μ-CONTINUOUS FUNCTIONSON GENERALIZED TOPOLOGYAND CERTAIN ALLIED STRUCTURES

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Abstract: The aim of this paper is to introduce (μ_1-μ_2)-continuous functions on generalized topology and to give some new results concerning generalized topological space. We obtain some characterizations and several properties of such function. This paper takes some investigations on generalized topological spaces with μ-open sets.

Keywords: Generalized topology, μ- open, μ-closed, μ-continuous, p- μ-open, R O (X, μ).

Introduction: In 1963, Levine[10] introduced and investigated the semi open sets and semi continuous functions. In 1987, Bhattacharyya and Lahiri [1] used semi open sets to define and investigate the notion of semi generalized closed sets. The origin for the development in the field of strong generalized topological spaces (X, μ) is N. Levine's work of [10].In topology weak forms of open sets play an important role in the generalization of various forms of continuity. Using various forms of open sets, many authors have introduced and studied various types of continuity.

Generalized topology (X, μ) was first introduced and studied by A.Csaszar [4]

2.Preliminaries: Let X be a set. A subset μ of exp X is called a generalized topology on X and (X, μ) is called a generalized topological spaces [4] (abbr.GTS) if μ has the following properties: (i) $\phi \in \mu$, (ii)Any union of elements of μ belongs to μ.

Generalized topological spaces is an important generalization of topological spaces, and many interesting results have been obtained. Throughout this paper, a space (X, μ) or simply X for short, will always mean a strong generalized topological spaces with strong generalized topology μ unless otherwise explicitly slated. Here, a generalized topology μ is said to be strong [5] if $X \in \mu$. For the space (X, μ), the elements of μ are called μ- open sets and the complement of μ- open sets are called μ- closed sets. For $A \subset X$, we denote by $c_\mu(A)$ the intersection of all μ-closed sets containing A, that is the smallest μ- closed set containing A, and by $i_\mu(A)$, the union of all μ-open sets contained in A, that is the largest μ- open set contained in A. It is easy to observe that $c_\mu$and i_μ are idempotent and monotonic, where γ:exp X→exp X is said to idempotent if and only if $A \subset B \subset X$ implies $\gamma(\gamma(A)) = \gamma(A)$ and monotonic if and only if $A \subset B \subset X$ implies $\gamma(A) \subset \gamma(B)$. It is also well known that from [6,7] that if μ is a generalized topology on X and $A \subset X$, $x \in X$ then $x \in c_\mu(A)$ if and only if $x \in M \in \mu \Rightarrow M \cap A \neq \phi$ and $c_\mu(X-A) = X- i_\mu(A)$.

Let $B \subset$exp X and $\phi \in B$. Then B is called a base [5] for μ if $\{\cup B' : B' \subset B\} = \mu$. Wealso say that μ is generated by B. A point $x \in X$ is called a μ-cluster point of B if$U \cap (B-\{x\}) \neq \phi$ for each $U \in \mu$ with $x \in U$. The set of all μ-cluster point of B is denoted by d(B).

Definition 2.1 Let X be a space and let $A \subset X$. The μ-cluster point of A in X is the set $A = c_\mu(A) = \cap\{K \subset X /$ K is μ- closed and $A \subset K\}$.

Definition 2.2 : If X is a space and $A \subset X$, the interior of A in X is the set $i_\mu(A) = U\{G \subset X /$ G isμ-open and $G \subset A\}$.

The notion of μ-closure and μ-interior are dual to each other, in much the same way that "μ-closed" and "μ-open" are. Thus any theorem about μ -closures in a space X can be translated to a theorem about μ-interiors.

Notation 2.1 Let X be a spaces. For any $x \in X$, we use the following notation and $K \subset$exp X.

(i) $\mu_x = \{U : x \in U \in \mu\}$
(ii) (ii) $\cap K = \cap\{K : K \in K\}$
(iii) (iii) $\cup K = \cup\{K : K \in K\}$
(iv) (iv) $c(\mu_x) = \{c(U) : U \in \mu_x\}$
(v) N(x) = $\cap \mu_x$
(v) (vi) $\overline{N(x)} = \cap\, c(\mu_x)$
(vi) following lemma is easy and we omit the proof.

