FINITE SERIES AND COMPLETE SOLUTION OF THIRD ORDER α-DIFFERENCE EQUATION

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Abstract: In this paper, we define third kind α-difference operator and its equation and derive the corresponding discrete version of Leibnitz's and Montmorte's Theorems. Also, we investigate the numerical and complete solutions of third kind α-difference equation for finding the values of the sum of second partial sums of various finite series on arithmetic- geometric progression in the field of finite difference methods.

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Introductionin 1984, Jerzy Popenda [1] introduced a particular type of difference operator Δ_α which is defined as

$$\Delta_\alpha u(k) = u(k+1) - \alpha u(k).$$

In 1989, Miller and Rose [2] introduced the discrete analogue of the Riemann-Liouville fractional derivative and proved some properties of the fractional difference operator. The general fractional h-difference Riemann- Liouville operator and its inverse $\Delta_h^{-\nu} f(t)$ were mentioned in [8, 9]. As an application of $\Delta_h^{-\nu}$ by taking $\nu = m$ (positive integer) and $h = \ell$, the sum of second partial sums of higher powers of arithmetic-geometric progression and products of n consecutive terms of arithmetic progression have been derived using $\Delta_\ell^{-3} u(k)$ [6]. In [10], M.M.S. Manuel, et.al., for any $\ell \in (0,\infty)$, the authors have extended from α-difference operator into the generalized α-difference operator as $\Delta_{\alpha(\ell)} v(k) = v(k+\ell) - \alpha v(k), \ell \in (0,\infty)$ (1) and obtained a numerical solution of the above equation in the form

$$v(k) = \Delta_{\alpha(\ell)}^{-1} u(k) \Big|_j^k = \sum_{r=1}^{[k/\ell]} \alpha^{r-1} u(k - r\ell). \qquad (2)$$

There are two type of solutions to the equation (1); one is numerical another one is closed form solution. If we are able to find a closed form solution of equation (1), which is coinciding with the numerical solution of that equation, then we can obtain formula for finding the values of several finite series. In this paper, we extend the theory and applications of $\Delta_{\alpha(\ell)}$ developed in [10] to generalized third kind α-difference equation

$$v(k+\ell_1+\ell_2+\ell_3) - \alpha[v(k+\ell_1+\ell_2) + v(k+\ell_1+\ell_3) + v(k+\ell_2+\ell_3)] + \alpha^2[v(k+\ell_1) + v(k+\ell_2) + v(k+\ell_3)] - \alpha^3 v(k) = u(k) \qquad (3)$$

where $\alpha, \ell_1, \ell_2, \ell_3 \in (0,\infty)$ and $k \in [0,\infty)$. Since $\Delta_{\alpha(\ell_i)} v(k) = v(k+\ell_i) - \alpha v(k)$, by defining $\Delta_{\alpha(\ell_1)}(\Delta_{\alpha(\ell_2)}(\Delta_{\alpha(\ell_3)}))$ as $\underset{L_3}{\Delta_\alpha}$, (3)can be expressed as

$$\underset{L_3}{\Delta_\alpha} v(k) = \sum_{t=0}^{3} \sum_{A \in t(J_3)} (-\alpha)^{3-t} v(k + \sum_{i \in A} \ell_i) \qquad (4)$$

where $L_3 = \{\ell_1, \ell_2, \ell_3\}$, $J_3 = \{1,2,3\}$ and $0(J_3) = \phi$ (empty set), $1(J_3) = \{\{1\},\{2\},\{3\}\}$, $2(J_3) = \{\{1,2\},\{1,3\},\{2,3\}\}$, $3(J_3) = \{1,2,3\}$. We call $\underset{L_3}{\Delta_\alpha}$ as third kind α-difference operator. Also investigate numerical-complete solution of (3) to obtain the sum of second partial sums of several finite series. When $\alpha = 1$, the results coincide with [5].

Preliminaries: Before stating and proving our results, we present some notations, basic definitions and preliminary results which will be used for further subsequent discussions. Let $\ell \in (0,\infty)$, $k \in [0,\infty)$, $\left[\frac{k}{\ell}\right]$ denotes the integer part of $\frac{k}{\ell}$, $j = k - \left[\frac{k}{\ell}\right]\ell$, $E^\ell v(k) = v(k+\ell)$ and $\alpha \neq 0$. The following lemmas can be obtained by simple algebraic derivations.

Lemma 2.1. Let $\alpha, \ell_1, \ell_2, \ell_3 \in (0,\infty)$. Then

$$(i)\ \underset{L_3}{\Delta_\alpha} = \sum_{t=0}^{3} \sum_{A \in t(J_3)} (-\alpha)^{3-t} E^{\sum_{i \in A} \ell_i}. \qquad (5)$$

$(ii)\ \underset{L_3}{\Delta_\alpha} = \sum_{t=0}^{3} \sum_{A\in t(J_3)} (-\alpha)^{3-t}$

$$\Delta_{\alpha(\sum_{i\in A}\ell_i)} + (\alpha - 3\alpha^2 + 3\alpha^3). \quad (6)$$

$(iii)\ \underset{L_3}{\Delta_\alpha} = \prod_{t=1}^{3}\sum_{s=1}^{\ell_t} \ell_t C_s \Delta^s - \alpha. \quad (7)$

Lemma 2.2. For the positive integer r,

$(i)\ \underset{L_3}{\Delta_\alpha} = \prod_{t=1}^{3}\left[\sum_{i=0}^{r} (-\alpha)^i rC_i E^{\ell_t(r-i)}\right]. \quad (8)$

$(ii)\ \underset{L_3}{\Delta_\alpha} = \prod_{t=1}^{3}\left[\sum_{i=0}^{r} (-\alpha)^i rC_i u(k+\ell_t(r-i))\right].$

