TWO PARAMETER ENTROPY OF UNCERTAIN VARIABLES

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Abstract: Uncertainty theory is a new branch of axiomatic mathematics for studying the subjective uncertainty. In uncertain theory, uncertain variable is a fundamental concept, which is used to represent imprecise quantities (unknown constants and unsharp concepts). Entropy of uncertain variable is an important concept in calculating uncertainty associated with imprecise quantities. This paper introduces the two parameter entropy of uncertain variable, and proves its several important theorems.

Keywords: Uncertain Distribution, Entropy of Uncertain Variable, Two Parameter Entropy.

Introduction: The concept of entropy was founded by Shannon [7] in 1948, which is a measurement of the degree of uncertainty of random variables. In 1972, De Luca and Termini [9] introduced the definition of fuzzy entropy by using Shannon function. Inspired by the Shannon entropy and fuzzy entropy, Liu [2] in 2009 proposed the concept of entropy of uncertain variable, where the entropy characterizes the uncertainty of uncertain variable resulting from information deficiency.
Wada and Suyari [11] proposed two parametric generalization of Shannon-Khinchin axioms and proved that the entropy function satisfying two parameter generalized Shannon Khinchin axioms is given by Eq.(1.1) ,

$$S_{\alpha,\beta}(p_1, p_2, \cdots p_n) = \sum_{i=1}^{n} \frac{p_i^{\alpha} - p_i^{\beta}}{C_{\alpha,\beta}} \qquad (1.1)$$

where $C_{\alpha,\beta}$ is a function of α and β satisfying certain conditions. One parametric Tsallis entropy [8] and Shannon entropy [7] recovered for specific values of α and β. Huge literature developed on generalization of entropy measures in last fifty years.
Uncertainty theory was founded by Liu [5] in 2007 and refined by Liu [6] in 2010, which is a branch of mathematics based on normality, monotonicity, self-duality, countable subadditivity, and product measure axioms. It is a effectively mathematical tool disposing of imprecise quantities in human systems. In recent years, Uncertainty theory was widely developed in many disciplines, such as uncertain process [1], uncertain calculus, uncertain differential equation [2], uncertain logic [14], uncertain inference [4], uncertain risk analysis [3], and uncertain statistics [6].
In order to provide a quantitative measurement of the degree of uncertainty in relation to an uncertain variable, Liu [2] proposed the definition of uncertain entropy resulting from information deficiency. Dai and Chen [12] investigated the properties of entropy of function of un-certain variables. The principle of maximum entropy for uncertain variables is introduced by Chen and Dai [12]. Besides, there is literature concerning the definition of entropy of uncertain variables, such as Dai [13], etc.
Inspired by two parametric probabilistic entropy, this paper introduces a new type two parameter entropy in the framework of uncertain theory and discusses its properties. Consequently, we generalize the entropy of uncertain variable. The rest of the paper is organized as follows. In Section 2, we recall some basic concepts and theorems of uncertain theory. In Section 3, the definition of two parameter entropy of uncertain variables is proposed. In addition, some examples of the two parameter entropy are illustrated. In Section 4, several properties of two parameter entropy are proved. At last, conclusion is presented.

Preliminaries: In this section, we will recall several basic concepts and theorems in the uncertain theory.
Let Γ be non-empty set and Ω is a $\sigma-$algebra over Γ. Each element $\Lambda \in \Omega$ is called an event. Uncertain measure M was introduced as a set function satisfying the following five axioms (Liu[5]):

Axiom 1. (Normality Axiom) M{Γ}=1 for universal set Γ.

Axiom2. (Monotonicity Axiom) M{Γ_1}≤ M{Γ_2} whenever $\Lambda_1 \subset \Lambda_2$.

Axiom 3. (Self-Duality Axiom) M{Λ}+ M{Λ^c} = 1 for any event Λ.