Lemma 2.1 For any $A \subset X$,
(i)$i_\mu(A) \subset A \subset c_\mu(A)$.
(ii) $i_\mu(i_\mu(A)) = i_\mu(A)$ and $c_\mu(c_\mu(A)) = c_\mu(A)$.
(iii) $i_\mu(A) = A$ if and only if A is μ-open.

(iv)$c_\mu(A) = A$ if and only if A is μ-closed.
(v) $i_\mu(X-A) = X-c_\mu(A)$ and) $c_\mu(X-A) = X- i_\mu(A)$.
(vi)$x \in c_\mu(A)$ if and only if $U \subset A$ for some $U \in \mu_x$.
(vii)$x \in c_\mu(A)$ if and only if $U \cap A \neq \phi$ for each$U \in \mu_x$.
(viii)$x \in i_\mu(A)$ if and only if $U \subset A$ for some $U \in \mu_x$.

Lemma 2.2 Let X be space. If $A \subset B \subset X$, then $c_\mu(A) \subset c_\mu(B)$.

proof: Since $B \subset c_\mu(B)$, if A is contained in B, we have $A \subset c_\mu(B)$; since $c_\mu(B)$is μ-closed, we must then have $c_\mu(A) \subset c_\mu(B)$.

Remark 2.1: (1) Theμ -closure of a subset A of a discrete space X is A itself, which is same as in the topological space (X,τ)(2) $c_\mu(\phi) = \phi$.

Example 2.1 : Let X = {a,b,c}and let $\mu = \{\phi,\{a\},\{b\},\{a,b\}\}$.Then μ-closed sets are X,{a,c},{b,c} and {c}. If A = {a,b} then A is notμ-closed.

Lemma 2.3:If $A\subset B\subset X$, then $i_\mu(A)\subset i_\mu(B)$.

proof: It is clear that $i_\mu(A)\subset A$, so if $A\subset B$, we have $i_\mu(A)\subset B$. Thus, $i_\mu(A)$ is aμ-open set contained in B,so$i_\mu(A)\subset i_\mu(B)$.

Remark 2.2: Let X be a space and let A_1,A_2,B_1,B_2 be subsets of X. Then in general, $i_\mu(A_1 \cap A_2) \neq i_\mu(A_1)\cap i_\mu(A_2)$and $c_\mu(B_1 \cup B_2) \neq c_\mu(B_1) \cup c_\mu(B_2)$.For, let X={a,b,c} and let$\mu=\{\phi,X,\{a,b\},\{b,c\},\{a,c\}\}$.Take A_1={a,b} and A_2={b,c}. then $i_\mu(\{A_1 \cap A_2\}) = i_\mu(\{b\}) = \phi$But $i_\mu(A_1)\cap i_\mu(A_2) = i_\mu(\{a,b\})\cap i_\mu(\{b,c\})$

$= \{a,b\} \cap \{b,c\} = \{b\}$

Now take B_1={a} and B_2={b}

Then $c_\mu(\{B_1 \cup B_2\}) = c_\mu(\{a,b\}) = X$

But $c_\mu(B_1) \cup c_\mu(B_2) = c_\mu(\{a\}) \cup c_\mu(\{b\})=\{a\}\cup\{b\}=\{a,b\}$.

The following definition come from [2,8]

Definition 2.3: Let X be a space.

i)Let $x\in X$ and $U\in\mu_x$.Then x is called a representative element of U if $U\subset V$ for each $V\in\mu_x$. ii)A Space X is called a C_o-space if C_o=X where C_o is the set of all representative elements of sets of μ.iii)Let $x\in X$. The set $Md(x)=\{U\in\mu_x:U\supset V\in\mu_x\Rightarrow U=V\}$ is called the minimal description of x.

