LOGARITHMIC ORDER OF ENTIRE MONOGENIC FUNCTIONS

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Abstract: In the present paper we study the logarithmic order of entire monogenic functions. The characterizations of logarithmic order of entire monogenic functions have been obtained in terms of their Taylor's series coefficients.

Keywords: Clifford analysis, Clifford algebra, Generalized Cauchy-Riemann system, Monogenic function, Logarithmic order.

Introduction: Clifford analysis offers possibility of generalizing complex function theory to higher dimensions. It considers Clifford algebra valued functions that are defined in open subsets of $\mathbf{i}^n$ for arbitrary finite n ∈ ¥ and that are solutions of higher dimensional Cauchy-Riemann systems. These are often called Clifford holomorphic or monogenic functions. In order to make calculations more concise we use following notations, where m = $(m_1,...,m_n)$ ∈$¥_o^n$ be $n-$ dimensional multi-index and x ∈ $\mathbf{i}^n$

$$\mathbf{x^m} = x_1^{m_1}...x_n^{m_n} \ , \ \mathbf{m!} = m_1!...m_n!,$$

$$|\mathbf{m}| = m_1 + ... + m_n .$$

Following Constales, Almeida and Krausshar ([1] and [2]), we give some definitions and associated properties. By $\{e_1, e_2, ..., e_n\}$ we denote the canonical basis of the Euclidean vector space $\mathbf{i}^n$ The associated real Clifford algebra Cl_{0n} is the free algebra generated by $\mathbf{i}^n$ modulo

$$\mathbf{x}^2 = -\|\mathbf{x}\|^2 \ e_0 ,$$

where e_0 is the neutral element with respect to multiplication of the Clifford algebra Cl_{0n}. In the Clifford algebra Cl_{0n} following multiplication rule holds:

$$e_i e_j + e_j e_i = -2\delta_{ij} e_0 , \quad i, j = 1, 2, ..., n ,$$

where δ_{ij} is Kronecker symbol. A basis for Clifford algebra Cl_{0n} is given by the set

$$\{e_A : A \subseteq \{1, 2, ..., n\}\}$$

with $e_A = e_{l_1} e_{l_2} ... e_{l_r}$, where

$$1 \le l_1 < l_2 < \cdots < l_r \le n, \ e_\phi = e_0 = 1.$$

Each $a \in Cl_{0n}$ can be written in the form $a = \sum_A a_A e_A$ with a_A ∈ $\mathbf{i}$ The conjugation in Clifford algebra Cl_{0n} is defined by $\overline{a} = \sum_A a_A \overline{e}_A$, where $\overline{e_A} = \overline{e}_{l_r} \overline{e}_{l_{r-1}} ... \overline{e}_{l_1}$ and $\overline{e}_j = -e_j$ for $j = 1, 2, ... n,\ \overline{e}_0 = e_0 = 1.$

The linear subspace

$span_R\{1, e_1, .., e_n\} = \mathbf{i} \oplus \mathbf{i}^n \subset Cl_{on}$

is the so called space of para vectors

$$z = x_0 + x_1 e_1 + x_2 e_2 + ... + x_n e_n$$

which we simply identify with $\mathbf{i}^{n+1}$ Here

$$x_0 = \mathrm{Sc}(z)$$

is scalar part and

$$\mathbf{x} = x_1 e_1 + x_2 e_2 + ... + x_n e_n = \mathrm{Vec}(z)$$

is vector part of para vector z. The Clifford norm of an arbitrary $a = \sum_A a_A e_A$ is given by

$$\|a\| = \left(\sum_A |a_A|^2 \right)^{1/2} .$$

Each para vector z ∈ $\mathbf{i}^{n+1} \setminus \{0\}$ has an inverse element in $\mathbf{i}^{n+1}$ which can be represented in the form $z^{-1} = \overline{z} / \|z\|^2$.

The generalized Cauchy-Riemann operator in $\mathbf{i}^{n+1}$ is given by

$$D \equiv \frac{\partial}{\partial x_0} + \sum_{i=1}^{n} e_i \frac{\partial}{\partial x_i} .$$

If U ⊆ $\mathbf{i}^{n+1}$ is an open set, then a function $g : U \to Cl_{0n}$ is called left (right) monogenic at a point $z \in U$ if $D g(z) = 0$ ($g D(z) = 0$).

The functions which are left (right) monogenic in the whole space are called left (right) entire monogenic functions.

