MULTIOBJECTIVE NON LINEAR PROGRAMMING PROBLEM IN INTUITIONISTIC FUZZY ENVIRONMENT

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Abstract: In this article, we formulate a multi-objective non linear programming problem in intuitionistic fuzzy environment. Using a linear ranking function we transform the problem into conventional multi-objective non linear programming then we have developed an algorithm to solve it, finally we illustrate our methodology by a numerical example.

Keywords: Complete solution, Intuitionistic fuzzy non-linear programming, Membership function

Introduction: Most of the real world problems are inherently characterized by multiple and conflicting aspects of evaluation. This evaluation is generally judged by optimizing multiple objective functions. Furthermore, when addressing real world problems, frequently the parameters are imprecise numerical quantities due to various factors. Fuzzy quantities are very adequate for modeling these situations. The application of fuzzy set theory to decision making has gained considerable attention by many author since the pioneering work of Bellman and Zadeh [6]. The most common approach to solve fuzzy linear programming problem is to change them into corresponding crisp linear programming. Zimmermann [2] has introduced fuzzy programming approach to solve crisp multi objective linear programming problem. Many authors transform the fuzzy programming problem into crisp by utilizing ranking function [1, 4] then solve it by conventional methods. In real world situation generally we have to deal with uncertainty as well as hesitation. In such situations intuitionistic fuzzy quantities are more reliable representatives of the data. Intuitionistic fuzzy set theory introduced by Atanassov [3] is one of the important generalizations of fuzzy set theory which has very pioneering applications in physical problems.

In this paper, we have formulated intuitionistic fuzzy multi objective non-linear programming problem (IFMONLPP) in which coefficients of all objective functions as well as the constraints are intuitionistic fuzzy in nature. We have developed an algorithm to solve this problem using a linear ranking function.

The paper is organized as follows: Section 2 deals with some definitions from literature and arithmetic operations on TIFNs [3, 5]. In Section 3, we define the ordering of TIFNs using the proposed accuracy function. In Section 4, we have formulated the multi-objective nonlinear programming problem in intuitionistic fuzzy environment and developed an algorithm to solve it. In Section 5, a numerical example is given to illustrate the methodology. Finally, conclusion is given in Section 6.

2. Some Definitions

2.1 Definition Let X be a universal set. Then a fuzzy set $\tilde{A}$ in X is defined by

$\tilde{A}=\{(x,\mu_{\tilde{A}}(x)):x\in X\}$, where $\mu_{\tilde{A}}:X\to[0,1]$.

2.2 Definition Let X be a universe of discourse. Then an intuitionistic fuzzy set (IFS) $\tilde{A}^I$ in X is defined by a set of ordered triples

$$\tilde{A}^I=\{<x,\mu_{\tilde{A}^I}(x),\upsilon_{\tilde{A}^I}(x)>:x\in X\},$$

where $\mu_{\tilde{A}^I},\upsilon_{\tilde{A}^I}:X\to[0,1]$ are functions such that $0\le\mu_{\tilde{A}^I}(x)+\upsilon_{\tilde{A}^I}(x)\le 1,\forall x\in X$. $\mu_{\tilde{A}^I}(x)$ represents the degree of membership and $\upsilon_{\tilde{A}^I}(x)$ represents the degree of non- membership of the element $x\in X$ being in $\tilde{A}^I$. $h(x)=1-\mu_{\tilde{A}^I}(x)-\upsilon_{\tilde{A}^I}(x)\ \forall x\in X$ is called degree of hesitation that $x\in X$ being in $\tilde{A}^I$.

2.3 Definition An intuitionistic fuzzy subset $\tilde{A}^I=\{<x,\mu_{\tilde{A}^I}(x),\upsilon_{\tilde{A}^I}(x)>:x\in X\}$ of the real line R is called an intuitionistic fuzzy number (IFN) if the following hold:

1. There exists $m\in R$ such that $\mu_{\tilde{A}^I}(m)=1$ and $\upsilon_{\tilde{A}^I}(m)=0$ (m is called the mean value of $\tilde{A}^I$).
2. $\mu_{\tilde{A}^I}$ is a piecewise continuous mapping from R to the closed interval [0, 1] and the relation $0\le\mu_{\tilde{A}^I}(x)+\upsilon_{\tilde{A}^I}(x)\le 1,\forall x\in R$ holds.

