_{n+1}(x)a(x)dx + \lambda \sum_{k=1}^{n} \chi_k \int_0^\infty P_{n-k+1}(x)dx + \lambda \chi_{n+1} P_0,\ n \geq 1 \tag{8}$$

$$\prod_{i,n}(0) = \theta_{i-1} \int_0^\infty \prod_{i,n}(x)\mu_i(x)dx,\ n \geq 1,\ (2 \leq i \leq k) \tag{9}$$

$$Q_0(0) = \sum_{i=1}^{k-1} q_i \int_0^\infty \prod_{i,0}(x)\mu_i(x)dx + (1-p_k)\int_0^\infty \prod_{k,0}(x)\mu_k(x)dx, n = 0 \tag{10}$$

$$Q_n(0) = 0,\ n \geq 1 \tag{11}$$

$$R_{i,n}(x,0) = \alpha_i P_{i,n}(x),\ n \geq 0,\ \text{for}(1 \leq i \leq k) \tag{12}$$

The normalizing condition is

$$P_0 + \sum_{n=1}^{\infty}\int_0^\infty P_n(x)dx + \sum_{n=0}^{\infty}\int_0^\infty Q_n(x)dx + \sum_{n=0}^{\infty}\sum_{i=1}^{k}\left(\int_0^\infty \prod_{i,n}(x)dx + \int_0^\infty\int_0^\infty R_{i,n}(x,y)dxdy\right) = 1 \tag{13}$$

To solve the above equations, then we define the generating functions for $|z| \leq 1$, for ($1 \leq \underline{i} \leq k$)

$$P(x,z) = \sum_{n=1}^{\infty} P_n(x)z^n;\ P(0,z) = \sum_{n=1}^{\infty} P_n(0)z^n;\ \prod_i(x,z) = \sum_{n=0}^{\infty}\prod_{i,n}(x)z^n;$$

$$\prod_i(0,z) = \sum_{n=0}^{\infty}\prod_{i,n}(0)z^n; Q(x,z) = \sum_{n=0}^{\infty} Q_n(x)z^n;\ Q(0,z) = \sum_{n=0}^{\infty} Q_n(0)z^n;$$

$$R_i(x,y,z) = \sum_{n=0}^{\infty} R_{i,n}(x,y)z^n;\ R_i(x,0,z) = \sum_{n=0}^{\infty} R_{i,n}(x,0)z^n$$

Now multiplying the steady state equation and steady state boundary condition (1)-(12) by z^n and summing over n, ($1 \leq i \leq k$)

$$\frac{dP(x,z)}{dx} + [\lambda + a(x)]P(x,z) = 0 \tag{14}$$

$$\frac{d\prod_i(x,z)}{dx} + [\lambda(1 - X(z)) + \alpha_i + \mu_i(x)]\prod_i(x,z) = \int_0^\infty \xi_i(y)R_i(x,y,z)dy \tag{15}$$

$$\frac{dQ(x,z)}{dx} + [\lambda(1 - X(z)) + \gamma(x)]Q(x,z) = 0 \tag{16}$$

$$\frac{dR_i(x,y,z)}{dy} + [\lambda(1 - X(z)) + \xi_i(y)]R_i(x,y,z) = 0 \tag{17}$$

The steady state boundary conditions at x = 0 and y = 0 are

$$P(0,z) = \sum_{i=1}^{k}\left(p_i z + q_i\right)\int_0^\infty \prod_i(x,z)\mu_i(x)dx + \int_0^\infty Q(x,z)\gamma(x)dx - \lambda P_0 - Q_0(0) \tag{18}$$

$$\prod_1(0,z) = \frac{1}{z}\int_0^\infty P(x,z)a(x)dx + \lambda X(z)\int_0^\infty P(x,z)dx + \frac{\lambda X(z)}{z}P_0 \tag{19}$$

$$\prod_i(0,z) = \theta_{i-1}\int_0^\infty \prod_i(0,z)\mu_i(x)dx, (i = 2,3,...k). \tag{20}$$

