SOLUTION OF PARTIAL DIFFERENTIAL EQUATIONS INVOLVING MIXED PARTIAL DERIVATIVES BY LAPLACE SUBSTITUTION METHOD

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Abstract: The main purpose of this article is to solve linear and nonlinear homogeneous and non-homogeneous partial differential equations involving mixed partial derivatives by using Laplace Substitution Method.

Key words: Laplace Substitution Method, Laplace transform, Mixed partial derivatives, Partial differential equations.

Introduction: The partial differential equations arising from engineering and scientific applications, which were previously intractable, can now, be routinely solved [2]. Finite difference methods approximate the differential operators and hence, solve the difference equations. Infinite element method the continuous domain is represented as a collection of a finite number N of subdomains known as elements. The collection of elements is called the finite element mesh. For time dependent problems, the differential equations are approximated by the finite element method to obtain a set of ordinary differential equations in time. These differential equations are solved approximately by finite difference methods or other methods. In all finite difference and finite elements it is necessary to have a boundary and initial conditions. But the Adomian decomposition method, which has been developed by George Adomian (Solving Frontier Problem of Physics: The Decomposition Method, Kluwer Academic Publishers, Boston, MA, 1994), depends only on the initial conditions to obtain solution in series form which almost converges to the exact solutions of the problem. In recently years, some other ansatz methods have been developed, such as, the tanh method [3, 4], the extended tanh-function method [5, 6], the modified extended tanh-function method [10], variational iteration variables method [11, 12] and the sine-cosine method [9, 10].

We know that in [1] Laplace Substitution Method is not useful to solve linear partial differential equations involving mixed partial derivatives in which general linear term operator not equal to zero i.e ($Ru(x;y) \neq 0$). The main goal of this paper is to use (LSM) to find exact or approximate solution of linear and nonlinear partial differential equations involving mixed partial derivatives with the help of Adomian polynomial. This powerful method will be proposed in section 2; in section 3 we will apply it to three coupled partial differential equations involving mixed partial derivatives out of them example 1 is linear nonhomogeneous partial differential equation involving mixed partial derivatives with $Ru(x, y) \neq 0$ and example 2 and 3 are of nonlinear nonhomogeneous partial differential equations involving mixed partial derivatives. In last section we give some conclusion.

Laplace Substitution Method for nonlinear partial differential equations involving mixed partial derivatives: The aim of this section is to discuss the Laplace substitution method for nonlinear partial differential equations involving mixed partial derivatives. Let us consider the consider the general form of nonlinear, nonhomogeneous partial differential equation involving mixed partial derivatives with initial conditions is given below

$$Lu(x, y) + Ru(x, y) + Nu(x, y) = h(x, y) \quad (2.1)$$

$$u(x, 0) = f(x),\ u_y(0, y) = g(y) \quad (2.2)$$

Where $L = \frac{\partial}{\partial x \partial y}$, Ru(x,y) is the remaining linear operator, Nu represents a general nonlinear differential operator and h(x, y) is the source term. We can write equation (2.1) in the following form

$$\frac{\partial u}{\partial x \partial y} + Ru(x, y) + Nu(x, y) = h(x, y)$$

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) + Ru(x; y) + Nu(x; y) = h(x; y) \quad (2.3)$$

Substituting $\frac{\partial u}{\partial y} = U$ in equation (2.3), we get

$$\frac{\partial U}{\partial x} + Ru(x; y) + Nu(x; y) = h(x; y) \quad (2.4)$$

Taking Laplace transform of equation (2.4) with respect to x, we get

$$sU(s, y) - U(0, y) = Lx\,[h(x, y) - Ru(x, y) - Nu(x, y)]$$

$$U(s; y) = \frac{1}{s} g(y) + \frac{1}{s} Lx\,[h(x; y) - Ru(x; y) - Nu(x; y)] \quad (2.5)$$

Taking inverse Laplace transform of equation (2.5) with respect to x, we get

$$U(x; y) = g(y) + L^{-1}_x\{[\frac{1}{s} L_x\,[h(x, y) - Ru(x, y)\ Nu(x, y)]\}$$

(2.6)

Re-substitute the value of U(x,y) in equation (2.6), we get

$$\frac{\partial u(x,y)}{\partial y} = g(y) + L_x^{-1}\left[\frac{1}{s}L_x[h(x,y) - Ru(x,y) - Nu(x,y)]\right] \quad (2.7)$$