Axiom 4. (Countable Subadditivity Axiom) For every countable sequence of events $\{\Lambda_i\}$, we have

$$\mathrm{M}\{\bigcup_{i=1}^{\infty} \Lambda_i\} \le \sum_{i=1}^{\infty} \mathrm{M}\{\Lambda_i\}$$

Axiom 5. (Product Measure Axiom) Let Γ_k be non empty sets on which M_k are uncertain measures $k = 1, 2, \ldots, n$, respectively. Then product uncertain measure on the product $\sigma-$algebra $\Omega_1 \times \Omega_2 \times \cdots \times \Omega_n$ satisfying

$$\mathrm{M}\{\prod_{k=1}^{n} \Lambda_k\} \le \min_{1\le k\le n} \sum_{i=1}^{\infty} \mathrm{M}_k\{\Lambda_k\} .$$

Where $\Lambda_k \in \Omega_k$, $k = 1, 2, \ldots, n$.

We will introduce the definitions of uncertain variable and uncertainty distribution as follows.

Definition 2.1 (Liu[5])Let Γ be non-empty set and Ω is a $\sigma-$algebra over Γ, and M an uncertain measure. Then the triplet $(\Gamma, \Omega, \mathrm{M})$ is called an uncertainty space.

Definition 2.2 (Liu[5]) An uncertain variable is a measurable function from an uncertainty space $(\Gamma, \Omega, \mathrm{M})$ to the set of real numbers.

Definition 2.3 (Liu[5]) The uncertainty distribution Φ of an uncertain variable ξ is defined by

$$\Phi(x) = \mathrm{M}\{\xi \le x\}.$$

Theorem 2.1 (Sufficient and Necessary Condition for Uncertainty distribution [15]) A function Φ is an uncertainty distribution if and only if it is an increasing function except $\Phi(x) = 0$ and $\Phi(x) = 1$.

Example 2.1 An uncertain variable ξ is called normal if it has a normal uncertainty distribution

$$\Phi(x) = \left(1 + \exp\left(\frac{\pi(e - x)}{\sqrt{3}\sigma}\right)\right)^{-1}$$

denoted by N (e, σ) where e and σ are real numbers with σ > 0. Then we recall the definition of inverse uncertainty distribution as follows.

Definition 2.4 (Liu[5]) The uncertainty distribution Φ is said to be regular if its inverse function Φ^{-1} exists and is unique for each $t \in (0,1)$.

Definition 2.5 (Liu[5]) Let ξ be uncertain variable with uncertainty distribution Φ. Then inverse function Φ^{-1} is called inverse uncertainty distribution of ξ.

Example 2.2 The inverse uncertainty distribution of normal uncertain variable N (e, σ) is

$$\Phi^{-1}(t) = e + \frac{\sqrt{3}\sigma}{\pi} \ln \frac{t}{1-t}$$

Definition 2.6 (Independence of uncertain variable Liu[5]) The uncertain variables $\xi_1, \xi_2, \cdots \xi_n$ are said to be independent if

$$\mathrm{M}\left\{\bigcap_{i=1}^{m} \xi \in B_i\right\} = \min_{1 \le i \le n} \mathrm{M} \left\{\xi \in B_i\right\}$$

for any Borel sets $B_1, B_2, \cdots, B_n$ of real numbers.

Theorem 2.2 (Liu[5]) Assume $\xi_1, \xi_2, \ldots, \xi_n$ are independent uncertain variables with regular uncertainty distribution $\Phi_1, \Phi_2, \ldots, \Phi_n$, respectively. If $f : \Re^n \to \Re$ is a strictly increasing function, then uncertain variable $\xi = f(\xi_1, \xi_2, \ldots, \xi_n)$ has inverse uncertainty distribution

$$\psi^{-1}(t) = f(\Phi_1^{-1}(t), \Phi_2^{-1}(t), \cdots, \Phi_n^{-1}(t)), \quad 0 < t < 1$$