$$Q(0,z) = Q_0(0) \tag{21}$$

$$R_i(x,0,z) = \alpha_i \prod_i(x,z) \tag{22}$$

Solving the partial differential equations (14)-(17), it follows that for ($1 \leq i \leq k$)

$$P(x,z) = P(0,z)[1 - R(x)]e^{-\lambda x} \tag{23}$$

$$\prod_i(x,z) = \prod_i(0,z)[1 - S_i(x)]e^{-A_i(z)x} \tag{24}$$

$$Q(x,z) = Q(0,z)[1 - V(x)]e^{-b(z)x} \tag{25}$$

$$R_i(x,y,z) = R_i(x,0,z)[1 - G_i(y)]e^{-b(z)y} \tag{26}$$

where $A_i(z) = b(z) + \alpha_i\left(1 - G_i^*(b(z))\right)$ and $b(z) = \lambda\left(1 - X(z)\right)$

Solving the above equations (14)-(26), then integrating the equations (23)-(25) with respect to x and define the partial probability generating functions as, for ($1 \leq i \leq k$)

$$P(z) = \int_0^\infty P(x,z)dx,\ \prod_i(z) = \int_0^\infty \prod_i(x,z)dx,\ Q(z) = \int_0^\infty Q(x,z)dx.$$

$$P(z) = \frac{\lambda P_0\left(1 - R^*(\lambda)\right)}{V^*(\lambda)} \times \frac{Nr(z)}{Dr(z)} \tag{27}$$

$$Nr(z) = \left(\begin{array}{l} V^*(\lambda)X(z)\left\{\sum_{i=1}^{k}\left\{\left(p_i z + q_i\right)\Theta_{i-1}\left(B_i^*\left[A_i(z)\right]\right)\right\}\right\} \\ + z\left(V^*\left[b(z)\right] - 1 - V^*(\lambda)\right) \end{array}\right)$$

$$Dr(z) = \begin{pmatrix} z - \sum_{i=1}^{k}\left\{(p_i z + q_i)\Theta_{i-1}\left(B_i^*\left[A_i(z)\right]\right)\right\} \\ \left(R^*(\lambda) + X(z)(1-R^*(\lambda))\right) \end{pmatrix}$$

$$\prod_i(z) = \frac{\lambda P_0\left(1-S_i^*(A_i(z))\right)}{V^*(\lambda)A_i(z)} \times \frac{Nr(z)}{Dr(z)} \quad (28)$$

$$Nr(z) = \Theta_{i-1}\left(B_{i-1}^*\left[A_{i-1}(z)\right]\right)\left\{\left(V^*\left[b(z)\right]-1\right)X(z) + \left(V^*\left[b(z)\right]-1-V^*(\lambda)\right)R^*(\lambda)(1-X(z))\right\}$$

$$Q(z) = \frac{\lambda P_0\left(1-V^*(b(z))\right)}{V^*(\lambda)b(z)} \quad (29)$$

Integrating the equation (26) with respect to x and y define the partial probability generating functions as, for ($1 \le i \le k$),

$$R_i(x,z) = \int_0^\infty R_i(x,y,z)dy,\ R_i(z) = \int_0^\infty R_i(x,z)dx.$$

$$R_i(z) = \frac{\alpha_i \lambda P_0\left(1-S_i^*(A_i(z))\right)\left(1-G_i^*(b(z))\right)}{V^*(\lambda)A_i(z)b(z)} \times \frac{Nr(z)}{Dr(z)} \quad (30)$$

$$Nr(z) = \Theta_{i-1}\left(B_{i-1}^*\left[A_{i-1}(z)\right]\right)\left\{\left(V^*\left[b(z)\right]-1\right)X(z) + \left(V^*\left[b(z)\right]-1-V^*(\lambda)\right)R^*(\lambda)(1-X(z))\right\}$$