The membership and non–membership functions of $\tilde{A}^I$ are of the following form:

$$\mu_{\tilde{A}^I}(x)=\begin{cases}\dot{g}_1(x), & m-a\le x\le m,\\ 1, & x=m,\\ h_1(x), & m\le x\le m+b,\\ 0, & \text{otherwise},\end{cases}$$

where $g_1(x)$ and $h_1(x)$ are piecewise continuous; strictly increasing and strictly decreasing function in $[m-a, m]$ and $[m, m+b]$ respectively.

$$\upsilon_{\tilde{A}^I}(x)=\begin{cases} g_2(x), & m-a' \le x \le m; 0 \le g_1(x)+g_2(x) \le 1, \\ 0, & x=m, \\ h_2(x), & m \le x \le m+b'; 0 \le h_1(x)+h_2(x) \le 1, \\ 1, & \text{otherwise.} \end{cases}$$

Here m is the mean value of $\tilde{A}^I$; a and b are left and right spreads of membership function $\mu_{\tilde{A}^I}(x)$ respectively, a' and b' are left and right spreads of non- membership function $\upsilon_{\tilde{A}^I}(x)$ respectively. The IFN $\tilde{A}^I$ is represented by $\tilde{A}^I=(m; a, b; a', b')$.

2.4 Definition A triangular intuitionistic fuzzy number (TIFN) $\tilde{A}^I$ is an intuitionistic fuzzy set in R with the following membership function $\mu_{\tilde{A}^I}(x)$ and non-membership function $\nu_{\tilde{A}^I}(x)$:

$$\mu_{\tilde{A}^I}(x)=\begin{cases} \dfrac{x-a_1}{a_2-a_1}, & a_1 < x \le a_2 \\ \dfrac{a_3-x}{a_3-a_2}, & a_2 \le x < a_3 \\ 0, & \text{otherwise,} \end{cases}$$

$$\nu_{\tilde{A}^I}(x)=\begin{cases} \dfrac{a_2-x}{a_2-a_1'}, & a_1' < x \le a_2 \\ \dfrac{x-a_2}{a_3'-a_2}, & a_2 \le x < a_3' \\ 1, & \text{otherwise.} \end{cases}$$

where $a_1' \le a_1 < a_2 < a_3 \le a_3'$. This TIFN is denoted by $\tilde{A}^I=(a_1,a_2,a_3;a_1',a_2,a_3')$. The set of all TIFNs is denoted by $IF(R)$.

2.5 Arithmetic operations on TIFNs: Let $\tilde{A}^I=(a_1,a_2,a_3;a_1',a_2,a_3')$ and $\tilde{B}^I=(b_1,b_2,b_3;b_1',b_2,b_3')$

Addition

$$\tilde{A}^I \oplus \tilde{B}^I=(a_1+b_1,a_2+b_2,a_3+b_3;a_1'+b_1',a_2+b_2,a_3'+b_3')$$

Subtraction

$$\tilde{A}^I \Theta \tilde{B}^I=(a_1-b_3,a_2-b_2,a_3-b_1;a_1'-b_3',a_2-b_2,a_3'-b_1')$$

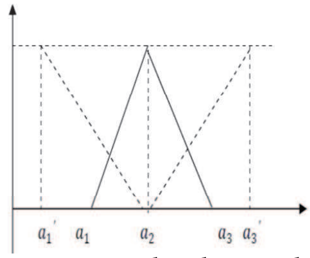

Figure-1: Membership and non-membership functions of TIFN.