Let A_{n+1} be n-dimensional surface area of $n+1$-dimensional unit ball and $q_{\mathbf{0}}(z) = \overline{z} / \|z\|^{n+1}$ be

Cauchy kernel function. Then every function g which is monogenic in a neighbourhood of closure $\overline{G}$ of domain G satisfies the following equation (see [2], pp. 766)

$$g(z)=\frac{1}{A_{n+1}}\int_{\partial G} q_{\mathbf{0}}(z-\zeta)\ d\tau(\zeta)\,g(\zeta)\ ,$$

for all $z\in G$, where

$$d\tau(\zeta)=\sum_{j=0}^{n}(-1)^{j}e_{j}\widehat{d\zeta}_{j}$$

with

$$\widehat{d\zeta}_{j}=d\zeta_{0}\wedge\ldots\wedge d\zeta_{j-1}\wedge d\zeta_{j+1}\wedge\ldots\wedge d\zeta_{n}$$

is the oriented outer normal surface measure. If g is a left monogenic function in a ball $\|z\|<R,$ then for all $\|z\|<r$ with $0<r<R$,

$$g(z)=\sum_{|\mathbf{m}|=0}^{\infty}V_{\mathbf{m}}(z)\,a_{\mathbf{m}}\ . \qquad (1.1)$$

In (1.1) $V_{\mathbf{m}}(z)$ are called Fueter polynomials and are given as

$$V_{\mathbf{m}}(z)=\frac{\mathbf{m}!}{|\mathbf{m}|!}\sum_{\pi\in perm(\mathbf{m})}z_{\pi(m_1)}\ldots z_{\pi(m_n)},$$

where $perm(\mathbf{m})$ is the set of all permutations of the sequence $(m_1,m_2\ldots,m_n)$ and $z_i=x_i-x_0e_i$ for $i=1,\ldots,n$ and $V_{\mathbf{0}}(z)=1$. Also in (1.1), $\{a_{\mathbf{m}}\}$ are Clifford numbers which are defined by

$$a_{\mathbf{m}}=\frac{1}{\mathbf{m}!\,A_{n+1}}\int_{\|\zeta\|<r}q_{\mathbf{m}}(\zeta)\ d\tau(\zeta)\,g(\zeta)$$

and satisfy the inequality

$$\|a_{\mathbf{m}}\|\le c(n,\mathbf{m})\frac{M(r,g)}{r^{|\mathbf{m}|}}.$$

Here $M(r,g)=\max_{\|z\|=r}\{\|g(z)\|\}$ denotes the maximum modulus of the function g in the closed ball of radius r and

$$q_{\mathbf{m}}(z)=\frac{\partial^{m_0+m_1+\ldots+m_n}}{\partial x_0^{m_0}\partial x_1^{m_1}\ldots\partial x_n^{m_n}}q_{\mathbf{0}}(z),$$

$$c(n,\mathbf{m})=\frac{n(n+1)\ldots(n+|\mathbf{m}|-1)}{\mathbf{m}!}.$$

Constales, Almeida and Krausshar have defined the order ρ of entire monogenic function $g(z)$ as (see [2], pp. 767)

$$\rho(g)=\limsup_{r\to\infty}\frac{\log\log M(r,g)}{\log r}.$$

Also for entire monogenic function having order $\rho(0<\rho<\infty)$, they defined the type σ of $g(z)$ as (see [1], pp. 155)

$$\sigma(g)=\limsup_{r\to\infty}\frac{\log M(r,g)}{r^{\rho}}.$$

If an entire monogenic function $g(z)$ is of zero order then the definition of type defined by Constales, Almeida and Krausshar has no meaning. Hence the comparison of growth of such entire monogenic functions cannot be made by confining to the above concepts. To overcome this difficulty following Iyer [3], we have introduced the concept of logarithmic order for entire monogenic functions. Thus for entire monogenic function $g(z)$ of order zero, we define the logarithmic order ρ_l of $g(z)$ as

$$\rho_l=\limsup_{r\to\infty}\frac{\log\log M(r)}{\log\log r}. \qquad (1.2)$$

Also for $1<\rho_l<\infty,$ we define the logarithmic type σ_l of entire monogenic function $g(z)$ as

$$\sigma_l=\limsup_{r\to\infty}\frac{\log M(r)}{(\log r)^{\rho_l}}.$$

In the present paper we have obtained the characterizations of logarithmic order of entire monogenic functions in terms of their Taylor's series coefficients.

