ANALYSIS OF A BEDDINGTON-DEANGELIS TYPE PREY-PREDATOR MODEL

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Abstract: In this paper, a prey-predator model with reserved area is proposed and analyzed. We assume that whole habitat is divided into two regions, namely reserved (predator prohibited region) and unreserved (free) zone. The feeding rate of consumers (predators) per consumer (i.e. functional response) is considered Beddington-DeAngelis type. Role of reserved region and degree of mutual interference among predators in the dynamics of system is investigated. Dynamics of the system is discussed mainly from the point of view permanence and stability. At the end, we have performed extensive numerical simulations to illustrate our theoretical results. Numerical simulations show the existence of periodic solutions.

Keywords*: Functional Response; Limit Cycle; Periodic Solution.*

Introduction: The dynamics of biological systems have been studied in various point of view from last many decades and prey-predator model is one of the most popular area in biological systems. The dynamics of Lotka-Volterra model and it's several modifications have received a lot of attention from last forty years (see e.g. [5], [6], [11]). Persistence (the survival of species for a longer time) becomes a fundamental concern of ecology and has been discussed in a great deal. Literature shows that in the recent years it has received a lot of attention (see e.g., [2], [4], [6], [11], [15], [16]). The term reserve zones/refuges has received considerable attention in the dynamics of prey-predator model. One can see in detail in [12]. Most of researches [6], [8], [18], show that refugia has stabilizing effect on prey-predator model. In particular prey-predator resource model is explained with optimum harvesting policy in [12], [19]. See [6], [8] and references therein for the stability of prey-predator model incorporating prey refuge. [19], [20] discuss stability analysis of three dimensional predator-prey model incorporating prey cover.

Functional response [18] determines stability and bifurcation dynamics of the systems. Traditional functional response, $g(X,Y) = aX$, where $a>0$, is prey dependent and based on principle of mass action (i.e. when feeding rate is proportional to the product of prey and predator populations). But this functional response has a conceptual shortcoming in case of superabundant supply of food, predators will feed at maximum rate [14] per individual predator and hence further increase in food supply will not be able to increase the feeding rate further. In [7] this modification was given in the form, $g(X, Y) = bX/(w+X)$, which is non-linear as well as bounded (Holling type II function or Michaelis-Menten function). Situation where an increase of consumer (predator) density implies the decrease in feeding rate of predator is due to mutual interference among individual of predators [14]. This was the key factor for the modification of Holing type II functional response in form of a predator dependent functional response, Beddington-DeAngelis functional response given as

$$g(X,Y) = \frac{bX}{\alpha + Y + \beta X} \qquad (1)$$

2. Mathematical Model: It is assumed that whole habitat (where prey and predator species are living together) is divided into two zones, namely reserved (predator prohibited area) and free (unreserved). It is also assumed that the predator species are not allowed to enter the reserved zone while the prey species are permissible for mixing from free zone to reserved zone and vice-versa. It is also considered that in the absence of predator population prey species follow logistic growth rate while predators are subjected to natural mortality. Further, we assumed a refuge protecting mX of prey , where m in $[0, 1)$, is constant, and hence, $(1-m)X$, is only prey available to predator. Under the above assumptions prey-predator system incorporating Holling type II feeding scheme, proposed and analyzed in [6] is:

$$\begin{aligned} \frac{dX}{dt} &= X(1-\frac{X}{K}) - \frac{B(1-m)Y}{C+F(1-m)X} \\ \frac{dY}{dt} &= Y(-D + \frac{EB(1-m)X}{C+F(1-m)X}), \end{aligned} \qquad (2)$$