Remark 2.3: [2,8] Let X be a space and let $x\in X$.If$\mu_x$is finite, then $x\in C_o$ if and only if$|Md(x)|>1$where$|Md(x)|$isthe cardinality of Md(x).The equality of remark 2.2follows by using sufficient condition, which is in the following lemma

Lemma 2.4 : Let A and B be subsets of a C_o-space X. Then i) $i_\mu(A\cap B) = i_\mu(A)\cap i_\mu(B)$ ii) $c_\mu(A\cap B) = c_\mu(A)\cup c_\mu(B)$**proof**: i)By lemma 2.3,$i_\mu(A\cap B)=i_\mu(A)\cap i_\mu(B)$,suppose $x\in i_\mu(A)\cap i_\mu(B)$.Then there are $U\in\mu_x$and $V\in\mu_x$such that $U\subset A$ and $V\subset B$. Since X is a C_o-space,we have $x\in C_o$. So there is $G\in\mu_x$ such that x is a representative element of G, and hence $G\subset U$ and $G\subset V$. Consequently, $x\in G\subset U\cap V \subset A\cap B$. This gives that $x\in i_\mu(A)\cap i_\mu(B)\subset i_\mu(A\cap B)$. Thus$i_\mu(A\cap B) = i_\mu(A)\cap i_\mu(B)$.

ii) Follows by (i) and lemma 2.1(v)

Lemma 2.5: [11]. Let X be a space. If μ is finite, then the following are equivalent.

i) X is a C_o-space

ii)$i_\mu(A\cap B)= i_\mu(A)\cap i_\mu(B)$ for each A,B in exp X.

iii) $c_\mu(A\cap B)=c_\mu(A)\cup c_\mu(B)$ for each A,Bin exp X.

Proof :(i)⇒(ii).It follows from lemma 2.4,

(ii)⇒(i).Suppose X is not a C_o-space .Then there is an $x\in X$ such that $x\notin C_o$.By remark 2.3, $|Md(x)|>1$.So there are U and V $\in Md(x)$ such that $U\neq V$, hence $x\in U\cap V= i_\mu(U)\cap i_\mu(V)$. On the other hand, for each $G\in\mu_x$,$G\not\subset U\cap V$ because $U,V\in Md(x)$. So $x\notin i_\mu(U\cap V)$. This contradicts $i_\mu(U\cap V) = i_\mu(U) \cap i_\mu(V)$.(ii)⇔(iii).It holds by lemma 2.1.

Theorem 2.1 Let X be a space and let $x \in X$. Then $c_\mu(\{x\}) = X - \cup(\mu - \mu_x)$.

Proof: Let $y\in c_\mu(\{x\}) = X - i_\mu(X - \{x\})$.Then $y\notin i_\mu(\{X - \{x\})$.Suppose$U\in \mu_y$. Then $U\not\subset X - \{x\}$, and hence $x\in U$. That is $U\in\mu_x$. So $\mu_y \subset \mu_x$, and hence $\mu - \mu_x\subset \mu - \mu_y$. It follows that$\cup(\mu-\mu_x) \subset\cup(\mu-\mu_y)$. It is easy to see that $y\notin\cup(\mu-\mu_y)$. So $y\notin\cup(\mu - \mu_x)$. Consequently $y\in X-\cup(\mu-\mu_x)$. On the other hand, let $y\in X-\cup(\mu- \mu_x)$. Then we have, $y\in c_\mu(\{x\})$ by reversing the proof above. This proves that $c_\mu(\{x\}) =X - \cup(\mu - \mu_x)$.

Definition 2.4 Let X be a space and let Y be a subset of X.Define $\mu_y=\{Y\cap U/U\in\mu\}$.Since $\phi = Y\cap \phi$ and $Y= Y\cap X$, we have ϕ and Y are contained in μ_y, where ϕ and X are elements of μ.Also $\bigcup_{\alpha\in J}(U_\alpha \cap Y) = (\bigcup_{\alpha\in J} U_\alpha)\cap Y$ contained in μ_Y, where α the index set. Thus,Yis a subspace of SGTS, with SGT μ_Y. Now we observe that, any subspace of a discrete space is discrete and any subspace of a trivial space is trivial.