2. Main Results: Now we prove

Theorem 2.1: *Let $g:R^{n+1}\to Cl_{0n}$ be an entire monogenic function whose Taylor's series representation is given by*

$$g(z)=\sum_{|\mathbf{m}|=0}^{\infty}V_{\mathbf{m}}(z)\,a_{\mathbf{m}}\ .$$

Then the logarithmic order $\rho_l\left(1<\rho_l<\infty\right)$ of $g(z)$ is given by

$$\rho_l-1=\limsup_{|\mathbf{m}|\to\infty}\frac{\log|\mathbf{m}|}{\log\left\{\log\left(\left\|a_{\mathbf{m}}/c(n,\mathbf{m})\right\|^{-1/|\mathbf{m}|}\right)\right\}}. \qquad (2.1)$$

Proof: Let

$$\theta=\limsup_{|\mathbf{m}|\to\infty}\frac{\log|\mathbf{m}|}{\log\left\{\log\left(\left\|a_{\mathbf{m}}/c(n,\mathbf{m})\right\|^{-1/|\mathbf{m}|}\right)\right\}}.$$

Now first we prove that $\theta\le\rho_l-1$. The coefficients of a monogenic Taylor's series satisfy Cauchy's inequality, that is

$$\left\|a_{\mathbf{m}}/c(n,\mathbf{m})\right\|\le M(r,g)\ r^{-|\mathbf{m}|}. \qquad (2.2)$$

Also from (1.2), for $\varepsilon>0$ and all $r>r_0(\varepsilon)$, we have

$$M(r,g) \le \exp\left[\exp\left\{\overline{\rho_l}\log(\log r)\right\}\right],$$

where $\overline{\rho_l} = \rho_l + \varepsilon$ provided r is sufficiently large. So from (2.2) we get

$$\|a_{\mathbf{m}} / c(n,\mathbf{m})\| \le r^{-|\mathbf{m}|} \times$$
$$\times \exp\left[\exp\left\{\overline{\rho}\log(\log r)\right\}\right]$$

or

$$\|a_{\mathbf{m}} / c(n,\mathbf{m})\| \le \exp\left[-|\mathbf{m}|\log r + \right.$$
$$\left. + \exp\left\{\overline{\rho_l}\log(\log r)\right\}\right] \qquad (2.3)$$

Since $\log x$ is an increasing function of x, we define $r = r(|\mathbf{m}|)$ as unique root of the equation

$$\log\left[\frac{|\mathbf{m}|\log r}{\overline{\rho_l}}\right] = \overline{\rho_l}\log(\log r). \qquad (2.4)$$

Then for large value of $|\mathbf{m}|$, we have

$$\log r \sim \exp\left\{\frac{1}{\overline{\rho_l}-1}\log(|\mathbf{m}|)\right\}$$
$$= F\left(|\mathbf{m}|, \frac{1}{\overline{\rho_l}-1}\right), \qquad (2.5)$$

where

$$F(x,y) = \exp\left(\frac{1}{y}\log x\right).$$

Using (2.4) and (2.5) in (2.3), we get

$$\|a_{\mathbf{m}} / c(n,\mathbf{m})\| \le \exp\left[-|\mathbf{m}|F + \left(\frac{|\mathbf{m}|}{\overline{\rho_l}}\right)F\right]$$

or

$$\frac{\overline{\rho_l}}{\overline{\rho_l}-1}\log\left\{\|a_{\mathrm{m}} / c(n,\mathrm{m})\|^{-1/|\mathbf{m}|}\right\}$$
$$\ge \exp\left\{\frac{1}{\overline{\rho_l}-1}\log(|\mathbf{m}|)\right\}$$

or

$$\frac{\log(|\mathbf{m}|)}{\log\left[\frac{\overline{\rho_l}}{\overline{\rho_l}-1}\log\left\{\|a_{\mathbf{m}} / c(n,\mathbf{m})\|^{-1/|\mathbf{m}|}\right\}\right]}$$
$$\le \overline{\rho_l} - 1$$

or

$$\frac{\log(|\mathbf{m}|)}{\log\left[\log\left\{\|a_{\mathbf{m}} / c(n,\mathbf{m})\|\right\}^{-1/|\mathbf{m}|}\right]}$$
$$\le \left(\overline{\rho_l} - 1\right) \times$$
$$\times \frac{\log\left[\frac{\overline{\rho_l}}{\overline{\rho_l}-1}\log\left\{\|a_{\mathbf{m}} / c(n,\mathbf{m})\|^{-1/|\mathbf{m}|}\right\}\right]}{\log\left[\log\left\{\|a_{\mathbf{m}} / c(n,\mathbf{m})\|\right\}^{-1/|\mathbf{m}|}\right]}.$$