This is the first order nonlinear, nonhomogeneous partial differential equation in the variables x and y. Taking the Laplace transform of equation (2.7) with respect to y, we get

$$su(x,s) = f + L_y\left[g(y) + L_x^{-1}\left[\frac{1}{s}L_x[h(x,y) - Ru(x,y) - Nu(x,y)]\right]\right]$$

$$u(x,s) = \frac{1}{s}f(x) + \frac{1}{s}L_y\left[g(y) + L_x^{-1}\left[\frac{1}{s}L_x[h(x,y) - Ru(x,y) - Nu(x,y)]\right]\right] \quad (2.8)$$

Taking the inverse Laplace transform of equation (2.8) with respect to y, we get

$$u(x,y) = f(x,y) + \frac{1}{s}L_y^{-1}\left[\frac{1}{s}L_y\left[g(y) + L_x^{-1}\left[\frac{1}{s}L_x[h(x,y) - Ru(x,y) - Nu(x,y)]\right]\right]\right] \quad (2.9)$$

For solving nonlinear, non homogeneous PDE involving mixed partial derivatives by LSM, let we consider solution of (2.1) is in series form. Therefore suppose that

$$u(x,y) = \sum_{n=0}^{\infty} u_n(x,y) \quad (2.10)$$

is a required solution of (2.1) in series form. We know that nonlinear term Nu(x, y) appear in equation (2.1), let we decompose it by using Adomian polynomial which is defined bythe formula [2]

$$A_n = \frac{1}{n!}\left[\frac{d^n}{d\gamma^n}\left[N\left[\sum_{i=0}^{\infty}\gamma^i u_i\right]\right]\right]_{\gamma=0} \quad (2.11)$$

$$Nu(x,y) = \sum_{n=0}^{\infty} A_n \quad (2.12)$$

Where An is an Adomian polynomial of u_0, u_1, u_2, u_3,................u_n. Substitute (2.10) and (2.12) in equation (2.9), we get

$$\sum_{n=0}^{\infty} u_n(x,y) = f(x,y) + L_y^{-1}\left[\frac{1}{s}L_y\left[g(y) + L_x^{-1}\left[\frac{1}{s}L_x[h(x,y) - Ru(x,y) - Nu(x,y)]\right]\right]\right]$$

On comparing both sides of above equation, we get

$$u_0(x,y) = f(x) + L_y^{-1}\left[\frac{1}{s}L_y\left[g(y) + L_x^{-1}\left[\left[\frac{1}{s}L_x[h(x,y)]\right]\right]\right]\right]$$

$$u_1(x,y) = L_y^{-1}\left[\frac{1}{s}L_y\left[L_x^{-1}\left[\left[\frac{1}{s}L_x[Ru_0(x,y) + A_0]\right]\right]\right]\right]$$

$$u_2(x,y) = L_y^{-1}\left[\frac{1}{s}L_y\left[L_x^{-1}\left[\left[\frac{1}{s}L_x[Ru_1(x,y) + A_1]\right]\right]\right]\right]$$

In general we get the, following required recursive relation

$$u_0(x,y) = f(x) + L_y^{-1}\left[\frac{1}{s}L_y\left[g(y) + L_x^{-1}\left[\left[\frac{1}{s}L_x[h(x,y)]\right]\right]\right]\right] \quad (2.13)$$

$$u_{n+1}(x,y) = L_y^{-1}\left[\frac{1}{s}L_y\left[L_x^{-1}\left[\left[\frac{1}{s}L_x[Ru_n(x,y) + A_n]\right]\right]\right]\right], \quad n \geq 0 \quad (2.14)$$

From this recursive relation we can calculate the factors u_i,i=1, 2, 3, 4,...........of u(x, y).Substitute all values of u_i in equation (2.10), we get the required solution of equation (2.1).

3.Applications: To illustrate this method for coupled linear and nonlinear partial differential equations involving mixed partial derivatives with Ru(x, y) ≠ 0. We take three examples in this section.