Theorem 2.2 Suppose A is a subspace of a space (X, μ). Then

(i) $H \subset A$ is μ-open if and only if $H = G \cap A$, where G is μ-open in X.

(ii) $F \subset A$ is μ-closed in A if and only if $F = K \cap A$, where K is μ-closed in X.(iii)If $E \subset A$, then $C_{A\mu}(E) =A \cap C_\mu(E)$.

Proof:

(i) is just the definition of the subspace of generalized topology on A.

(ii) (ii) follows from (i)

(iii) follows from (ii) and the definition of E as the intersection of all μ-closed sets containing E.

Definition 2.5: A space(X, μ) is a generalized door space if every subset of (X, μ) is either μ-open or μ-closed.

3. μ-open and μ-closed mappings

Definition 3.1: Let X and Y be strong generalized topological spaces. A mapping $f: (X, \mu_1)\rightarrow (Y, \mu_2)$ is μ_1-open (resp., μ_1-closed) if the image under f of every μ_1-open (resp., μ_1-closed) subset of X is μ_2-open (resp., μ_2-closed) subset of Y. If no confusion arise, we can say, $f: X \rightarrow Y$ is μ-open (resp., μ-closed) if the image under f of every μ-open (resp., μ-closed) subset of X is μ-open (resp., μ-closed) subset of Y.

Example 3.1: Let X = {a, b, c}, $\mu_1 = \{\phi, X, \{a, c\},\{b, c\}\}$ and let Y = {p, q, r}, $\mu_2 = \{\phi, Y,\{p, r\},\{q, r\}\}$. Define $f: (X, \mu_1)\rightarrow(Y, \mu_2)$ by f(a) = p, f(b) = q and f(c) = r.Then μ_1-open subset of X are f(X) = Y, $f(\phi) = \phi$, f({a, c}) = {p, r}, f({b, c}) = {q, r}. Thus f is μ_1-open mapping of X onto Y.

Example 3.2: On the real line R, define $f: R\rightarrow R$ by $f(x)=x^2$ for all $x\in R$ is not μ-open, since we observe that (-1,1) is a μ-open set, but f((-1,1)) = [0,1) which is not μ-open, but f is obviously μ-closed.

Definition 3.2 Let X and Y be the strong generalized topological spaces with strong generalized topologies μ_1 and μ_2 respectively. A function f: X → Y is said to be ($\mu_1 - \mu_2$) continuous (or simplyμ-continuous if no confusion arise) if the inverse image of every μ_2-open set of Y is μ_1-open in X.

Example 3.3 Let X = {a, b, c}, μ_1 = {ϕ, X, {a, c},{b, c}} and let Y = {p, q, r}, μ_2 = {ϕ, Y, {p, r},{q, r}}. Define f: (X, μ_1) → (Y, μ_2) by f(a) = p, f(b) =q, f(c) = r. Then f is (μ_1 - μ_2)-continuous function.

Result 3.1 (i) A (μ_1 - μ_2)-continuous map need not send μ_1-open sets to μ_2-open sets.(ii) A μ-open map need not be μ-continuous. For, if f: (X, μ_1) → (Y, μ_2) is μ-open iff $\mu_1 \subset \mu_2$, but it is not (μ_1 - μ_2)-continuous whenever $\mu_1 \neq \mu_2$.(iii) Constant map is always μ-continuous, because the inverse image of any μ_2-open set in Y is either ϕ or X, which are μ_1-open in X.**Lemma 3.1** Let X and Y be SGTS. If f: X → Y is μ-continuous and μ-open then $f^{-1}(c_Y(A)) = c_X(f^{-1}(A))$ for every A ⊂Y.

Proof is easy and so is omitted.

Definition 3.3 A subset A of a space X is pre-μ-open denoted by PO(X, μ) if A $\subset i_\mu(c_\mu(A))$, and is regular μ-open denoted by RO(X, μ) if A = $i_\mu(c_\mu(A))$.