Since $\log(cx) \sim \log(x)$ as $x \to \infty$, proceeding to limits as $|\mathbf{m}| \to \infty$ we get

$$\theta \le \overline{\rho_l} - 1$$

Since $\varepsilon > 0$ is arbitrarily small, we finally get

$$\theta \le \rho_l - 1. \qquad (2.6)$$

Now we will prove that $\rho_l - 1 \le \theta$. If $\theta = \infty$, then there is nothing to prove. So let us assume that $0 \le \theta < \infty$. Therefore for a given $\varepsilon > 0$ there exist $n_0 \in N$ such that for all multi –indices $\mathbf{m}$ with $|\mathbf{m}| > n_0$, we have

$$0 \le \frac{\log(|\mathbf{m}|)}{\log\left[\log\left\{\|a_{\mathbf{m}} / c(n,\mathbf{m})\|\right\}^{-1/|\mathbf{m}|}\right]}$$
$$\le \theta + \varepsilon = \overline{\theta}$$

or

$$\|a_{\mathbf{m}} / c(n,\mathbf{m})\|$$
$$\le \exp\left[-|\mathbf{m}|\exp\left\{\log(|\mathbf{m}|)/\overline{\theta}\right\}\right].$$

Now from the property of maximum modulus, we have

$$M(r,g) \le \sum_{|\mathbf{m}|=0}^{\infty} \|a_{\mathbf{m}}\| r^{|\mathbf{m}|}$$

or

$$M(r,g) \le \sum_{|\mathbf{m}|=0}^{n_0} \|a_{\mathbf{m}}\| r^{|\mathbf{m}|}$$
$$+ \sum_{|\mathbf{m}|=n_0+1}^{\infty} c(n,\mathbf{m})\, r^{|\mathbf{m}|} \times$$
$$\times \exp\left[-|\mathbf{m}|\exp\left\{\log(|\mathbf{m}|)/\overline{\theta}\right\}\right].$$

Now for

$$r = \max\left\{1, \exp\left\{\frac{\exp\left(\left\{\log\left(n_0+1\right)\right\}/\overline{\theta}\right)}{\left(n+1\right)}\right\}\right\},$$

we have

$$M(r,g) \le A_1 r^{n_0} + \sum_{|\mathbf{m}|=n_0+1}^{\infty} c(n,\mathbf{m})\, r^{|\mathbf{m}|} \times \exp\left[-|\mathbf{m}|\exp\left\{\log\left(|\mathbf{m}|\right)/\overline{\theta}\right\}\right], \qquad (2.7)$$

where A_1 is a positive real constant. We take

$$N(r) = \left[\exp\left\{\overline{\theta}\log\left[\log\left\{(n+1)r\right\}\right]\right\}\right],$$

where $[x]$ denotes the integer part of x. Now if r is sufficiently large, then from (2.7), we have

$$M(r,g) \le A_1 r^{n_0} + r^{N(r)} \times \sum_{n_0+1\le|\mathbf{m}|\le N(r)} c(n,\mathbf{m}) \times \exp\left[-|\mathbf{m}|\exp\left\{\log\left(|\mathbf{m}|\right)/\overline{\theta}\right\}\right] + \sum_{|\mathbf{m}|>N(r)} c(n,\mathbf{m}) \times r^{|\mathbf{m}|}\exp\left[-|\mathbf{m}|\exp\left\{\log\left(|\mathbf{m}|\right)/\overline{\theta}\right\}\right]$$

or

$$M(r,g) \le A_1 r^{n_0} + r^{N(r)} \times \sum_{|\mathbf{m}|=1}^{\infty} c(n,\mathbf{m}) \times \exp\left[-|\mathbf{m}|\exp\left\{\log\left(|\mathbf{m}|\right)/\overline{\theta}\right\}\right] + \sum_{|\mathbf{m}|>N(r)} c(n,\mathbf{m}) \times r^{|\mathbf{m}|}\exp\left[-|\mathbf{m}|\exp\left\{\log\left(|\mathbf{m}|\right)/\overline{\theta}\right\}\right]. \qquad (2.8)$$

Now the first series in (2.8) can be rewritten as

$$\sum_{p=1}^{\infty}\left(\sum_{|\mathbf{m}|=p} c(n,\mathbf{m})\right) \times \exp\left[-p\exp\left\{\log\left(p\right)/\overline{\theta}\right\}\right]. \qquad (2.9)$$