Example 1: Consider the following linear partial differential equation with Ru(x, y) ≠ 0 [1]

$$\frac{\partial^2 u}{\partial x\,\partial y} + \frac{\partial u}{\partial x} + u = 6x^2y \quad (3.15)$$

with initial conditions

$$u(x, 0) = 0, \; u(0, y) = 0, \; u_y(0, y) = 0 \quad (3.16)$$

In the above example $Ru(x,y) = \frac{\partial u(x,y)}{\partial x} + u(x,y)$. Use the substitution $\frac{\partial u(x,y)}{\partial y} = U(x,y)$ inequation (3.26), we get

$$\frac{\partial U}{\partial x} + \frac{\partial u}{\partial x} + u = 6x^2y \quad (3.17)$$

Taking Laplace transform of equation (3.17) with respect to x, we get

$$sU(s,y) - u(0,y) + su(s,y) - u(0,y) + L_x[u(x,y)] = \frac{12y}{s^3}$$

$$U(s,y) = \frac{12y}{s^4} - u(s,y) - \frac{1}{s}L_x[u(x,y)] \quad (3.18)$$

Taking inverse Laplace transform of equation (3.18) with respect to x, we get

$$U(x,y) = 2yx^3 - u(x,y) - L_x^{-1}\left[\frac{1}{s}L_x[u(x,y)]\right]$$

$$\frac{\partial u(x,y)}{\partial x} = 2yx^3 - u(x,y) - L_x^{-1}\left[\frac{1}{s}L_x[u(x,y)]\right] \quad (3.19)$$

Taking Laplace transform of equation (3.19) with respect to y, we get

$$su(x,s) - u(x,0) = \frac{2x^3}{s^2} - L_y\left[u(x,y) + L_x^{-1}\left[\frac{1}{s}L_x[u(x,y)]\right]\right]$$

$u(x,s) = \frac{2x^3}{s^3} - \left[\frac{1}{s}L_y\left[u(x,y) + L_x^{-1}\left[\frac{1}{s}L_x[u(x,y)]\right]\right]\right]$

(3.20)

Taking inverse Laplace transform of equation (3.20) with respect to y, we ge

$u(x,y) =$

$x^3y^2 - L_y^{-1}\left[\frac{1}{s}L_y\left[u(x,y) + L_x^{-1}\left[\frac{1}{s}L_x[u(x,y)]\right]\right]\right]$ (3.21)

In equation (3.15) does not contain nonlinear term Nu(x, y). Therefore there is not required to use Adomian polynomial. Let we suppose that,

$u(x,y) = \sum_{n=0}^{\infty} u_n(x,y)$ (3.22)

is a required solution of equation (3.15). Substitute (3.22) in equation (3.21), we have

$\sum_{n=0}^{\infty} u_n(x,y) = x^3y^2 - L_y^{-1}\left[\frac{1}{s}L_y\left[\sum_{n=0}^{\infty} u_n(x,y) + L_x^{-1}\left[\frac{1}{s}L_x\sum_{n=0}^{\infty} u_n(x,y)\right]\right]\right]$ (3.23)

On comparing both sides of above equation, we have a recursive relation

$u_0(x,y) = x^3y^2$ (3.24)

$u_{n+1}(x,y) = -L_y^{-1}\left[\frac{1}{s}L_y\left[\sum_{n=0}^{\infty} u_n(x,y) + L_x^{-1}\left[\frac{1}{s}L_x\sum_{n=0}^{\infty} u_n(x,y)\right]\right]\right]$, n≥0 (3.25)

From above recursive relation, we get

$u_0(x,y) = x^3y^2$, $u_1(x,y) = -\left[\frac{x^3y^3}{3} + \frac{x^4y^3}{12}\right]$,

$u_2(x,y) = \left[\frac{x^3y^4}{12} + \frac{x^4y^4}{24} + \frac{x^5y^4}{240}\right]$,

$u_3(x,y) = -\left[\frac{x^3y^5}{60} + \frac{x^4y^5}{60} + \frac{x^5y^5}{400} + \frac{x^5y^4}{240} + \frac{x^5y^6}{7200}\right]$

and so on.

Substitute all these values in equation (3.22), we get

$u(x,y) = x^3y^2 - \frac{x^3y^3}{3} - \frac{x^4y^3}{12} + \frac{x^3y^4}{12} + \frac{x^4y^4}{24} + \frac{x^5y^4}{240} - \frac{x^3y^5}{60} - \frac{x^4y^5}{60} - \frac{x^5y^5}{400} - \frac{x^5y^4}{240} - \frac{x^5y^6}{7200}$ (3.26)

This is the required solution of equation (3.15). Which can be verifying through the substitution?