Definition 3.4 A function f: (X, μ_1) → (Y, μ_2) is said to be p-μ-open if f(A) ∈ PO(Y, μ_2) for all A ∈ PO(X, μ_1).

Definition 3.5 A function f: (X, μ_1)→ (Y, μ_2) is said to be pre-μ-continuous (resp. semi-μ-continuous) if the inverse image of every μ_2-open set of Y is pre -μ-open (resp. semi - μ -open) in X.

Theorem 3.1 If f: (X, μ_1)→ (Y, μ_2) is (μ_1-μ_2) continuous and μ-open then f is p-μ-open.

Proof: Let A ∈ PO(X, μ_1).Then A$\subset i_{\mu 1}(c_{\mu 1}$(A)).
By μ-openness and (μ_1-μ_2) continuity of f, it then follows that f(A)⊂f($i_{\mu 1}(c_{\mu 1}$(A)))⊂$i_{\mu 2}$(f($c_{\mu 1}$(A))) ⊂$i_{\mu 2}(c_{\mu 2}$(f(A))). This shows that f(A) ∈ PO(Y, μ_2).

Theorem 3.2 Let X, Y and Z be SGTS and f: (X, μ_1) → (Y, μ_2) and g: (Y, μ_2) → (Z, μ_3) are μ-continuous functions, then g ∘ f: (X, μ_1) → (Z, μ_3) is μ-continuous.

Proof: If G is μ_3-open in Z, then $g^{-1}(G)$ is μ_2-open in Y, by continuity of g. Hence by μ_1- continuity of f, $f^{-1}[g^{-1}(G)] = (g \circ f)^{-1}(G)$ is μ_1-open in X. Thus, g ∘ f is ($\mu_1 - \mu_3$)-continuous or simply μ-continuous.

Definition 3.6 Let X and Y be SGTS. If f: X → Y and A ⊂ X, the restriction of f to A denote the map of A into Y defined by (f/A)(a) = f(a) for each a ∈ A.

Lemma 3.2 If A⊂ X and f: X → Y is (μ_1 - μ_2) continuous, then (f/A): A → Y is (μ_1 - μ_2) continuous.

Proof: If G is μ_2-open in Y, then $(f/A)^{-1}(G) = f^{-1}(G) \cap A$. Clearly $f^{-1}(G) \cap A$ is μ_1-open in the subspace of strong generalized topology on A.

Lemma 3.3 If A is a subset of a space (X, μ), then
(i) $i_\mu(c_\mu(A)) \subset s.c_\mu(A)$
(ii) (ii) $i_\mu(s.c_\mu(A)) = i_\mu(c_\mu(A))$
Proof is trivial.

Theorem 3.3 Let A ⊂ (X, μ).
Then
(i) A ∈ PO(X, μ) if and only if $sc_\mu(A) = I_\mu(c_\mu(A))$
(ii) A ∈ PO(X, μ) if and only ifs.$c_\mu(A)$ ∈ RO(A, μ)
(iii) RO(X, μ) = PO(X, μ) ∩s.c_μ(X).

Proof: (i) Let A ∈ PO(X, μ). Then s.$c_\mu(A)$ ⊂s.$c_\mu(i_\mu(c_\mu(A)))$ and since $i_\mu(c_\mu(A))$ ∈s.c_μ(X), s.$c_\mu(A) \subset i_\mu(c_\mu(A)$. By lemma 3.3(i) s.$c_\mu(A) \supset i_\mu(c_\mu(A))$.The converse is obvious.
(ii) Let s.$c_\mu(A)$ ∈ RO(X, μ). Then s.$c_\mu(A) = i_\mu(c_\mu(s.c_\mu(A)))$
and hence s.$c_\mu(A) \subset i_\mu(c_\mu(c_\mu(A))) = i_\mu(c_\mu(A))$. By Lemma 3.3(i), it follows that s.$c_\mu(A) = i_\mu(c_\mu(A))$.By part(i), A∈ PO(X, μ). The converse follows from part (i)
(iii) This follows from part(i) and (ii)