where X and Y denote prey and predator densities at time t respectively. All the parameters, B, E, C, D, E, K, F, in the model system (2), assumes only positive values and will be considered as constants throughout our discussion. R and K represent intrinsic rate of growth (per capita rate of change) and environmental carrying capacity for prey respectively. D stands for natural mortality rate for predator population. E and F stand for the conversion factor denoting the newly born predators for each captured prey and effect of handling time for predators. The quantities B and C represent maximum number of prey that can be eaten by each predator in unit time (maximum predator attack rate) and the prey density where the attack rate is half saturated (i.e. half saturation constant) respectively. Introducing both predator dependent (Bedington-DeAngelis) functional response as well as prey cover protecting, mX of the prey, the model system to be investigated is:

$$\frac{dX}{dT} = X(R(1-\frac{X}{K}) - \frac{B(1-m)Y}{C+F(1-m)X+GY})$$
$$\frac{dY}{dt} = Y(-D+\frac{EB(1-m)X}{C+F(1-m)X+GY}), \quad (3)$$

where, initial conditions are $X(0)>0$, $Y(0)>0$ which are biologically meaningful. To reduce the complexity arising in dynamical analysis of the model system (3), the non-dimensional form of (3) with the following set of variables and parameters $t = RT$, $x = X/K$, $aY = GA/C$, $d = D/R$, $e = EBK/RC$, is:

$$\frac{dx}{dt} = x((1-x) - \frac{a(1-m)y}{1+b(1-m)x+cy})$$
$$\frac{dy}{dt} = y(-d+\frac{e(1-m)}{1+b(1-m)x+cy}). \quad (4)$$

Boundedness and existence of equilibria

Lemma III.I *The positive quadrant (R^2_+) is invariant for the model system (4)*.

Proof: (Hint) One can easily show the positivity of solutions of the model system (eq4) by integrating the model system (4) and putting the initial condition. There are various ways to show positivity of solutions. One can also see [17].

Lemma III.2 *Show that all the solutions of the model system (4) that starts in (R^2_+) are confined to the region* $D = \{(x,y)\in R^2_+ : x(t)\leq 1,\ 0\leq \eta(t)\leq 1+\frac{1}{4d}\}$ *as* $t\longrightarrow\infty$ *for all positive initial value* $(x(0), y(0))\in R^2_+$ *where* $\eta(t) = x(t)+\frac{a}{e}y(t)$.

Proof: Since all the species and parameters associated with model systems (4) are positive and $m\in[0,1)$, hence from the first equation of the model system (eq4), we have

$\frac{dx}{dt} \leq x(1-x)$. From Lemma III.1, any solution of the system (4) must satisfies $x(t)\leq 1$ for all $t \geq 0$.

Define a function, $\eta = x + \frac{a}{e}y$.

Hence, $\frac{d\eta}{dt} + d\eta \leq d + \frac{1}{4}$.

Using comparison lemma [17] for sufficiently large time T *such that* $t\geq T \geq 0$, we have

$$\eta(t) \leq 1+\frac{1}{4d} - (1+\frac{1}{4d} - \eta(T))e^{-d(t-T)}.$$

Taking $T = 0$ and letting $t\longrightarrow\infty$, *since* $x(0)>0, y(0)>0$, hence we find

$$\eta(t) \leq 1+\frac{1}{4d}.$$

We observe that all the solutions of the model system (4) initiating in R^2_+ eventually lies in the region S defined by

$$S = \{(x,y)\in R^2_+ :\ \eta(t)\leq 1+\frac{1}{4d}+\delta\}.$$

In other word we can say that all the trajectories of the system (4) initiating from any point in R^2_+ are ultimately lie in the region S. Hence the flow/dynamical systems associated with system (4) and defined on R^2_+ is dissipative.

