TOTAL COLORING AND (k, d)- TOTAL COLORING OF PRISMS

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Abstract: A total coloring of a graph was discussed by Borodin [1]. A (k, d)-total coloring of a graph was defined and discussed by Hackmann and Kemnitz [2] for the cycle, complete graphs and certain multipartite graphs. In this paper, we have discussed the total coloring and (k, d)-total coloring for the Prisms, $\boldsymbol{Y_n}$.

Introduction: A total coloring is a coloring on the vertices and edges of a graph such that
(i) no two adjacent vertices have the same color
(ii) no two adjacent edges have the color and
(iii) no edge and its end vertices are assigned with the same color.

Total coloring of graphs was discussed by many researchers since late 1990's. Borodin [1] has discussed the total coloring of graphs. Hackmann and Kemnitz [2] defined the (k, d)-total coloring of graphs and discussed the chromatic number of the cycle, complete graphs and certain multipartite graphs. Prisms Y_n with $2n$ nodes are characterized as generalized Petersen graphs $P(n, 1)$. We have discussed the total coloring of prisms Y_n and deduced the (k, d)-total coloring for the same.

If a simple graph G has two adjacent vertices of maximum degree, then $\chi_{tc}(G) \geq \Delta(G) + 1$. Otherwise $\chi_{tc}(G) \leq \Delta(G) + 2$. The truth of the total coloring conjecture would imply that $\chi_{tc}(G)$ attains one of the two values for every graph G. A graph G is called type-1 graph if $\chi_{tc}(G) = \Delta(G) + 1$ and a type-2 graph if $\chi_{tc}(G) = \Delta(G) + 2$.

A (k, d)-total coloring of a graph G, with $k \geq 2d$, k and d are positive integers, is an assignment of colors C_i $(1 \leq i \leq n)$ to the vertices and edges of G such that

$d \leq |C(x_i) - C(x_j)| \leq k - d$, for every adjacent vertices x_i and x_j and for every adjacent edges x_i and x_j of G.

Theorem 1: The prism Y_n $(n \geq 4)$ admits total colouring.

Proof:

Let the prism Y_n has the inner vertices $v_1, v_2, \ldots\ldots\ldots v_{n-1}, v_n$ and let its outer vertices be $u_1, u_2, \ldots\ldots\ldots u_{n-1}, u_n$. Let the edges of the prism Y_n be represented as follows:

$e_i = (v_i, v_{i+1})$, $1 \leq i \leq n$ and (v_n, v_1).

$f_i = (v_i, u_i)$, $1 \leq i \leq n$

$g_i = (u_i, u_{i+1})$, $1 \leq i \leq n$ and (u_n, u_1).

Let the coloring be denoted by 1, 2, 3,…. We assign the colors for the vertices and edges from the set $\{1, 2, 3, \ldots\}$ satisfies the following conditions:
(i) no two adjacent vertices have the same color
(ii) no two adjacent edges have the color and
(iii) no vertex and its incident edge has the same color.

If all these conditions are satisfied then the coloring is total in the prism Y_n.

We define the functions f_1 and f_2 as follows:

$f_1: V \to \{1, 2, 3, \ldots\}$ and $f_2: E \to \{1, 2, 3, \ldots\}$

For each even n and odd n, the proof is given below separately.

Case (1): Let n be even we need only five colors for total coloring of Y_n. We assign the colors 1, 2, 3, 4 and 5 to the vertices and edges as follows:

$$f_1(v_i) = \begin{cases} 1, & 1 \leq i \leq n \text{ and odd } i \\ 2, & 1 \leq i \leq n \text{ and even } i \end{cases}$$

$$f_1(u_i) = \begin{cases} 2, & 1 \leq i \leq n \text{ and odd } i \\ 1, & 1 \leq i \leq n \text{ and even } i \end{cases}$$

$$f_2(e_i) = \begin{cases} 3, & 1 \leq i \leq n \text{ and odd } i \\ 4, & 1 \leq i \leq n \text{ and even } i \end{cases}$$

$$f_2(f_i) = 5, 1 \leq i \leq n$$

$$f_2(g_i) = \begin{cases} 3, & 1 \leq i \leq n \text{ and odd } i \\ 4, & 1 \leq i \leq n \text{ and even } i \end{cases}$$

The prism Y_n has total coloring with this type of coloring and hence its chromatic number $\chi_{tc}(Y_n)$ *is* 5.

