THE ANALYTIC PROPERTIES OF STRONG INVARIANT APPROXIMATION PROPERTY

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Abstract: A countable exact discrete group G has the strong invariant approximation property (SIAP) if and only if for any Hilbert space H and closed subspace $S \subseteq H$

$$C_u^*(G,S)^G = C_\lambda^*(G) \otimes S$$

for any Hilbert space H and closed subspace $S \subseteq H$. We shall use results of Haagroup and Kraus on the approximation property (AP) to investigate some permanence properties of the SIAP for discrete groups. This can be done most efficiently for exact groups. In this paper we describe that the analytic properties of the SIAP property pass to free direct products.

Keywords: *Strong Invariant Approximation Property, Uniform Roe algebras, Invariant Approximation Property.*

Introduction: This purpose of this paper is to provide an illustration of an interesting and nontrivial interaction between analytic and geometric properties of a group. We provide approximation property of operator algebras associated with discrete groups. There are various notations of finite dimensional approximation properties for C^*–algebras and more generally operator algebras. Some of these (approximation properties) notations will be defined in this paper, the reader is referred to [2], [9], [8] and [13] for these a beautiful concept: Haagerup discovery that the reduced C^* – algebra $\mathbf{F}^n$ has the metric approximation property, Higson and Kasparov's resolution of the Baum-connes conjecture for the Haagerup groups. Roe considered the discrete group of the reduced group C^* – algebra of $C^*{}^r(G)$ is the fixed point algebra $\{Ad\rho(t) : t \in G\}$ acting on the uniform Roe algebra $C^*{}^u(G)$ [13]. A discrete group G has natural coarse structure which allows us to define the uniform Roe algebra, $C^*{}^u(G)$ [13]. We say that the uniform Roe algebra, $C^*{}^u(G)$, is the C^*–algebra completion of the algebra of bounded operators on $\ell^2(X)$ which have finite propagation. The reduced C^*–algebra $C^*_r(G)$ is naturally contained in $C^*_u(G)$. According to Roe [13] G has the invariant approximation property (IAP) if

$$C^*_\lambda(G) = C^*_u(G)^G$$

According to [16] G has the strong invariant approximation property (SIAP) if and only if

$$C^*_u(G,S)^G = C^*_\lambda(G) \otimes S$$

for any Hilbert space H and closed subspace $S \subseteq$ H . We give a general exposition of invariant approximation property (IAP), which was initiated by Roe [13]. Our interest in these properties comes from a link to the strong invariant approximation property (SIAP) of Zacharias, which implies the IAP. We shall use results of Haagerup and Kraus [6] on the AP to investigate some permanence properties of the IAP and the SIAP for discrete groups. This can be done most efficiently for exact groups. In this paper we describe that the analytic properties of the SIAP property pass to free products (see Proposition 3.12).

PRELIMINARIES: The reduced C^*–algebra $C^*_\lambda(G)$ of a group G (which we shall assume to be discrete) arises from study of the left regular representation λ of the group ring $\mathbb{C}[G]$ on the Hilbert space of square-summable functions on the group. The reduced C^*–algebra $C^*_p(G)$ of a group (which we shall assume to be discrete) arises from the study of the right regular representation ρ of the group ring $\mathbb{C}[G]$ on the Hilbert space of square-summable functions on the group.

A discrete group G has a natural coarse structure which allows us to define the uniform Roe algebra$C^*_u(G)$. A group G can be equipped with either the left or right-invariant of the metric. A choice of one of the determines whether $C^*_\lambda(G)$ or $C^*_p(G)$ is a sub-algebra of the uniform Roe algebra $C^*_u(G)$ of G.

We now define the invariant approximation property (IAP).

Definition 2.9. [13] We say that G has the *invariant approximation property (IAP)* if

$$C^*_\lambda(G) = C^*_u(G)^G$$

Roe [13] is proved in that amenable groups and finitely generated free groups have the IAP. Haagerup and Kraus [7] defined and studied the AP properties (AP) for a group. We begin with a some definition of Haagerup and Kraus [7].