Example 2: Consider the following nonlinear homogeneous partial differential equation with given initial conditions in which Ru(x; y) = 0

$\frac{\partial^2 u}{\partial x \partial y} = \left[\frac{\partial u}{\partial y}\right]^2$ (3. 27)

$u(x, 0) = 0$, $u_y(0, y) = 1$ (3.28)

Above equation (3.27) we can write in the form of $u_{xy} = u_y^2$. Let we use the substitution $\frac{\partial u(x,y)}{\partial y} = U(x,y)$ in equation (3.27), we get

$\frac{\partial U}{\partial x} = U^2$ (3.29)

This is the first order nonlinear partial differential equation with initial condition U (0, y) = 1.Taking Laplace transforms of above equation (3.29) with respect to x, we get

$sU(s,y) - U(0,y) = L_x[U^2]$

$U(s,y) = \frac{1}{s} + \frac{1}{s}L_x[U^2]$

Taking inverse Laplace transform of above equation on both sides with respect to x, we get

$U(s,y) = 1 + L_x^{-1}\left[\frac{1}{s}L_x[U^2]\right]$

$\frac{\partial u}{\partial y} = 1 + L_x^{-1}\left[\frac{1}{s}L_x\left[\left(\frac{\partial u}{\partial y}\right)^2\right]\right]$ (3.30)

Taking Laplace transform on both sides of equation (3.30) with respect to y, we get

$u(x,s) = \frac{1}{s^2} + \frac{1}{s}L_y\left[L_x^{-1}\left[\frac{1}{s}L_x\left[\left(\frac{\partial u}{\partial y}\right)^2\right]\right]\right]$

(3.31)

Taking inverse Laplace transform of equation (3.31) with respect to y, we get

$u(x,y) = y + L_y^{-1}\left[\frac{1}{s}L_y\left[L_x^{-1}\left[\frac{1}{s}L_x\left[\left(\frac{\partial u}{\partial y}\right)^2\right]\right]\right]\right]$

(3.32)

From section (2), we know that in LSM, we represent solution in infinite series form . Letwe suppose that

$u(x,y) = \sum_{n=0}^{\infty} u_n(x,y)$ (3.33)

be the required solution of given equation (3.27). The nonlinear term appear in equation(3.27), we can decompose it by using Adomian polynomial defined by the equation (2.11)

$\left(\frac{\partial u}{\partial y}\right)^2 = \sum_{n=0}^{\infty} A_n$ (3.34)

Where An is an Adomian polynomial of components u_0, u_1, u_2, ...u_n. Let we find the few Adomian polynomials,

$A_0 = u_{0y}^2, A_1 = 2u_{0y}u_{1y}, A_2 = 2u_{0y}u_{2y} + u_{1y}^2$, ...

From equations (3.32), (3.33) and (3.34), we get

$\sum_{n=0}^{\infty} u_n(x,y) = y + L_y^{-1}\left[\frac{1}{s}L_y\left[L_x^{-1}\left[\frac{1}{s}L_x\left[\sum_{n=0}^{\infty} A_n\right]\right]\right]\right]$

(3.35)

Comparing on both sides of above equation, we get a recursive relation

$u_0(x, y) = y$,

$u_{n+1}(x,y) = L_y^{-1}\left[\frac{1}{s}L_y\left[L_x^{-1}\left[\frac{1}{s}L_x[A_n]\right]\right]\right]$ (3.37)

From the above recursive relation, we get the following few components of u(x, y)

$u_0(x,y) = y$, $u_1(x,y) = xy$, $u_2(x,y) = x^2y$, $u_3(x,y) = x^3y$ … … … … … … … … … … … …

and so on. Thus the solution of give equation (3.27) is,

$u(x,y) = u_0(x,y) + u_1(x,y) + u_2(x,y) + u_3(x,y) +$ … … … … … … … … … … … …

$u(x,y) = y + xy + x^2y + x^3 +$ … …

$u(x,y) = y\sum_{n=0}^{\infty} x^n$ (3.38)

This is the solution of given equation (3.27). Which is convergent only when $|x| < 1$. Thus the equation (3.27) has a convergent and exact solution

$$u(x, y) = \frac{y}{1-x}, \quad |x| < 0 \qquad (3.39)$$

Which can be verifying through the substitution?