The model system (4) possesses at least three equilibrium solutions namely, trivial equilibrium, $E_0 = (0,0)$, predator free equilibrium, $E_1 = (1,0)$, and coexistence (interior equilibrium point), $E_* = (x^*,y^*)$. The existence and uniqueness of first two trivial equilibrium are obvious. We discuss only existence and uniqueness of the interior equilibrium solution. There exist an interior equilibrium solution, $E_* = (x^*,y^*)$ if and only if

$$m < 1 - \frac{(b+1)d}{e}. \quad (5)$$

One can see all the possible configurations of both the isoclines with respect to m,the prey reserve in the Figures, 1, 2, 3 and 4. These Figures illustrate that slope of predator isocline decrease while the prey isocline becomes convex from concave as the parameter m, (the prey refuge) increases. As we increase value of prey reserve (m) further than a threshold value (i.e. $m \geq 1 - \frac{(b+1)d}{e}$), we loose the positive equilibria of the system (4). See Figure 4 for m = 0.70.

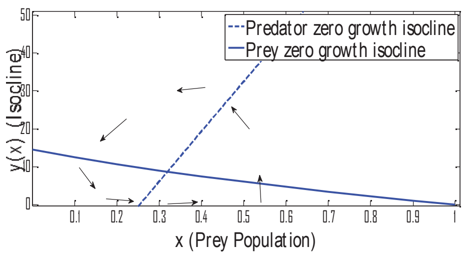

Fig.1 For m = 0.01

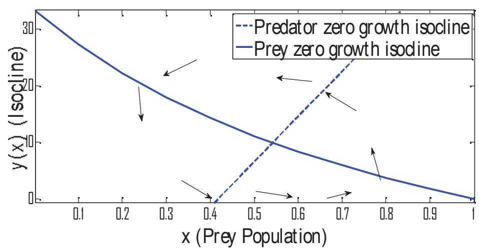

Fig.2. For m = 0.40

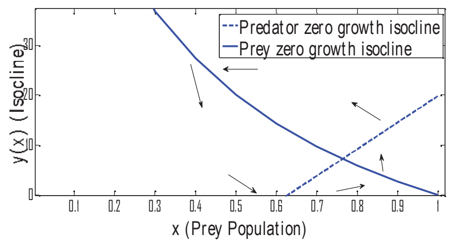

Fig. 3. For m = 0.60

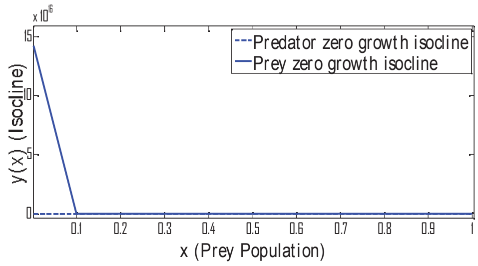

Fig.4. For m = 0.70

Iv. Stability Analysis And Permanence : The jacobian matrix corresponding to the linear system of the system (4) around the positive equilibrium E_i after simplification using the equilibrium equations for (4) is

$$J_{E_i} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

where, $a_{11} = -x^* + (1-x^*)\dfrac{bd}{e}$,

$$a_{12} = \frac{d}{e}\left(-a + \frac{c}{(1-m)}(1-x^*)\right),$$

$$a_{21} = \frac{(1-x^*)}{a}(e-bd),\quad a_{22} = -\frac{cd}{a(1-m)}(1-x^*).$$

We observe that J_{E_0}, the jacobian of the model system (4) at equilibrium point E_0, has eigenvalues 1, and $-d$. As $d>0$, therefore, the system (4) is always unstable around E_0, which is, in fact, a saddle point and whose stable manifold is $y-axis$ while unstable subspace as sell as manifold is $x-axis$. One can notice that the interference coefficient c and the prey reserve m play no role in the stability of trivial equilibrium E_0.

From the variational matrix J_{E_1}, it is observed that it has a negative eigenvalue -1 and other eigenvalue is positive if $-d + \dfrac{e(1-m)}{1+b(1-m)} < 0$. Hence, the system (4) is always unstable around E_1. which is, in fact, a saddle point and whose stable manifold is $x-axis$ while the unstable space as well as manifold is $y-axis$ for $m \ge 1 - \dfrac{d}{e-bd}$.