Case (2): Let n be odd. We need only five colors for total coloring of Y_n. We assign the colors 1, 2, 3, 4 and 5 to the vertices and edges as follows:

$$f_1(v_i) = \begin{cases} 1, & 1 \leq i \leq n-1 \text{ and odd } i \\ 2, & 1 \leq i \leq n-1 \text{ and even } i \end{cases}$$

$f_1(v_n) = 3$ and $f_1(u_1) = 3$ $\quad f_1(u_i) = \begin{cases} 2, & 2 \leq i \leq n \text{ and odd } i \\ 1, & 2 \leq i \leq n \text{ and even } i \end{cases}$

$$f_2(e_i) = \begin{cases} 3, & 1 \leq i \leq n-1 \text{ and odd } i \\ 4, & 1 \leq i \leq n-1 \text{ and even } i \end{cases}$$

$f_2(e_n) = 5$ and $f_2(f_1) = 2$

$f_2(f_i) = 5, \quad 2 \leq i \leq n-1$

$f_2(f_n) = 1$ and $f_2(g_1) = 5$

$$f_2(g_i) = \begin{cases} 3, & 2 \leq i \leq n \text{ and odd } i \\ 4, & 2 \leq i \leq n \text{ and even } i \end{cases}$$

The prism Y_n has total coloring with this type of coloring and hence its chromatic number $\chi_{tc}(Y_n)$ *is* 5.

Deduction from total coloring to (k, d)- total coloring

In the above theorem, if we replace 5 by 0, the conditions required for (k, d)- total coloring is satisfied.

Illustration 1

Consider the prism Y_6 for even n with inner vertices $v_1, v_2, v_3, v_4, v_5, v_6$ and its outer vertices be $u_1, u_2, u_3, u_4, u_5, u_6$. By the above theorem-1 for even n, the colors 1, 2, 3, 4, 5 are assigned to the vertices and edges as shown in fig.1.

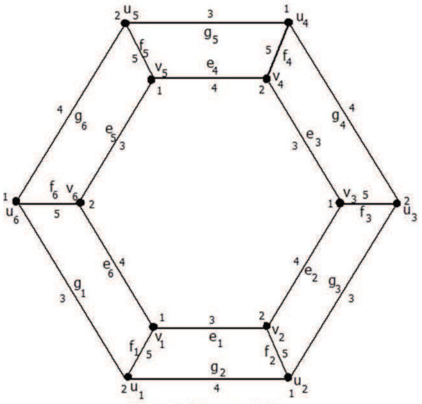


Figure 1. Prism graph Y_6

The inner vertices $v_1, v_2, v_3, v_4, v_5, v_6$ are colored with colors 1,2,1,2,1,2 respectively; the outer vertices $u_1, u_2, u_3, u_4, u_5, u_6$ are colored with colors 2,1,2,1,2,1 respectively. The inner edges $e_1, e_2, e_3, e_4, e_5, e_6$ are colored with colors 3,4,3,4,3,4 respectively; the middle edges $f_1, f_2, f_3, f_4, f_5, f_6$ are colored with colors 5 and the edges $g_1, g_2, g_3, g_4, g_5, g_6$ are colored with colors 4, 3, 4, 3, 4, 3 respectively.

The chromatic number of the total coloring of Y_6 is 5 i.e. $\chi_{tc}(Y_6) = 5$.

Illustration 2

Consider the prism Y_5 for odd n, with inner vertices v_1, v_2, v_3, v_4, v_5 and its outer vertices be u_1, u_2, u_3, u_4, u_5 . By theorem-1 the colors 1,2,3,4,5 are assigned to the vertices and edges as shown in fig. 2.

