ON FUZZY LINEAR SPACES OVER VALUED FIELDS

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Abstract : *In this paper, we establish some concepts of fuzzy linear spaces over valued fields. We will also establish convex sets in fuzzy linear spaces over any valued fields.*

Definition 1.1:

Let X be a field, and F a fuzzy set in X with membership function μ_F. Suppose the following conditions hold:

(i) $\mu_F(x+y) \geq \min\{\mu_F(x), \mu_F(y)\}, \quad x, y \in X$;

(ii) $\mu_F(-x) \geq \mu_F(x), \quad x \in X$;

(iii) $\mu_F(xy) \geq \min\{\mu_F(x), \mu_F(y)\}, \quad x, y \in X$;

(iv) $\mu_F(x^{-1}) \geq \mu_F(x), \quad x(\neq 0) \in X$.

Then we call F a fuzzy field in X and denote it by (F, X). Also, (F, X) is called a fuzzy field of X.

Definition 1.2

Let X be a field, and (F, X) be a fuzzy field of X. Let Y be a linear space over X and V a fuzzy set of Y with membership function μ_V. Suppose the following conditions hold:

(i) $\mu_V(x+y) \geq \min\{\mu_V(x), \mu_V(y)\}, \quad x, y \in Y$;

(ii) $\mu_V(-x) \geq \mu_V(x), \quad x \in Y$;

(iii)

$\mu_V(\lambda x) \geq \min\{\mu_F(\lambda), \mu_V(x)\}, \quad \lambda \in X \text{ and } x \in Y$;

(iv) $\mu_F(1) \geq \mu_V(0)$.

Then (V, Y) is called a fuzzy linear space over (F, X).

Theorem 1.3 : If (V, Y) is a fuzzy linear space over (F, X), then

(i) $\mu_F(0) \geq \mu_V(0)$;

(ii) $\mu_V(0) \geq \mu_V(x), \quad x \in Y$;

(iii) $\mu_F(0) \geq \mu_V(x), \quad x \in Y$.

Theorem 1.4 : Let (F, X) be a fuzzy field of X, and Y a linear space over X. Let V be a fuzzy set of Y. Then (V, Y) is a fuzzy linear space over (F, X) iff

(i) $\mu_V(\lambda x + \mu y) \geq \min\{\mu_F(\lambda) \wedge \mu_V(x), \mu_F(\mu) \wedge \mu_V(y)\}$, $\lambda, \mu \in X \text{ and } x, y \in Y$;

(ii) $\mu_F(1) \geq \mu_V(x), \quad x \in Y$.

Proof : If (V, Y) is a fuzzy linear space over (F, X), then for $\lambda, \mu \in X \text{ and } x, y \in Y$,

$$\mu_V(\lambda x + \mu y) \geq \min\{\mu_V(\lambda x), \mu_V(\mu y)\},$$
$$\geq \min\{\mu_F(\lambda) \wedge \mu_V(x), \mu_F(\mu) \wedge \mu_V(y)\}.$$

By Theorem 1.3 (ii)and Definition 1.2 (iv),

$$\mu_F(1) \geq \mu_V(x) \quad \text{for all } x \in Y.$$

On the other hand, if the inequality of the Theorem 1.4 hold for all x, y$\in$Y, then

$$\mu_V(x+y) = \mu_V(1x+1y)$$
$$\geq \min\{\mu_F(1) \wedge \mu_V(x), \mu_F(\lambda) \wedge \mu_V(y)\}, \quad \lambda, \mu \in Y$$
$$= \min\{\mu_V(x), \mu_V(y)\}$$

So we have (i). Futher, since (F, X) is a fuzzy field of X,

$$\mu_F(0) \geq \mu_F(1) \geq \mu_V(x) \quad \text{and} \quad \mu_F(-1) \geq \mu_F(1) \geq \mu_V(x), \quad x \in Y.$$

Hence,

$$\mu_V(-x) = \mu_V(0x + (-1)x)$$
$$\geq \min\{\mu_F(0) \wedge \mu_V(x), \mu_F(-1) \wedge \mu_V(x)\}$$
$$= \min\{\mu_V(x), \mu_V(x)\}$$
$$= \mu_V(x),$$

i.e. condition (ii) of Definition 1.2 holds.