Now from ([2], Lemma 1), we have

$$\limsup_{p\to\infty}\left(\sum_{|\mathbf{m}|=p} c(n,\mathbf{m})\right)^{1/p} = n.$$

Hence we have

$$\limsup_{p\to\infty}\left[\left(\sum_{|\mathbf{m}|=p} c(n,\mathbf{m})\right) \times \exp\left[-p\exp\left\{\log\left(p\right)/\overline{\theta}\right\}\right]\right]^{1/p}$$
$$= n \limsup_{p\to\infty}\exp\left[-\exp\left\{\log\left(p\right)/\overline{\theta}\right\}\right]$$
$$= 0.$$

Hence the series (2.9) converges to a positive real constant A_2. So from (2.8), we get

$$M(r,g) \le A_1 r^{n_0} + A_2 r^{N(r)} + \sum_{|\mathbf{m}|>N(r)} c(n,\mathbf{m}) \times r^{|\mathbf{m}|}\exp\left[-|\mathbf{m}|\exp\left\{\log\left(|\mathbf{m}|\right)/\overline{\theta}\right\}\right]$$

or

$$M(r,g) \le A_1 r^{n_0} + A_2 r^{N(r)} + \sum_{|\mathbf{m}|>N(r)} c(n,\mathbf{m}) \times r^{|\mathbf{m}|}\exp\left[-|\mathbf{m}|\log\left\{(n+1)r\right\}\right]$$

or

$$M(r,g) \le A_1 r^{n_0} + A_2 r^{N(r)} + \sum_{|\mathbf{m}|>N(r)} c(n,\mathbf{m})\left(\frac{1}{n+1}\right)^{|\mathbf{m}|}$$

or

$$M(r,g) \le A_1 r^{n_0} + A_2 r^{N(r)} + \sum_{|\mathbf{m}|=1}^{\infty} c(n,\mathbf{m})\left(\frac{1}{n+1}\right)^{|\mathbf{m}|}. \qquad (2.10)$$

The series in (2.10) can we rewritten as

$$\sum_{p=1}^{\infty}\left(\sum_{|\mathbf{m}|=p} c(n,\mathbf{m})\right)\left(\frac{1}{n+1}\right)^{p}. \qquad (2.11)$$

So we have

$$\limsup_{p\to\infty}\left[\left(\sum_{|\mathbf{m}|=p} c(n,\mathbf{m})\right)\left\{\frac{1}{n+1}\right\}^{p}\right]^{1/p}$$
$$= \frac{n}{n+1} < 1.$$

Hence the series (2.11) converges to a positive real constant A_3. Therefore from (2.10), we get

$$M(r,g) \le A_1 r^{n_0} + A_2 r^{N(r)} + A_3.$$

Since $N\left(r\right)\to\infty$ as $r\to\infty$, so we can write above inequality as

$$\log M(r,g) \le [1+o(1)]\, N(r)\, \log r$$

or

$$\log M(r,g) \le [1+o(1)]\times \left[\exp\left\{\overline{\theta}\log\left[\log\left\{(n+1)r\right\}\right]\right\}\right] \log r$$

$$\le [1+o(1)]\times$$

$$\times\left[\exp\left\{\overline{\theta}\log\left[\log\left\{(n+1)r\right\}\right]\right\}\right]$$

$$\times\left[\exp\left\{\log\left[\log\left\{(n+1)r\right\}\right]\right\}\right]$$

or

$$\log M(r,g) \le [1+o(1)]\times$$

$$\times\left[\exp\left\{\left(\overline{\theta}+1\right)\log\left[\log\left\{(n+1)r\right\}\right]\right\}\right]$$

or

$$\log[\log M(r,g)] \le \left(\overline{\theta}+1\right)\log\left[\log\left\{(n+1)r\right\}\right].$$

or

$$\frac{\log\log M(r,g)}{\log(\log r)} \le \left(\overline{\theta}+1\right)\times \frac{\log\left[\{1+o(1)\}\log r\right]}{\log(\log r)}.$$

Proceeding to limits as $r\to\infty$, we get

$$\rho_l \le \overline{\theta}+1$$

Since $\varepsilon > 0$ is arbitrarily small, therefore finally we get

$$\rho_l - 1 \le \theta. \qquad (2.12)$$

Combining (2.6) and (2.12), we get (2.1). Hence Theorem 2.1 is proved.

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