Multiplication $\tilde{A}^I \otimes \tilde{B}^I=(l_1,l_2,l_3;l_1',l_2,l_3')$,

where

$l_1=\min\{a_1b_1,a_1b_3,a_3b_1,a_3b_3\}, l_1'=\min\{a_1'b_1',a_1'b_3',a_3'b_1',a_3'b_3'\}$

$l_3=\max\{a_1b_1,a_1b_3,a_3b_1,a_3b_3\}, l_3'=\max\{a_1'b_1',a_1'b_3',a_3'b_1',a_3'b_3'\}$

$l_2=a_2b_2$.

Scalar multiplication

1. $k\tilde{A}^I=(ka_1,ka_2,ka_3;ka_1',ka_2,ka_3'):k>0$
2. $k\tilde{A}^I=(ka_3,ka_2,ka_1;ka_3',ka_2,ka_1'):k<0.$

3. Ranking of TIFNs

3.1 Score function and Accuracy function of a TIFN

Let $\tilde{A}^I=(a_1,a_2,a_3;a_1',a_2,a_3')$ be a TIFN. The score function for the membership function $\mu_{\tilde{A}^I}$ is denoted by $S(\mu_{\tilde{A}^I})$ and is defined by $S(\mu_{\tilde{A}^I})=\dfrac{a_1+2a_2+a_3}{4}$. The score function for the non-membership function $\nu_{\tilde{A}^I}$ is denoted by $S(\nu_{\tilde{A}^I})$ and is defined by $S(\nu_{\tilde{A}^I})=\dfrac{a_1'+2a_2+a_3'}{4}$.

The accuracy function of $\tilde{A}^I$ is denoted by $f(\tilde{A}^I)$ and defined by

$$f(\tilde{A}^I)=\frac{S(\mu_{\tilde{A}^I})+S(\nu_{\tilde{A}^I})}{2}=\frac{(a_1+2a_2+a_3)+(a_1'+2a_2+a_3')}{8}.$$

Theorem: The accuracy function $f: IF(R) \to R$ is a linear function.

Proof: Let $\tilde{A}^I=(a_1,a_2,a_3;a_1',a_2,a_3')$ and $\tilde{B}^I=(b_1,b_2,b_3;b_1',b_2,b_3')$. Then for $k>0$, we have

$$f(k\tilde{A}^I+\tilde{B}^I)=f((ka_1,ka_2,ka_3;ka_1',ka_2,ka_3')+(b_1,b_2,b_3;b_1',b_2,b_3'))$$

$$=f(ka_1+b_1,ka_2+b_2,ka_3+b_3;ka_1'+b_1',ka_2+b_2,ka_3'+b_3')$$

$$=((ka_1+b_1+2(ka_2+b_2)+ka_3+b_3)+(ka_1'+b_1'+2(ka_2+b_2)+ka_3'+b_3'))/8$$

$$= \frac{k((a_1+2a_2+a_3)+(a_1^{'}+2a_2+a_3))}{8} + \frac{(b_1+2b_2+b_3)+(b_1^{'}+2b_2+b_3^{'})}{8} = kf(\tilde{A}^I)+f(\tilde{B}^I).$$

Similarly, it can be proved for $k<0$.

3.2 Ordering of TIFNs using accuracy function

Let $\tilde{A}^I=(a_1,a_2,a_3;a_1',a_2,a_3')$ and $\tilde{B}^I=(b_1,b_2,b_3;b_1',b_2,b_3')$ be two TIFNs. Then

a) $f(\tilde{A}^I)\ge f(\tilde{B}^I) \Rightarrow \tilde{A}^I \ge \tilde{B}^I$

b) $f(\tilde{A}^I)\le f(\tilde{B}^I) \Rightarrow \tilde{A}^I \le \tilde{B}^I$

c) $f(\tilde{A}^I)= f(\tilde{B}^I) \Rightarrow \tilde{A}^I = \tilde{B}^I$

d) $Min(\tilde{A}^I,\tilde{B}^I)=\tilde{A}^I \; if \; \tilde{A}^I \le \tilde{B}^I \; or \; \tilde{B}^I \ge \tilde{A}^I$

e) $Max(\tilde{A}^I,\tilde{B}^I)=\tilde{A}^I \; if \; \tilde{A}^I \ge \tilde{B}^I \; or \; \tilde{B}^I \le \tilde{A}^I.$