Theorem 2.3 (Liu[5]) Assume $\xi_1, \xi_2, \ldots, \xi_n$ are independent uncertain variables with regular uncertainty distribution $\Phi_1, \Phi_2, \ldots, \Phi_n$, respectively. If $f : \Re^n \to \Re$ is a strictly decreasing function, then uncertain variable $\xi = f(\xi_1, \xi_2, \ldots, \xi_n)$ has inverse uncertainty distribution

$$\psi^{-1}(t) = f(\Phi_1^{-1}(1-t), \Phi_2^{-1}(1-t), \cdots, \Phi_n^{-1}(1-t)), \quad 0 < t < 1$$

Theorem 2.4 (Liu [5]) Let $\xi_1, \xi_2, \cdots \xi_n$ be independent uncertain variables with regular uncertainty distribution $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively. If $f : \Re^n \to \Re$ be strictly decreasing function with respect to $x_1, x_2, \cdots x_m$ and strictly increasing function with $x_{m+1}, x_{m+2}, \cdots x_n$, then $\xi = f(\xi_1, \xi_2, \cdots \xi_n)$ is uncertain variable with uncertainty distribution with inverse uncertainty distribution

$$\psi^{-1}(t) = f(\Phi_1^{-1}(t), \Phi_2^{-1}(t), \cdots, \Phi_m^{-1}(t), \Phi_{m+1}^{-1}(1-t), \cdots, \Phi_n^{-1}(1-t)), \quad 0 < t < 1.$$

3. Two Parameter Entropy : In this section, we will introduce the definition and theorem of two parameter entropy of uncertain variable. For the purpose, we recall the entropy of uncertain variable proposed by Liu [2].

Definition 3.1 (Liu[2]) Suppose that ξ is an uncertain variable with uncertainty distribution Φ. Then its entropy is defined by

$$\mathrm{H}(\xi) = \int_{-\infty}^{\infty} S(\Phi(x))\, dx$$

Where

$$S(\Phi(x)) = -\Phi(x) \ln \Phi(x) - (1 - \Phi(x)) \ln(1 - \Phi(x)).$$

We set $0 \ln 0 = 0$ throughout this paper. By the enlightenment of Tsallis entropy [8], Liu *et al.* [10] defined the single parameter entropy.

Definition 3.2 (Liu *et al.*[10]) Suppose that ξ is an uncertain variable with uncertainty distribution Φ. Then its entropy is defined by

$$H_q(\xi) = \int_{-\infty}^{\infty} S_q(\Phi(x))\, dx$$

Where

$$S_q(\Phi(x)) = \frac{(1 - (\Phi(x))^q - (1 - \Phi(x))^q}{q(q-1)}$$

q is a

positive real number . Further, it can be seen that as $q \to 1$ we have

$$H_q(\xi) \to \mathrm{H}(\xi).$$

Now, in the light of two parametric probabilistic entropy function for case $C_{\alpha,\beta} = \beta - \alpha$ given in Eq.(1.1) a new two parameter entropy of uncertain variable can be proposed.

Definition 3.3 Suppose that ξ is an uncertain variable with uncertainty distribution Φ. Then its entropy is defined by

$$H_{\alpha,\beta}(\xi) = \int_{-\infty}^{\infty} S_{\alpha,\beta}(\Phi(x))\,dx$$

Where

$$S_{\alpha,\beta}(\Phi(x)) = \frac{\Phi^{\alpha}(x) + (1-(\Phi(x))^{\alpha} - \Phi^{\beta}(x) - (1-(\Phi(x))^{\beta}}{\alpha\beta(\beta-\alpha)}$$

α, β are positive real number . Further, it can be seen that as $\alpha = \beta \to 1$ we have

$$H_{\alpha,\beta}(\xi) \to \mathrm{H}(\xi).$$

In subsequent examples two parameter entropy formulae for uncertain variable ξ with different uncertainty distributions are obtained.