We have the following definition of weak amenability,CBAP,operator approximation property (OAP) and approximation property (AP).

Definition 2.10. [1] A C^*–*algebra* A is regular if and only if it has the following *completely positive approximation property* (CIAP): The identity map on A can be approximated in the point norm topology by finite rank completely positive contractions.

A C^*–algebra A has the metric approximation property (MAP) of Grothendieck if and only if the identity map on A can be approximated in the point-norm topology by a net of finite rank contractions.
Comparing the definitions we see that CPAP implies MAP (see for example [1]). Lance [11] has shown that Γ is amenable if and only if it's reduced C^*–algebra A has the CPAP which is equivalent to $C^*_r(\Gamma)$ being nuclear. Completely positive maps are in particular completely bounded, which suggest the following weakening of the CPAP.

Definition 2.11. [1] A C^*–algebra A is said to have the *completely bounded approximated* (CBAP) if there is a positive number C such that the identity map on A can be approximated in the point norm topology by a net $\{\phi_\alpha\}$ of finite rank completely bounded maps whose completely bounded norms are bounded by C.

Definition 2.12 [1] An approximate identify on G is a sequence (ϕ_n) of finitely supported functions such that ϕ_n uniformly converge to constant function 1. We say that discrete G is *weakly amenable* if there is an approximate identity (ϕ_n) such that

$$C : \sup \left\| M\phi_n \right\|_{cb} < \infty.$$

We have the following important result by Haagerup [6]

Theorem 2.13. Let G be discrete group. *The following are equivalent:*

I. *G is weakly amenable,*

II. *$C^*_r(G)$ has the CBAP*

Definition 2.14. [7] We say that C^*–algebra, A has the *operator approximation property* (OAP) if there exists a net of finite-rank maps $: A \to A_\alpha$ such that $T_\alpha \to id_A$ in the stable point-norm topology.

The Fourier algebra

$A(G) \coloneqq \{f : f(t) = \langle \lambda(t)\zeta, \eta\rangle$ for some $\zeta, \eta \in \ell_2(G)\}$ is the space of all coefficient function of the left regular representation λ. Given $f \in A(G)$, its norm is given by $\|f\| = \inf\{\|\zeta\|\|\eta\| : f(t) = \langle \lambda(t)\zeta|\eta\rangle\}$.

With this norm, $A(G)$ is Banach algebra with the point-wise multiplication [7].

Definition 2.15. [7] A complex-valued function ϕ on G is a *multiplier* for $A(G)$ if the linear map

$$M_\phi(f) = \phi f$$

Sends $A(\Gamma)$ to $A(\Gamma)$. If the map M_ϕ is completely bounded on $A(G)$, we call ϕ a completely bounded multiplier of $A(G)$. The set of multipliers of $A(G)$ is denoted by $M_0A(G)$. If $\phi \in A(G)$ then ϕ is a bounded continuous function and M_ϕ is a bounded operator on the space $A(G)$.

Definition 2.16. [7] The discrete group G has the *approximation property* (AP) if there is a net $\{\phi_\alpha\}$ in $A(G)$ such that $M_{\phi_\alpha} \to id_{A(G)}$ in the stable point-norm topology on $A(G)$.

In this section we will give definition of the strong invariant approximation property. Let $S \subseteq B(H)$ be a closed subspace. Next, we define the set of fixed points of $C^*_u(G,S)^G$:

Definition 2.17. We define

$$C^*_u(G,S)^G = \left\{T \in C^*_u(G,S); Ad\left(\rho t \otimes id\right)T = T \right\}. \text{ for all } t \in G.$$

We now define Joachim Zacharias's IAP with coefficients (SIAP):

Definition 2.18. [16] We say that a discrete group G has the *strong invariant approximation property* (SIAP) if for any closed subspace S of the compact operators K (on $\ell^2(\mathbb{N})$). We have an isomorphism $C^*_u(G,S)^G = C^*_\lambda(G) \otimes S$ holds.

3.The Analytic Properties of Strong IAP: In this section, we show some of the stability properties of the strong invariant approximation property for discrete exact groups.