After computing the variational matrix at E_*, one can show that interior equilibrium point E_* for the model system (4) is locally asymptotically stable if $x^* > \left(\dfrac{bd}{e} - \dfrac{cd}{a(1-m)}(1-x^*)\right)$,

where, $m > 1 + \dfrac{ec}{a(e-bd)}(2x^* - 1)$.

Theorem IV.1. *The model system (4) is permanent if and only if the condition given in (5) holds.*

Proof: Obviously the system (4) is dissipative (see Lemma III.2). Let us take Y_0 to be the first quadrant (nonnegative cone) and ∂Y_0 denotes it's boundary. Obviously $\omega(\partial Y_0)$ consists of only two axial equilibria namely E_0 and E_1. The unstable manifold of E_0 is the $x-axis$ while the stable manifold for E_1 is $x-axis$ under certain conditions of section IV. Clearly, $\omega(\partial Y_0)$ is isolated. It is obvious that all the trajectories that starts on $y-axis$ approaches to

E_0 except E_1. Further we show that stable manifold of E_0 as well as that of E_1 can not intersect the interior of Y_0. i.e. $W^s(E_i)\bigcap Y_0=\phi$. For this, on contrary suppose that $W^s(E_i)\bigcap Y_0=\phi$ is not true. Then, naturally there exist some $x\in W^s(E_i)\bigcap Y_0$. Now, denote $o(x)$ as the orbit through x and $\Omega(x)$ the limit set of the orbit through x. Since under our condition we have seen that E_0 and E_1 are saddle points. Hence by Butler-Mc Ghee lemma (see [9]) there exist a point $q\in\Omega(x)\bigcap W^s(E_i)$. Since $q\in\Omega(x)$, hence $o(q)$ lies in $\Omega(x)$. Further, since, $W^s(E_i)$ for $i=1,2$ are $y-axis$ and $x-axis$ respectively, we conclude that $o(q)$ is unbounded which is a contradiction. Therefore, our assumption that $W^s(E_i)\bigcap Y_0\neq\phi$ is wrong. Thus, taking all the points into account discussed above, permanence of the model system (4) is established.

V. Numerical Simulations:

We simulate with our model system (3). Let *R = 9.5, K = 99, B = 0.6, C = 1, D = 0.09, F=0.02, E=0.15, G=0.01,* then the system (3) becomes

$$\frac{dX}{dT}=X(9.5(1-\frac{X}{99})-\frac{0.6(1-m)Y}{1+0.02(1-m)X+0.01Y}$$

$$\frac{dY}{dt}=Y(-0.09+\frac{0.090(1-m)X}{1+0.02(1-m)X+0.01Y}),$$

The parameters satisfy the permanence condition $m<1-\frac{(b+1)d}{e}$, given by (5), and a bound on prey reserve *m*, for the permanence of the model system (4) is $m<m_1$ where $m_1=1-\frac{(b+1)d}{e}=0.9390.$

Fig 5. shows that the distinct solutions initiating either in the interior of limit cycle or outside the limit cycle approach to the limit cycle. In fact this shows an existence of a stable limit cycle. In this case the interior equilibrium solution (E_*) becomes unstable.