The probability generating function of number of customers in the system is

$$K(z) = P_0 + P(z) + Q(z) + z\sum_{i=1}^{k}\left(\prod_i(z) + R_i(z)\right)$$

$$K(z) = \frac{Nr1(z)}{Dr1(z)} \quad (31)$$

$$\begin{aligned} Nr1(z) = P_0 \Bigg\{ & z\sum_{i=1}^{k}\Theta_{i-1}\left(B_{i-1}^*\left[A_{i-1}(z)\right]\right)\left(1-S_i^*(A_i(z))\right) \\ & \left\{\left(V^*\left[b(z)\right]-1-V^*(\lambda)\right)R^*(\lambda)(1-X(z)) + \left(V^*\left[b(z)\right]-1\right)X(z)\right\} \\ & + \left(1-V^*(b(z))\right)\left(z - \left[R^*(\lambda)+X(z)\left(1-R^*(\lambda)\right)\right]\sum_{i=1}^{k}\left\{\begin{matrix}(p_i z+q_i)\Theta_{i-1} \\ \left(B_i^*\left[A_i(z)\right]\right)\end{matrix}\right\}\right) \\ & + \left[1-X(z)\right]\left(V^*(\lambda)\left(z - R^*(\lambda)\sum_{i=1}^{k}\left\{(p_i z+q_i)\Theta_{i-1}\left(B_i^*\left[A_i(z)\right]\right)\right\}\right) \\ & + z\left(1-R^*(\lambda)\right)\left(V^*\left[b(z)\right]-1-V^*(\lambda)\right)\right)\Bigg\} \end{aligned}$$

$$Dr1(z) = V^*(\lambda)\left[1-X(z)\right]Dr(z)$$

The probability generating function of number of customers in the orbit is

$$H(z) = P_0 + P(z) + Q(z) + \sum_{i=1}^{k}\left(\prod_i(z) + R_i(z)\right) \quad (32)$$

Since, the only unknown is P_0 the probability that the server is idle when no customer in the orbit and it can be found using the normalizing condition. Thus, by setting z = 1 in (27) – (30) and applying L–Hospitals rule whenever necessary and we get

$$P_0 + P(1) + Q(1) + \sum_{i=1}^{k}\left(\prod_i(1) + R_i(1)\right) = 1.$$

$$P_0 = \frac{V^*(\lambda)\times(1-\rho)}{Dr} \quad (33)$$

$$\begin{aligned} Dr = \left(R^*(\lambda)V^*(\lambda) + \lambda E(V)\right)\Bigg\{ & 1 - \sum_{i=1}^{k}\Theta_{i-1}(p_i + M_{1i}) + \sum_{i=1}^{k-1}\Theta_{i-1}M_{1i} \\ & - \sum_{i=1}^{k}\Theta_{i-1}\lambda E(X)E(S_i)\left(1-\alpha_i E(G_i)\right)\Bigg\} \end{aligned}$$

Note that, $P(z)$ ($Q(z)$, $\prod_i(z)$ and $R_i(z)$) are the PGF of number of customers in the system when server being idle, (busy on i^{th} stage, on vacation and under repair on i^{th} stage) respectively (for $1 \le i \le k$).

Performance measures: In this section, we derive system performance measures like, the mean number of customers in the orbit (L_q), the mean number of customers in the system (L_s), the average time a customer spends in the system (W_s) and the average time a customer spends in the queue (W_q) respectively.

$$L_q = \lim_{z\to 1}\frac{d}{dz}H(z) = H'(1) = \frac{P_0}{V^*(\lambda)}\left[\frac{Nr_q'''(1)Dr_q''(1) - Dr_q'''(1)Nr_q''(1)}{3\left(Dr_q''(1)\right)^2}\right]$$

$$Dr_q''(1) = -2E(X)\left(1 - E(X)(1-R^*(\lambda)) - \omega\right)$$

$$Dr_q''' = 3\left\{\begin{matrix} E(X)\left(\tau + 2E(X)(1-R^*(\lambda))\omega\right) \\ +E(X(X-1))\left(2E(X)(1-R^*(\lambda)) + \omega - 1\right)\end{matrix}\right\}$$

$$Nr_q''(1) = -2\left\{\begin{matrix} E(X)\left(R^*(\lambda)V^*(\lambda) + \lambda E(V)\right) \\ \left\{1 - \omega + \sum_{i=1}^{k}\Theta_{i-1}\lambda E(X)E(S_j)\left(1-\alpha_j E(G_j)\right)\right\}\end{matrix}\right\}$$