For all $\lambda \in X \text{ and } x \in Y$,

$$\mu_V(\lambda x) = \mu_V(0x + \lambda x)$$
$$\geq \min\{\mu_F(0) \wedge \mu_V(x), \mu_F(\lambda) \wedge \mu_V(x)\}$$
$$= \min\{\mu_V(x), \mu_F(\lambda) \wedge \mu_V(x)\}$$
$$= \min\{\mu_F(\lambda), \mu_V(x)\}.$$

Hence condition (iii) of the definition holds.

That condition (iv) of Definition 1.2 is obvious. Hence, (V, Y) is a fuzzy linear space over (F, X).

This completes the proof.

We state two propositions without proof which are useful in over further study.

Theorem 1.5 : Let Y and Z be linear spaces over the field X, and f a linear transformation of Y into Z. Let (F, X) be a fuzzy field of X, and (W, Z) be a fuzzy linear space over (F, X). Then $(f^{-1}(W), Y)$ is a fuzzy linear space over (F, X).

Theorem 1.6 : Let Y and Z be linear spaces over the field X, and f a linear transformation of Y into Z. Let (F, X) be a fuzzy field of X and (V, Y) be a fuzzy linear space over

(F, X). Then (f (V), Z) is a fuzzy linear space over (F, X).

Proof : For all $\lambda, \mu \in X \text{ and } u, v \in Z$ if either $f^{-1}(u) \text{ or } f^{-1}(v)$ is empty, then the inequality (i) of

Theorem 1.4 is satisfied. Suppose neither $f^{-1}(u)\, nor\, f^{-1}(v)$ is empty, then $f^{-1}(\lambda u+\mu v)\neq \phi$. Let $r\in f^{-1}(u), s\in f^{-1}(v)$. Then

$$f(\lambda r+\mu s)=\lambda f(r)+\mu f(s)=\lambda u+\mu v.$$

So

$$\mu_{f(v)}(\lambda u+\mu v)=\sup_{w\in f^{-1}(\lambda u+\mu v)}\mu_v(w)$$

$$\geq \sup_{r\in f^{-1}(u), s\in f^{-1}(v)}\mu_v(\lambda r+\mu s)$$

$$\geq \sup_{r\in f^{-1}(u), s\in f^{-1}(v)}\{\min \mu_F(\lambda)\wedge\mu_V(r), \mu_F(\mu)\wedge\mu_V(s)\}$$

$$=\min\left\{\min\left\{\mu_F(\lambda), \sup_{r\in f^{-1}(u)}\mu_V(r)\right\}, \min\left\{\mu_F(\mu), \sup_{s\in f^{-1}(v)}\mu_V(s)\right\}\right\}$$

$$=\min\{\min\{\mu_F(\lambda), \mu_{f(V)}(u)\}, \min\{\mu_F(\mu), \mu_{f(V)}(v)\}\}$$

$$=\min\{\mu_F(\lambda)\wedge\mu_{f(V)}(u), \mu_F(\mu)\wedge\mu_{f(V)}(v)\}.$$

Obviously, for any $x\in Z$, $\mu_F(1)\geq\mu_{f(V)}(x)$.

Thus (f(V), Z) is a fuzzy linear space over (F, X), which ends the proof.

Let X be any set and L a lattice (in particular L could be [0, 1]). Then a fuzzy set A in X is characterised by a mapping A : X → L. We now quote the following definitions which will be needed in the sequel.

Definition 1.7

Let E be a field and Y a linear space over E. Let X be a fuzzy subset of Y. Then X is a fuzzy linear space (fls in short) in Y if for all $x, y\in Y$ *and* $\lambda\in E$,

(i) X (x+y) ≥ min {X (x), X (y)}.

(ii) X (λx) ≥ X (x).

Definition 1.8

Let E be a field. The map | | : E → R (where R denotes the real numbers) is called a valuation on E if or all $\alpha, \beta\in E$,

(i) $|\alpha|\geq 0$ and $=0$ iff $\alpha=0$,

(ii) $|\alpha\beta|=|\alpha||\beta|$,

(iii) $|\alpha+\beta|=|\alpha|+|\beta|$.