In Fig 6, the existence of a limit cycle is shown for m = 0.25. On the other hand Fig. 7 shows that the solution starting from any point of the phase plane ultimately converges to the interior equilibrium solution (7.801, 51.56). In this case the limit cycle disappears and the interior equilibrium solution becomes stable. Both the population converges to their equilibrium $x^*=7.018, y^*=51.56.$ (see Fig 8). When the value of prey reserve *m,* crosses a threshold value, the predator species extinct. Fig. 9 shows that the equilibrium solution (0.9897, 0.0000) is a global attractor. **VII. Conclusion :** At many places, many effort have been put to manage subpopulation of vulnerable cheetas (Acinonyx Jubatus) in small reserves. Like vulnerable cheetas there are many other species those are either driven to extinction or at the verge of extinction, due to overexploitation, over predation environmental pollution, etc. This motivate us for the creation of reserve zones/refuges and to consider prey-predator models incorporating prey refuge with Beddington-DeAngelis type functional response. In the analysis, we found that the prey reserve *m* plays no role in determining the stability of the trivial equilibrium solution E_0. Interference among predators, *c* also plays no role in determining the local stability of the system (4) in the small neighbourhood of E_0 and E_1. The condition $-d+\frac{e(1-m)}{1+b(1-m)}$, for the asymptotic stability of E_1, shows that as $m\to 1$, the E_1 becomes stable for all values of other parameters. Thus m has positive effect for the stability of E_1. A small computation on the stability condition of E_* and numerical simulations show that the interior equilibrium solution E_* remains unstable and the system exhibit a stable limit cycle up to a threshold value of m and when m crosses threshold value, E_* becomes stable and the stable limit cycle disappear. In other words, we can say that addition of small refuge to the model system (4) does not alter the stable oscillatory behaviour of the system (4). On the other hand the addition of large value of prey refuge to the system (4) changes the system's oscillatory behavior into stable equilibrium solution. In brief we can say that prey refuge has a stabilizing effect into the prey predator model with Beddington DeAngelis functional response.

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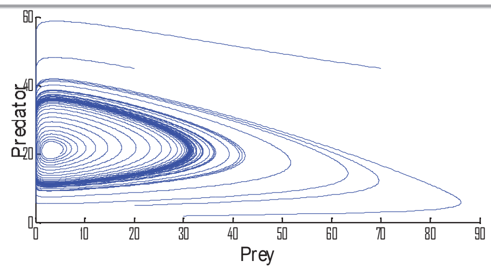

Fig 5. Phase portrait for m = 0.08

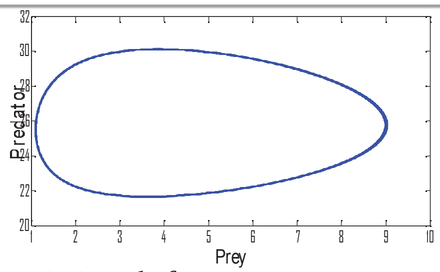

Fig. 6. Limit cycle for m = 0.25

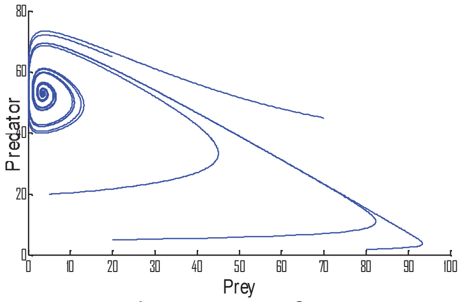

Fig. 7. Phase portrait for m=0.55

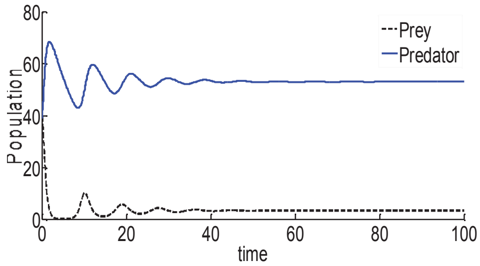

Fig. 8. Solution curves for m = 0.48

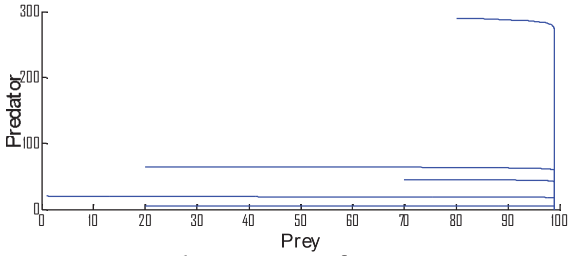

Fig. 9. Phase portrait for m = 0.978

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