Example 3.1 Let ξ be uncertain variable with uncertain distribution

$$\Phi(x) = \begin{cases} 0 & ,x < a \\ 1 & ,x \geq a \end{cases}$$

we have

$$H_{\alpha,\beta}(\xi) = 0$$

Example 3.2 Let ξ be uncertain variable with uncertain distribution

$$\Phi(x) = \begin{cases} 0, & x < a \\ \dfrac{x-a}{b-a}, & a \leq x \leq b \\ 1, & x > a \end{cases}$$

then ξ is called linear uncertain variable and is denoted by L (a, b) where a and b are real numbers a< b.

Then two parameter entropy of linear uncertain variable is

$$H_{\alpha,\beta}(\xi) = \frac{1}{\alpha\beta(\beta-\alpha)}\int_a^b \left[\begin{array}{l} \left(\dfrac{x-a}{b-a}\right)^{\alpha} - \left(\dfrac{x-a}{b-a}\right)^{\beta} \\ + \left(1-\dfrac{x-a}{b-a}\right)^{\alpha} - \left(1-\dfrac{x-a}{b-a}\right)^{\beta} \end{array}\right] dx$$

$$= \frac{2(b-a)}{\alpha\beta(\alpha+1)(\beta+1)}$$

Example 3.3 Let ξ be uncertain variable with uncertain distribution

$$\Phi(x) = \begin{cases} 0, & x < a \\ \dfrac{x-a}{2(b-a)} & a \leq x < b \\ \dfrac{x+c-2b}{2(c-b)} & b \leq x < c \\ 1, & x \geq c \end{cases}$$

then ξ is called Zig-Zag uncertain variable and is denoted by Z (a, b, c) where a, b and c are real numbers a< b<c.

Then two parameter entropy of Zig-Zag uncertain variable is

$$H_{\alpha,\beta}(\xi) = \frac{1}{\alpha\beta(\beta-\alpha)}\int_a^b \left[\begin{array}{l} \left(\dfrac{x-a}{2(b-a)}\right)^{\alpha} - \left(\dfrac{x-a}{2(b-a)}\right)^{\beta} \\ + \left(1-\dfrac{x-a}{2(b-a)}\right)^{\alpha} - \left(1-\dfrac{x-a}{2(b-a)}\right)^{\beta} \end{array}\right] dx$$

$$= \frac{1}{\alpha\beta(\beta-\alpha)}\int_b^c \left[\begin{array}{l} \left(\dfrac{x-(c-2b)}{2(c-b)}\right)^{\alpha} - \left(\dfrac{x-(c-2b)}{2(c-b)}\right)^{\beta} \\ + \left(1-\dfrac{x-(c-2b)}{2(c-b)}\right)^{\alpha} - \left(1-\dfrac{x-(c-2b)}{2(c-b)}\right)^{\beta} \end{array}\right] dx$$

$$= \frac{2(c-a)}{\alpha\beta(\alpha+1)(\beta+1)}$$

4. Properties of Two Parameter Entropy : Assuming the uncertain variable with regular distribution, we obtain some theorems of two parameter entropy as follows.

Theorem 4.1 Let ξ be uncertain variable and 'c' be a real number. Then

$$H_{\alpha,\beta}(\xi + c) = H_{\alpha,\beta}(\xi)$$

that is two parameter entropy is invariant under arbitrary translations.

Proof: If Φ is uncertainty distribution of ξ , We have

$$\Phi(x) = M\{\xi \le x\}$$

Now,

$$M\{\xi + c \le x\} = M\{\xi \le x - c\} = \Phi(x - c)$$

Using the definition of the two parameter entropy, we find

$$H_{\alpha,\beta}(\xi + c)$$

$$= \int_{-\infty}^{\infty} S_{\alpha,\beta}(\Phi(x-c))\,dx$$

$$= \int_{-\infty}^{\infty} \frac{\Phi^{\alpha}(x-c) + (1-(\Phi(x-c))^{\alpha} - \Phi^{\beta}(x-c) - (1-(\Phi(x-c))^{\beta}}{\alpha\beta(\beta-\alpha)}\,dx$$

$$= H_{\alpha,\beta}(\xi)$$

The theorem is proved.