4. **Multiobjective non-linear programming problem**

The problem to optimize multiple conflicting non linear objective functions simultaneously under given constraints is termed as multi objective non-linear programming problem and can be formulated as the following optimization problem.

$$Z=Min(f_1(x),f_2(x),\dots,f_K(x))$$
$$s.t\; x\in X=\{x\in R^n: g_j(x)\le 0, j=1,2,\dots,m_1,$$
$$g_j(x)\ge 0, j=m_1+1,m_1+2,\dots,m_2, \qquad (1)$$
$$g_j(x)=0, j=m_2+1,m_2+2,\dots,m\}.$$

where $f_1(x),f_2(x),\dots,f_K(x)$ are k distinct non linear objective functions of the decision variable and X is the feasible set.

Intuitionistic fuzzy multi-objective nonlinear programming problem (IFMONLPP) can be formulated as below:

$$Z=Min(\tilde{f}_1^I(x),\tilde{f}_2^I(x),\dots,\tilde{f}_K^I(x))$$
$$s.t\; x\in X=\{x\in R^n: \tilde{g}_j^I(x)\le \tilde{b}_j^I, j=1,2,\dots,m_1,$$
$$\tilde{g}_j^I(x)\ge \tilde{b}_j^I, j=m_1+1,m_1+2,\dots,m_2, \qquad (2)$$
$$\tilde{g}_j^I(x)= \tilde{b}_j^I, j=m_2+1,m_2+2,\dots,m\}.$$

where

$$\tilde{f}_i^I(x)=\sum_{k=1}^{K_i}\tilde{c}_{ik}^I\prod_{l=1}^{n}x_l^{\alpha_l}, i=1,2,\dots,K \text{ and}$$
$$\tilde{g}_j^I(x)=\sum_{k=1}^{K_j}\tilde{a}_{jk}^I\prod_{l=1}^{n}x_l^{\beta_l}, j=1,2,\dots,m.$$

Here $\tilde{c}_{ik}^I$ and $\tilde{a}_{jk}^I$ are TIFNs.

Definition 4.1 x^* is said to be a complete optimal solution for (1) if $f_i(x^*)\le f_i(x), \forall x\in X$, $i=1,2,\dots,K$.

Definition 4.2 x^* is said to be optimal solution to the IFMONLP problem (2) if there does not exist another $x\in X$ such $\tilde{f}_i^I(x)\le \tilde{f}_i^I(x^*)$, $i=1,2,\dots,K$.

Using the linear accuracy function problem (2) can be transformed to following crisp MONLPP.

$$\text{Min } f_i(x)=\sum_{k-1}^{K_i} f(\tilde{c}_{ik}^I)\prod_{l-1}^{n}x_l^{\alpha_l}, i=1,2,\dots,K$$
$$s.t \sum_{k-1}^{K_j} f(\tilde{a}_{jk}^I)\prod_{l-1}^{n}x_l^{\beta_l}\le f(\tilde{b}_j^I), j=1,2,\dots,m_1, \qquad (3)$$
$$\sum_{k-1}^{K_j} f(\tilde{a}_{jk}^I)\prod_{l-1}^{n}x_l^{\beta_l}\ge f(\tilde{b}_j^I), j=m_1+1,m_1+2,\dots,m_2,$$
$$\sum_{k-1}^{K_j} f(\tilde{a}_{jk}^I)\prod_{l-1}^{n}x_l^{\beta_l}= f(\tilde{b}_j^I), j=m_2+1,m_2+2,\dots,m.$$
$$x=(x_1,x_2,\dots,x_n)\ge 0.$$