$$\begin{aligned} Nr_q'''(1) = 3\Bigg\{ & E(X)\left(R^*(\lambda)V^*(\lambda) + \lambda E(V)\right) \\ & \left\{\tau - \sum_{i=1}^{k}\Theta_{i-1}\begin{pmatrix} \lambda E(X(X-1))E(S_i)\left(1-\alpha_i E(G_i)\right) \\ +\alpha_i\left(\lambda E(X)\right)^2 E(S_i)E(G_i^2) \\ +\left(\lambda E(X)\right)^2 E(S_i^2)\left(1-\alpha_i E(G_i)\right)^2 \\ +2\lambda E(X)E(S_i)\left(1-\alpha_i E(G_i)\right)M_{1i-1}\end{pmatrix}\right. \\ & -(1-\omega)\begin{pmatrix} 2\lambda\left(E(X)\right)^2 E(V)(1-R^*(\lambda)) + \left(\lambda E(X)\right)^2 E(V^2) \\ + E(X(X-1))\left(R^*(\lambda)V^*(\lambda) + \lambda E(V)\right)\end{pmatrix} \\ & -\sum_{i=1}^{k}\Theta_{i-1}\lambda E(X)E(S_i)\left(1-\alpha_i E(G_i)\right) \\ & \left.\begin{pmatrix} 2\lambda E(X)E(V)(1-R^*(\lambda)) + \left(\lambda E(X)\right)^2 E(V^2) \\ +E(X)R^*(\lambda)V^*(\lambda) + \lambda E(V)E(X(X-1))\end{pmatrix}\right\} \end{aligned}$$

where $\tau = \sum_{i=1}^{k}\Theta_{i-1}M_{2i} + 2\sum_{i=1}^{k}p_i\Theta_{i-1}M_{1i} - \sum_{i=1}^{k-1}\Theta_i M_{2i}$

$$L_s = \lim_{z\to 1}\frac{d}{dz}K(z) = K'(1)\frac{P_0}{V^*(\lambda)}\left[\frac{Nr_s'''(1)Dr_q''(1) - Dr_q'''(1)Nr_q''(1)}{3\left(Dr_q''(1)\right)^2}\right]$$

$$Nr_s'''(1) = Nr_q'''(1) - 6\sum_{i-1}^{k} \Theta_{i-1}\lambda E(X)E(S_i)\big(1-\alpha_i E(G_i)\big)$$
$$\big(V^*(\lambda)R^*(\lambda)E(X) + \lambda E(X)E(V)\big)$$

$$W_s = \frac{L_s}{\lambda E(X)} \text{ and } W_q = \frac{L_q}{\lambda E(X)}$$

Conclusion: In this paper, we have studied a batch arrival feedback retrial queue with multi stage service and single vacation, where the server is subject to server breakdowns and repair. The probability generating functions of the number of customers in the system and orbit are found by using the supplementary variable technique. The performance measures like, the mean number of customers in the system/orbit, the average waiting time of customer in the system/orbit are also obtained.

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References:

1. J. R. Artalejo, "Accessible bibliography on retrial queues", Mathematical and Computer Modelling, vol. 30, 1999, pp 1-6.
2. J.E.A. Bagyam and K. U. Chandrika, "Multi- stage retrial queueing system with Bernoulli feedback", International Journal of Scientific & Engineering Research, vol. 4, 2013, pp-496-499.
3. G. Choudhury and K. Deka, "A single server queueing system with two phases of service subject to server breakdown and Bernoulli vacation", Applied Mathematical Modelling, vol. 36, 2012, pp- 6050-6060.
4. M. R. Salehurad and Badamchizadeh, A, "On the multi-phase M/G/1 queueing system with random feedback", Central European Journal of Operation Research, vol. 17, 2009, pp. 131-139.
5. M.C.Saravanarajan and V. M. Chandra-sekaran, "Analysis of MX/G/1 feedback queue with two phase service, compulsory server vacation and random breakdowns", OPSEARCH, DOI 10.1007/s12597-013-0141-6.

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