Theorem 4.2 Assume ξ is an uncertain variables with regular uncertainty distribution Φ. If entropy $H_{\alpha,\beta}(\xi)$ exists, then

$$H_{\alpha,\beta}(\xi) = -\int_0^1 \Phi^{-1}(t)\; S'_{\alpha,\beta}(t)\,dt$$

Where

$$S'_{\alpha,\beta}(t) = \frac{1}{(\beta-\alpha)}\left[\frac{t^{\alpha-1} - (1-t)^{\alpha-1}}{\beta} + \frac{t^{\beta-1} - (1-t)^{\beta-1}}{\alpha}\right]$$

Proof: Since

$$S_{\alpha,\beta}(\Phi(x)) = \int_0^{\Phi(x)} S'_{\alpha,\beta}(t)\,dt = -\int_{\Phi(x)}^{1} S'_{\alpha,\beta}(t)\,dt$$

and noting that the uncertain variable ξ has a regular uncertain distribution Φ, we have

$$H_{\alpha,\beta}(\xi) = \int_{-\infty}^{\infty} S_{\alpha,\beta}(\Phi(x))\,dx$$

$$= \int_{-\infty}^{+\infty}\int_0^{\Phi(x)} S'_{\alpha,\beta}(t)\,dt\,dx$$

$$= \int_{-\infty}^{0}\int_0^{\Phi(x)} S'_{\alpha,\beta}(t)\,dt\,dx + \int_0^{+\infty}\int_0^{\Phi(x)} S'_{\alpha,\beta}(t)\,dt\,dx$$

$$= \int_{-\infty}^{0}\int_0^{\Phi(x)} S'_{\alpha,\beta}(t)\,dt\,dx - \int_0^{+\infty}\int_{\Phi(x)}^{0} S'_{\alpha,\beta}(t)\,dt\,dx$$

By Fubini theorem, we have

$$H_{\alpha,\beta}(\xi) = \int_0^{\Phi(0)}\int_{\Phi^{-1}(t)}^{0} S'_{\alpha,\beta}(t)\,dx\,dt$$

$$-\int_{\Phi(0)}^{1}\int_0^{\Phi^{-1}(t)} S'_{\alpha,\beta}(t)\,dt\,dx$$

$$= -\int_0^{\Phi(0)}\int_0^{\Phi^{-1}(t)} S'_{\alpha,\beta}(t)\,dx\,dt - \int_{\Phi(0)}^{1}\int_0^{\Phi^{-1}(t)} S'_{\alpha,\beta}(t)\,dt\,dx$$

$$= -\int_0^1 \Phi^{-1}(t)\, S'_{\alpha,\beta}(t)\,dt$$

The theorem is proved.

Theorem 4.3 Let $\xi_1, \xi_2, \ldots, \xi_n$ are independent uncertain variables with regular uncertainty distribution $\Phi_1, \Phi_2, \ldots, \Phi_n$, respectively. If $f:\Re^n \to \Re$ is a strictly monotone function, then the uncertain variable $\xi = f(\xi_1, \xi_2, \ldots, \xi_n)$ has an entropy

$$H_{\alpha,\beta}(\xi) = \left|\int_0^1 f(\Phi_1^{-1}(t), \Phi_2^{-1}(t), \cdots, \Phi_n^{-1}(t))\; S'_{\alpha,\beta}(t)\,dt\right|.$$

Proof: If f is a strictly increasing function, letting ψ be distribution function of $f(\xi)$, it is easily shown from Theorem 2.2 that

$$\psi^{-1}(t) = f(\Phi_1^{-1}(t), \Phi_2^{-1}(t), \cdots, \Phi_n^{-1}(t)).$$

Then it follows from Theorem 4.2 that

$$H_{\alpha,\beta}(f(\xi)) =$$

$$\int_0^1 f(\Phi_1^{-1}(t), \Phi_2^{-1}(t), \cdots, \Phi_n^{-1}(t))\; S'_{\alpha,\beta}(t)\,dt.$$