Problem in (3) transformed to a crisp MONLPP as follow:

$$\text{Min } f_i(x)=\sum_{k=1}^{K_i} c_{ik}^{'}\prod_{l=1}^{n}x_l^{\alpha_l}, i=1,2,\dots,K$$
$$s.t \sum_{k=1}^{K_j} a_{jk}^{'}\prod_{l=1}^{n}x_l^{\beta_l}\le b_j^{'}, j=1,2,\dots,m_1,$$
$$\sum_{k=1}^{K_j} a_{jk}^{'}\prod_{l=1}^{n}x_l^{\beta_l}\ge b_j^{'}, j=m_1+1,m_1+2,\dots,m_2, \qquad (4)$$
$$\sum_{k=1}^{K_j} a_{jk}^{'}\prod_{l=1}^{n}x_l^{\beta_l}= b_j^{'}, j=m_2+1,m_2+2,\dots,m,$$
$$x=(x_1,x_2,\dots,x_n)\ge 0,$$

where $f(\tilde{c}_{ik}^I)=c_{ik}^{'}, f(\tilde{a}_{ik}^I)=a_{ik}^{'}$ and $f(\tilde{b}_j^I)=b_j^{'}$.

Now we have to solve a crisp MONLPP for this we have developed an algorithm which is summarized in following steps.

Algorithm

Step-1 Solve the multi objective nonlinear programming problem by considering one objective function at a time and ignoring all others. Repeat the process K times for K different objective functions. Let the solutions obtained are $X_1,X_2,\dots,X_K$ respectively.

	X_1	X_2	$\cdots$	X_K
f_1	$f_1(X_1)$	$f_1(X_2)$	$\cdots$	$f_1(X_K)$
f_2	$f_2(X_1)$	$f_2(X_2)$	$\cdots$	$f_2(X_K)$
$\vdots$	$\vdots$	$\vdots$		$\vdots$
f_K	$f_K(X_1)$	$f_K(X_2)$	$\cdots$	$f_K(X_K)$

Step-2 Find the values of each objective at all solutions obtained in Step-1. Form the pay-off matrix of order $K \times K$ as under:Find the lower bound L_i and upper U_i for each objective $f_i,\ i = 1, 2, ..., K$ from the pay-off matrix obtained in Step-2.

Step-3 Define the membership function μ_{f_i} for i^{th} objective f_i as under-

$$\mu_{f_i}(x) = \begin{cases} 1, \ f_i(x) < L_i, \\ \dfrac{U_i - f_i(x)}{U_i - L_i}, \ L_i \le f_i(x) \le U_i, \\ 0, f_i(x) > U_i. \end{cases}$$

Step-4 Using the above membership function the problem can be described as how to make a reasonable plan so that the degree of satisfaction for the decision maker is higher. That is the value of membership functions are to be maximized. Thus problem in (4) can be transformed to the following single objective NLPP.

Let $\lambda = \min\{\mu_{f_i}\}, i = 1, 2, ..., K$ then the problem transformed to

$$\begin{aligned} &\text{Max } \lambda \\ &\text{s.t. } \ \mathrm{U}_i - \sum_{k=1}^{K_i} c'_{ik} \prod_{l=1}^{n} x_l^{\alpha_l} \ge \lambda(U_i - L_i) \ , i = 1, 2, ..., K \\ &\sum_{k=1}^{K_j} a'_{jk} \prod_{l=1}^{n} x_l^{\beta_l} \le b'_j \, , j = 1, 2, ..., m_1, \qquad (5) \\ &\sum_{k=1}^{K_j} a'_{jk} \prod_{l=1}^{n} x_l^{\beta_l} \ge b'_j \, , j = m_1 + 1, m_1 + 2, ..., m_2, \\ &\sum_{k=1}^{K_j} a'_{jk} \prod_{l=1}^{n} x_l^{\beta_l} = b'_j \, , j = m_2 + 1, m_2 + 2, ..., m, \\ &x = (x_1, x_2, ..., x_n) \ge 0. \end{aligned}$$

Problem in (5) is a single objective NLPP which can be solved easily by a suitable crisp NLPP method.