Definition 3.7: Let X and Y be SGTS and let f: (X, μ_1) → (Y, μ_2). Then f is (μ_1 - μ_2)-continuous or simply μ-continuous at x_0∈ X if and only if for each μ_2-open set V of f(x_0) in Y, there is a μ_1-open set U containing x_0 in X such that f(U) ⊂ V. We say that f is continuous at X if and only if f is continuous at each x_i ∈ X. Clearly this definition is equivalent to the Definition 3.2.

Theorem 3.4 If X and Y are spaces and f: X → Y, then the following are equivalent:
(i) f is μ-continuous.
(ii) for each μ_2-open set H in Y, $f^{-1}(H)$ is μ_1-open in X.
(iii) for each μ_2-closed set K in Y, $f^{-1}(K)$ is μ_1-closed in X.
(iv) for each E ⊂ X, $f(c_{\mu_1}(E)) \subset c_{\mu_2}(f(E))$

Proof: (i) ⇒ (ii) Suppose H is μ_2-open in Y. Then for each x ∈ $f^{-1}(H)$, H is a μ_2-open set of f(x). Hence by μ-continuity of f, there is a μ_1-open set U of x such that f(U) ⊂ H; that is U ⊂ $f^{-1}(H)$. Thus, $f^{-1}(H)$ contains a μ_1-open set of each of its points and is therefore μ_1-open.

(ii) ⇒ (iii) Suppose K is μ_2-closed in Y. Then $f^{-1}(Y-K)$ is μ_1-open in X, by part(ii). Hence, since $f^{-1}(K) = X - f^{-1}(Y-K)$, $f^{-1}(K)$ is μ_1-closed in

X.

(iii) ⇒ (iv) Let K be any μ_2-closed set in Y containing f(E). By part(iii), $f^{-1}(K)$ is μ_1-closed set in X containing E. Hence, $c_{\mu_1}(E) \subset f^{-1}(K)$, and it follows that $f(c_{\mu_1}(E)) \subset K$. Since this is true for any μ_2-closed set K containing f(E), we have, $f(c_{\mu_1}(E)) \subset c_{\mu_2}(f(E))$

(iv) ⇒ (i) Let x ∈X and let V be a μ_2-open set of f(x). Set E = $X - f^{-1}(V)$ and let U = $X - c_{\mu_1}(E)$. It is easy to verify that, since $f(c_{\mu_1}(E)) \subset c_{\mu_2}(f(E))$, we have x Є U. It is even clear that f(U) ⊂ V. Hence f is (μ_1 - μ_2)-continuous or simply μ-continuous at x.

Lemma 3.4 If A and B are subsets of a space (X, μ), A ⊂ B ⊂ c_μ(A) and B ∈ PO(X, μ), then A ∈ PO(X, μ).

Proof: Since A ⊂ B ⊂ c_μ(A), we have s.c_μ(A) ⊂s.c_μ(B) ⊂ c_μ(A). Theorem 3.3 implies that s.c_μ(A) ⊂ i_μ(c_μ(B)) ⊂ c_μ(A) since B ∈ PO(X, μ). Since, i_μ(c_μ(A)) = i_μ(c_μ(B)), we have A∈PO(X,μ).

Theorem 3.5 If a function f:(X,μ_1)→(Y, μ_2) is pre-μ-open and pre-μ-continuous, then f is p-μ-open.

Proof: Let U ∈ PO(X, μ). Since f is semi-μ-continuous, f(U) ⊂ f(s.c_μ(U)) ⊂ c_μ(f(U)).

By theorem 3.3(ii), s.c_μ(U) ∈ RO(X, μ), so that $f(s.c_{\mu_1}(U))$ ∈ PO(Y, μ_2), because f is pre μ-open. Lemma 3.4 implies thatf(U) ∈ PO(Y, μ_2), and the result follows.

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