Otherwise, if f is a strictly deccreasing function, letting ψ be distribution function of $f(\xi)$, it is easily shown from Theorem 2.3 that

$$\psi^{-1}(t) = f(\Phi_1^{-1}(1-t), \Phi_2^{-1}(1-t), \cdots, \Phi_n^{-1}(1-t)).$$

Then it follows from Theorem 4.2 that

$$H_{\alpha,\beta}(f(\xi)) =$$

$$\int_0^1 f(\Phi_1^{-1}(1-t), \Phi_2^{-1}(1-t), \cdots, \Phi_n^{-1}(1-t))\; S'_{\alpha,\beta}(t)\,dt.$$

Also it can be seen that $S'_{\alpha,\beta}(t) = -S'_{\alpha,\beta}(1-t)$, Therefore,

$$H_{\alpha,\beta}(f(\xi)) =$$

$$\int_0^1 f(\Phi_1^{-1}(1-t), \Phi_2^{-1}(1-t), \cdots, \Phi_n^{-1}(1-t))\; S'_{\alpha,\beta}(t)\,dt.$$

$$= -\int_0^1 f(\Phi_1^{-1}(t), \Phi_2^{-1}(t), \cdots, \Phi_n^{-1}(t))\; S'_{\alpha,\beta}(t)\,dt.$$

Thus, we have

$$H_{\alpha,\beta}(\xi) = \left|\int_0^1 f(\Phi_1^{-1}(t),\Phi_2^{-1}(t),\cdots,\Phi_n^{-1}(t))\, S'_{\alpha,\beta}(t)\,dt\right|.$$

The theorem is proved.

Theorem 4.4 Let ξ and η be independent uncertain variables. Then for any real numbers a and b,

$$H_{\alpha,\beta}[a\xi + b\eta] = |a|\, H_{\alpha,\beta}[\xi] + |b|\, H_{\alpha,\beta}[\eta].$$

Proof: Suppose that ξ and η are independent uncertain variables with regular distributions Φ and ψ, respectively. Otherwise, we may give them a small perturbation such that the uncertainty distributions are regular. It follows from Theorem 4.2 that

$$H_{\alpha,\beta}[\xi] = -\int_0^1 \Phi^{-1}(t)\; S'_{\alpha,\beta}(t)\,dt\,,$$

$$H_{\alpha,\beta}[\eta] = -\int_0^1 \psi^{-1}(t)\; S'_{\alpha,\beta}(t)\,dt$$

Since function $f(x, y) = x + y$ is strictly monotone, it follows from Theorem 4.3

$$H_{\alpha,\beta}[\xi + \eta] = -\int_0^1 (\Phi^{-1}(t) + \psi^{-1}(t))\, S'_{\alpha,\beta}(t)\,dt$$

$$= H_{\alpha,\beta}[\xi] + H_{\alpha,\beta}[\eta]$$

We prove H[aξ] = aH[ξ]. If a = 0, then the equation holds trivially. If a > 0, it follows from Theorem 4.3 that

$$H_{\alpha,\beta}[a\xi] = \left|\int_0^1 a\,\Phi^{-1}(t)\, S'_{\alpha,\beta}(t)\,dt\right|$$

$$= |a|\, H_{\alpha,\beta}[\xi].$$

Finally, for any real numbers a and b, it follows that

$$H_{\alpha,\beta}[a\xi + b\eta] = |a|\, H_{\alpha,\beta}[\xi] + |b|\, H_{\alpha,\beta}[\eta].$$

The theorem is proved.

Conclusion:. In this paper, the entropy of uncertain variable and its properties are recalled. On the basis of the entropy of uncertain variable and inspired by the two parameter probabilistic entropy (1.1), two parameter entropy of uncertain variable is introduced and explored its several important properties. Proposed two parameter entropy of uncertain variable makes the calculation of uncertainty of uncertain variable more general and flexible by choosing appropriate values of parameters α and β. Further, the results obtained by Liu *et al.* [10] for single parameter entropy of uncertain variable and Dai and Chen [12] for entropy of uncertain variable are the special cases of results obtained in this paper.

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