5. Numerical Example

Let us consider the following IFMONLPP:

$$\begin{aligned} &\text{Min } [\tilde{2}^I x_1^2 + \tilde{3}^I x_2 x_3, \tilde{1}^I x_1 x_2 + \tilde{4}^I x_2 x_3 + x_1, \tilde{3}^I x_1 x_3 + \tilde{2}^I x_1 x_2^2] \\ &\text{s.t.} \quad \tilde{3}^I x_1^2 + \tilde{4}^I x_2^2 \le \tilde{15}^I, \\ &\qquad \tilde{1}^I x_1 x_2 + \tilde{2}^I x_3 \ge \tilde{7}^I, \\ &\qquad \tilde{1}^I x_2 x_3 \le \tilde{7}^I, \\ &\qquad x_1, x_2, x_3 \ge 0. \end{aligned}$$

where $\tilde{2}^I = (1.5, 2, 2.3; 1.5, 2, 2.5), \tilde{3}^I = (2.8, 3, 3.2; 2.5, 3, 3.5),$
$\tilde{1}^I = (0.8, 1, 1.4; 0.7, 1, 1.4), \tilde{4}^I = (3.6, 4, 4.5; 3.5, 4, 4.6),$
$\tilde{5}^I = (4.3, 5, 5.3; 4, 5, 5.5), \tilde{15}^I = (14, 15, 16; 13.5, 15, 16.5),$
$\tilde{7}^I = (6.5, 7, 7.5; 6, 7, 8), \tilde{16}^I = (15, 16, 17; 14.5, 16, 17.5).$

Using the accuracy functional value this problem transformed to the following crisp MONLPP

$$\begin{aligned} &\text{Min}[1.975x_1^2 + 3x_2x_3, 1.0375x_1x_2 + 4.025x_2x_3 + x_1, 3x_1x_3 + 1.975x_1x_2^2] \\ &\text{s.t.} \quad 3x_1^2 + 4.025x_2^2 \le 15, \\ &\qquad 1.0375x_1x_2 + 1.975x_3 \ge 7, \\ &\qquad 1.0375x_2x_3 \le 7, \\ &\qquad x_1, x_2, x_3 \ge 0. \end{aligned}$$

On solving as a single objective NLPP we have
$X_1 = (0, 0, 3.5443), X_2 = (1.8001, 1.1451, 5.8915),$
$X_3 = (0, 3.5443, 0).$
$L_1 = 0, U_1 = 26.6388, L_2 = 0, U_2 = 31.0955,$
$L_3 = 0, U_3 = 36.4776.$

Using the membership function the single objective equivalent problem is as under.

$$\begin{aligned} &\text{Max } \lambda \\ &\text{s.t.} \quad 26.6388 - 1.975x_1^2 - 3x_2x_3 \ge 26.6388\lambda, \\ &\qquad 31.0955 - 1.0375x_1x_2 - 4.025x_2x_3 - x_1 \ge 31.0955\lambda, \\ &\qquad 36.4776 - 3x_1x_3 - 1.975x_1x_2^2 \ge 36.4776\lambda \\ &\qquad 3x_1^2 + 4.025x_2^2 \le 15, \\ &\qquad 1.0375x_1x_2 + 1.975x_3 \ge 7, \\ &\qquad 1.0375x_2x_3 \le 7, \\ &\qquad x_1, x_2, x_3 \ge 0. \end{aligned}$$

Solving by LINGO, $x_1 = 0,$ $x_2 = 0,$ $x_3 = 3.5443$ with satisfaction level $\lambda = 1$ and $\tilde{f}_1 = \tilde{f}_2 = \tilde{f}_3 = 0.$

Remark: Here $\lambda = 1$. So, the decision maker is fully satisfied for some other problem λ can be lesser than 1. In that case the value of objectives may not be reached exactly as the desired value.

6. Conclusion

In this paper we have formulated an IFMONLPP with parameters as TIFNs. Using a linear ranking function called accuracy function we converted the intuitionistic fuzzy problem into crisp MONLPP. Then we developed an algorithm to solve it. Our approach is very comfortable for solving the formulated problem. In future this approach may be very useful in solving MONLPP in intuitionistic fuzzy environment. Thus it can be applied in many real world problems such as in manufacturing, scheduling, planning etc.

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