Theorem 3.1. [16] *For a discrete exact group G the following are equivalent.*

1. *G has the AP.*
2. *$C^*_r(G)$ has the OAP.*
3. *G has the SIAP (Zacharias's IAP with coefficients)*

Theorem 3.2. [16] *For a discrete exact group$G$. G has the SIAP if for any closed subspace $S \subseteq B(H)$ the equality*

$$C^*_u(G,S)^G = C^*_\lambda(G) \otimes S \text{ holds.}$$

Theorem 3.3. [16] *SIAP implies IAP for discrete exact groups.*

Proposition 3.6. [9] *Let G be a discrete group. Let*

$$1 \to H \to G \xrightarrow{\pi} G/H \to 1.$$

Let us assume that H is a normal subgroup in G, and that H and G/H have the $SIAP$, then G has $SIAP$.

Proposition 3.7[7].. *The semidirect product of two discrete groups with the AP has the AP.*

We propose a simple method to establish that the following group have approximation property (AP).

Example 3.9. [7] We have the following short exact sequence of groups

. $1 \to \mathbb{Z}^2 \to \mathbb{Z}^2 \rtimes SL(2,\mathbb{Z}) \to SL(2,\mathbb{Z}) \to 1$.

Indeed $\mathbb{Z}^2$ and $SL(2,\mathbb{Z})$ are weakly amenable groups [7]. But $\mathbb{Z}^2 \rtimes SL(2,\mathbb{Z})$ (where ρ is the standard action of $\mathbb{Z}^2 \rtimes SL(2,\mathbb{Z})$ on $\mathbb{Z}^2$) can be written as an amalgamated free product

$$\mathbb{Z}^2 \rtimes SL(2,\mathbb{Z}) = \mathbb{Z}^2 \rtimes \left(\mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_6\right)$$
$$= \mathbb{Z}^2 \rtimes \mathbb{Z}_4 *_{\mathbb{Z}^2 \rtimes \mathbb{Z}_2} \mathbb{Z}^2 \rtimes \mathbb{Z}_6,$$

Whose factors are amenable and also $\Lambda(\mathbb{Z}^2 \rtimes SL(2,\mathbb{Z})) = \infty$ [7]. So, the class of groups C^*–algebra that are weakly amenable is not closed under taking arbitrary amalgamated free product. The semidirect product of two discrete groups with the AP has the AP [7]. Thus, $\mathbb{Z}^2 \rtimes SL(2,\mathbb{Z})$ has AP [7]. By using Proposition 3.7. Therefore $\mathbb{Z}^2 \rtimes SL(2,\mathbb{Z})$ has AP. In other word: $\mathbb{Z}^2$ and $SL(2,\mathbb{Z})$ have CBAP. Then by using 2, $\mathbb{Z}^2$ and $SL(2,\mathbb{Z})$ have AP. We also note the following.

Proposition 3.10. *Let $\{G_i, i \in I\}$ be a family of amenable groups, and let H be an open compact subgroup of G_i for each $i \in I$. Then $G = *_H G_i$ has the strong invariant approximation property.*

Proof. Amalgamated products of amenable groups are weakly amenable [3]. Then G is weakly amenable. By using Theorem 2.13, $C^*_r(G)$ has the CBAP. But Zacharias show that CBAP implies AP impliesIAP. Then G has the invariant approximation property.

Next, we describe that, if two discrete exact groups have SIAP, then their free product has SIAP. We have the following important result [14]:

Proposition 3.11. *Let G, H be discrete exact groups, then $G * H$ is exact.*

The following proposition has a direct proof using Proposition 3.11.

Proposition 3.12. *Let G and H be discrete exact groups. If G, H have $SIAP$, then $G * H$ has $SIAP$.*

Proof: Let G and H be discrete exact groups. If G and H have IAP, by Theorem 3.1 then G and H have AP and also we have proved $G * H$ has AP. It has been shown [14] $G * H$ is exact (see Proposition 3.11). AP if and only if SIAP for discrete exact groups. Therefore $G * H$ has S IAP.

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