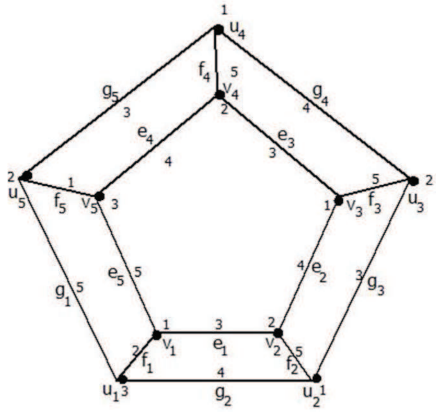


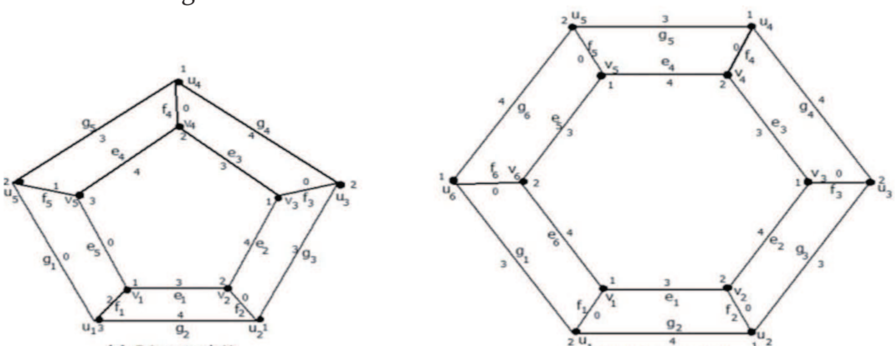
Figure 2. Prism graph Y_5

The inner vertices v_1, v_2, v_3, v_4, v_5 are colored with colors 1,2,1,2,3 respectively; the outer vertices u_1, u_2, u_3, u_4, u_5 are colored with colors 3,1,2,1,2 respectively. The inner edges e_1, e_2, e_3, e_4, e_5 are colored with colors 3,4,3,4,5 respectively; the middle edges f_1, f_2, f_3, f_4, f_5 are colored with colors 2,5,5,5,1 and the edges g_1, g_2, g_3, g_4, g_5 are colored with colors 5,4,3,4,3 respectively.

The chromatic number of the total coloring of Y_5 is 5 i.e. $\chi_{tc}(Y_5) = 5$.

Illustration 3

Consider the prisms Y_5 *and* Y_6 given in fig.1 and fig.2. Replace the color 5 by 0 as shown in fig.3(a) and fig.3(b).



(a). Prism graph Y_5 (b). Prism graph Y_6

Figure 3. (k,d)-total coloring of prism graph

In fig3(a), the values for d are 0, 1 and k value is 5. $(k \geq 2d)$ consider the vertices v_1 *and* u_1 which is colored with colors 1 and 3 respectively that satisfy the condition $0 \leq |2| \leq 4$. In fig3(b), the values for d are 0, 1 and k value is 5. $(k \geq 2d)$ consider the vertex v_4 and edge e_3 which is colored with colors 2 and 3 respectively. This satisfy the condition $1 \leq |1| \leq 3$.

The condition for (k, d)- total coloring is thus satisfied for the prisms Y_6 and Y_5.

Total coloring and (k, d)- total coloring of the prism Y_3.

For the prism Y_3, the above general method is not applicable. So we discuss it here.

Let the inner vertices of Y_3 be v_1, v_2, v_3 ; the outer vertices of Y_3 be u_1, u_2, u_3 ; the inner edges e_1, e_2, e_3 ; the middle edges f_1, f_2, f_3 and the outer edges g_1, g_2, g_3.

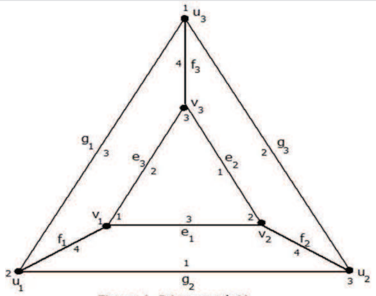


Figure 4. Prism graph Y_3

The inner vertices $v_{1,}v_{2,}$ v_3 of Y_3 colored with colors 1,2,3 respectively and the outer vertices $u_{1,}u_{2,}$ u_3 are colored with colors 2,3,1 respectively. The inner edges taken the colors $e_{1,}e_{2,}$ e_3 are colored with colors 3,2,1; the middle edges $f_{1,}f_{2,}$ f_3 are colored with the color 4 and the outer edges $g_{1,}g_{2,}$ g_3 are colored with colors 3,1,2 respectively. The colors required for the total coloring of Y_3 is 4 further for (k,d)-total coloring of Y_3 is also 4 and its chromatic number χ_{tc} $(Y_3) = 4$.

Conclusion: We have considered the prisms Y_n (n ≥ 4) with colors using the definition of total coloring and obtained the following results:

The total and the (k,d) −total chromatic number χ_{tc} $(Y_n) = 5$, for all n ≥ 4 and the total chromatic number χ_{tc} $(Y_3) = 4$.

References:

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