A field E along with a valuation is called a valued field. The valuation is said to be non-Archimedean if (iii) is replaced by

(iii)′ $|\alpha+\beta|\leq\max(|\alpha|, |\beta|)$,

Otherwise it is called Archimedean. The set $V=\{\alpha\in E: |\alpha|\leq 1\}$ is a subring of E and is called the valuation ring of E or the ring of integers of E if the valuation is non-Archimedean.

Definition 1.9

Let Y be a linear space over a non-Archimedean valued field E and let X be a fuzzy linear space in Y. Let S be a fuzzy set in X. Then S is said to be balanced if

$$S(kx)=S(x) \quad \textit{for all } k\in V \textit{ and } x\in X.$$

S is said to be convex if

$$S(k_1x+k_2y)\geq\min\{S(x), S(y)\} \quad \textit{for all } k_1, k_2\in V \textit{ and } x, y\in X.$$

Theorem 1.10 : Let G_1,G_2 be linear spaces over E and f : $G_1\to G_2$ a linear map. Let F_1 and F_2 be fuzzy linear spaces in G_1 and G_2, respectively. If A is a convex (balanced) fuzzy set in F_1, then f(A) is a convex (balanced) fuzzy set in F_2. Similarly if B is a convex (balanced) fuzzy set in F_2, then $f^{-1}(B)$ is a convex (balanced) fuzzy set in F_1.

Proof : We shall prove the result only for the convex case. Let $k_1, k_2\in V$, the valuation ring of E. Let A be a convex fuzzy set in F_1. Then

$$k_1f(A)+k_2f(A)=f(k_1A+k_2A)\subset f(A)$$

which implies that f(A) is convex.

Let B be a convex fuzzy set in F_2 and let $k_1, k_2\in V$.

Set

$$M=k_1f^{-1}(B)+k_2f^{-1}(B).$$

Then

$$f(M)=k_1f(f^{-1}(B))+k_2f(f^{-1}(B))\subset k_1B+k_2B\subset B.$$

Hence $M\subset f^{-1}(B)$ and this completes the proof.

Theorem (Lowen) 1.11 : Let L be a complete lattice and let $\{A_i\}$ be a family of convex (balanced) fuzzy sets in X. Then $A=\bigcap_{i\in I}A_i$ is a convex (balanced) fuzzy set in X.

Proof: For $k_1, k_2\in V$,

$$A(k_1x+k_2y)=\inf_{i\in I}A_i(k_1x+k_2y)$$

$$\geq\inf_{i\in I}[\min\{A_i(x), A_i(y)\}] \quad \textit{...(from Definition 1.3)}$$

$$=\min\left[\inf_{i\in I}A_i(x), \inf_{i\in I}A_i(y)\right]$$

= min [A(x), A(y)].

For $k\in V$,

$$\{x\in X: kA(x)\geq d\}\subset\{x\in X: A(x)\geq d\}.$$

This completes the proof.

Definition 1.12

Let A be a fuzzy set in X. The convex (balanced) hull of A is the intersection of all convex (balanced) fuzzy sets in X which contain A.

It follows from Theorem 1.11 that the convex (balanced) hull of A is the smallest convex (balanced) fuzzy set in X which contains A.

Theorem 1.13 : Let A be a fuzzy set in X. Then the balanced hull of A is the fuzzy set $\bigcup_{\lambda\in V}\lambda A$.

Proof : It is easy to see that the fuzzy set

$B = \bigcup_{\lambda \in V} \lambda A$ is included in any balanced fuzzy set which contains A. Since $B \supset A$, it suffices to show that B is balanced. Let $a \in V$ *and* $x \in X$. Then

$$B(x) = \sup_{\lambda \in V} \lambda A(x) \leq \sup_{\lambda \in V} a\lambda A(x)$$

$$= \sup_{\lambda \in V} \lambda A(ax) = B(ax).$$

Hence, a $B \subset B$ and this completes the proof.

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