Example 3: Consider the following nonlinear non-homogeneous partial differential equation with R(x; y) ≠ 0

$$\frac{\partial^2 u}{\partial x \partial y} + \left[\frac{\partial u}{\partial y}\right]^2 = 1 \qquad (3.40)$$

$$u(x, 0) = 0, \quad u_y(0, y) = 0 \qquad (3.41)$$

Above equation (3.240) we can write in the form of $u_{xy} + u_y^2 = 1$. Let we use the substitution $\frac{\partial u(x,y)}{\partial y} = U(x, y)$ in equation (3.40), we get

$$\frac{\partial U}{\partial x} + U^2 = 1 \qquad (3.42)$$

This is the first order nonlinear partial differential equation with initial condition U (0, y) = 0.Taking Laplace transform on both sides of equation (3.42) with respect to x, we get

$$sU(s, y) - U(0, y) = L_x[1 - U^2]$$

$$U(s, y) = \frac{1}{s^2} - \frac{1}{s} L_x[U^2] \quad \text{since } U(0, y) = 0 \qquad (3.43)$$

Taking inverse Laplace transform of above equation on both sides with respect to x, we get

$$U(s, y) = x - L_x^{-1}\left[\frac{1}{s} L_x[U^2]\right] \qquad (3.44)$$

$$\frac{\partial u}{\partial y} = x - L_x^{-1}\left[\frac{1}{s} L_x\left[\left(\frac{\partial u}{\partial y}\right)^2\right]\right] \qquad (3.45)$$

Taking Laplace transform on both sides of equation (3.45) with respect to y, we get

$$u(x, s) = \frac{x}{s^2} - \frac{1}{s} L_y\left[L_x^{-1}\left[\frac{1}{s} L_x\left[\left(\frac{\partial u}{\partial y}\right)^2\right]\right]\right] \qquad (3.46)$$

Taking inverse Laplace transform of equation (3.46) with respect to y, we get

$$u(x, y) = yx - L_y^{-1}\left[\frac{1}{s} L_y\left[L_x^{-1}\left[\frac{1}{s} L_x\left[\left(\frac{\partial u}{\partial y}\right)^2\right]\right]\right]\right] \qquad (3.47)$$

From section (2), we know that in LSM, we represent solution in infinite series form. Let we suppose that

$$u(x, y) = \sum_{n=0}^{\infty} u_n(x, y) \qquad (3.48)$$

be the required solution of given equation (3.40). The nonlinear term appear in equation(3.27), we can decompose it by using Adomian polynomial defined by the equation (2.11)

$$\left(\frac{\partial u}{\partial y}\right)^2 = \sum_{n=0}^{\infty} A_n \qquad (3.49)$$

Where An is an Adomian polynomial of u0, u1, u2... un. The Adomian polynomial An is defined by equation (2.11). We have calculated the few Adomian polynomials in example2. From equations (3.47), (3.48) and (3.49), we get

$$\sum_{n=0}^{\infty} u_n(x, y) = xy - L_y^{-1}\left[\frac{1}{s} L_y\left[L_x^{-1}\left[\frac{1}{s} L_x\left[\sum_{n=0}^{\infty} A_n\right]\right]\right]\right] \qquad (3.50)$$

Comparing on both sides of above equation, we get a recursive relation

$$u_0(x, y) = xy,$$

$$u_{n+1}(x, y) = -L_y^{-1}\left[\frac{1}{s} L_y\left[L_x^{-1}\left[\frac{1}{s} L_x[A_n]\right]\right]\right], \quad n \geq 0 \qquad (3.51)$$

From this recursive relation we get the following components of u(x,y)

$$u_0(x, y) = xy, \quad u_1(x, y) = \frac{-x^3 y}{3}, \quad u_2(x, y) = \frac{2x^5 y}{15},$$

$$u_3(x, y) = \frac{-17x^7}{315}, \ldots\ldots\ldots\ldots\ldots\ldots$$

and so on. Put all the values of un(x; y); n _ 0 in equation (3.48), we get

$$u(x, y) = xy - \frac{x^3 y}{3} + \frac{2x^5 y}{15} - \frac{17x^7}{315} + \ldots \qquad (3.52)$$

$$u(x, y) = y \tanh x \qquad (3.53)$$

This is the required exact solution of equation (3.40). Which can be verify through the substitution.

Conclusion: In this paper, we successfully apply the proposed Laplace Substitution Method (LSM) to solve linear and nonlinear homogeneous , non-homogeneous partial differential equations in which involves mixed partial derivatives with general linear term is either Ru(x, y) = 0 or Ru(x, y) ≠ 0. In comparison with the existing method to obtain the exact solution our (LSM) find the exact solution with less computation. One more advantage of this method is that it gives the solution in series form. In the future, we plan to generalize our method to apply for the higher order nonlinear partial differential equations involving mixed partial derivatives in nonlinear terms.

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