$(iii)\ \underset{L_3}{\Delta_\alpha} = \prod_{t=1}^{3}\left[\prod_{i=1}^{n}(\Delta_{\alpha(\ell_{t_i})}+\alpha)-\alpha\right]. \quad (9)$

(iv) Let n be positive integer and $nL_3 = \{n\ell_1, n\ell_2, n\ell_3\}$. Then

$(a)\ \underset{L_3}{\Delta_\alpha} = \sum_{t=0}^{3}\sum_{A\in t(J_3)} (-\alpha)^{3-t} E^{n\sum_{i\in A}\ell_i}.$

$(b)\ \underset{L_3}{\Delta_\alpha} \sum_{t=0}^{3}\sum_{A\in t(J_3)} (-\alpha)^{3-t} (\alpha + \Delta_{\alpha(\sum_{i\in A}\ell_i)})^n.$

$(c)\ \underset{L_3}{\Delta_\alpha} \sum_{r=1}^{n} nC_r\alpha^{n-r} \sum_{t=0}^{3} (-\alpha)^{3-t} \sum_{A\in t(J_3)} \Delta^r_{\alpha(\sum_{i\in A}\ell_i)}.$

$(d)\ \underset{L_3}{\Delta_\alpha} \prod_{t=1}^{3}\left(\sum_{i=0}^{n-1}(-\alpha)^i nC_i \Delta_{(n-i)\ell_t}\right).$

Theorem 2.3. Let $u(k)$ and $v(k)$ be two real valued functions. Then

$$\underset{L_3}{\Delta_\alpha} u(k)v(k) = \sum_{r=0}^{n} (n-r)C_r \alpha^{n-r}$$

$$\Delta^{n-r}_{\alpha(\ell_1)}[\Delta^r_{\alpha(\ell_2)} u(k)\Delta^{n-r}_{\alpha(\ell_3)} v(k+r\ell_3)].$$

Proof: Proof follows from the generalized Leibnitz's Theorem (Theorem 2.5, [10]).

Corrolory 2.4. Let $u(k)$ and $v(k)$ be two real valued functions. Then

$$\Delta_{\alpha(\ell,\ell,\ell)}[u(k)v(k)] = \sum_{r=0}^{n} (n-r)C_r$$

$$\alpha^{n-r}\Delta^{n-r}_{\alpha(\ell)}[\Delta^r_{\alpha(\ell)}u(k)\Delta^{n-r}_{\alpha(\ell)}v(k+r\ell)].$$

Proof. The proof follows by taking $\ell_1 = \ell_2 = \ell_3 = \ell$ in Theorem 2.3.

Corollary 2.5. For the positive integer n, then

$$\sum_{t=1}^{3}\sum_{A\in t(J_3)} (-\alpha)^{3-t} E^{n\sum_{i\in A}\ell_i} = \sum_{r_1=1}^{n}\sum_{r_2=0}^{n} \alpha^{r_1+r_2}$$

$$nC_{r_1} r_1 C_{r_2} \sum_{t=1}^{3}\sum_{A\in t(J_3)} (-\alpha)^{3-t} \Delta_{\alpha(r_1-r_2)(\sum_{i\in A}\ell_i)}. \quad (10)$$

Theorem 2.6. For positive integers n and p,

$$\sum_{t=1}^{3}\sum_{A\in t(J_3)} (-\alpha)^{3-t}(k+n\sum_{i\in A}\ell_i)^p = \sum_{r=0}^{n}(-\alpha)^{n-r}$$

$$nC_r\sum_{s=0}^{r-1} rC_s \sum_{t=0}^{3}\sum_{A\in t(J_3)} (-\alpha)^{3+s-t}(k+(r-s)\sum_{i\in A}\ell_i)^p. \quad (11)$$

Proof. The proof follows by operating (10) on $u(k) = k^p$.

Lemma 2.7. Let $u(k)$ be real valued function and x is real. Then,

$$\sum_{j=0}^{\infty}\sum_{t=1}^{3}\sum_{A\in t(J_3)} (-\alpha)^{3-t} \frac{x^{j\sum_{i\in A}\ell_i} u(j\sum_{i\in A}\ell_i)}{j!(\sum_{i\in A}\ell_i)^j}$$

$$= \sum_{t=1}^{3}\sum_{A\in t(J_3)} (-\alpha)^{3-t} e^{\frac{\alpha x^{\sum_{i\in A}\ell_i}}{\sum_{i\in A}\ell_i}} e^{\frac{x^{\sum_{i\in A}\ell_i}\Delta_{\alpha(\sum_{i\in A}\ell_i)}}{\sum_{i\in A}\ell_i}} u(0).$$

Proof: The proof follows from (5), $u(k) = E^k u(0)$ and $E^\ell = \alpha + \Delta_{\alpha(\ell)}$.

Corollory 2.8. If ℓ is a positive real, then

$$\sum_{j=0}^{\infty}\sum_{t=1}^{3}(-\alpha)^{3-t}3C_t \frac{x^{jt\ell}u(jt\ell)}{j!(t\ell)^j}u(0)$$

$$= \sum_{t=1}^{3}(-\alpha)^{3-t}3C_t e^{\frac{\alpha x^{t\ell}}{t\ell}} e^{\frac{x^{t\ell}\Delta_{\alpha(t\ell)}}{t\ell}} u(0).$$

Numerical – Complete Solutions Of (3): In this section, we derive a numerical-complete solution of the equation (3) and obtain the value of several finite series in the finite difference methods.

Definition 3.1. A function $v(k)$ satisfying third kind α-difference equation (3) is called solution of that equation and is denoted as $v(k) = \underset{L_3}{\Delta_\alpha^{-1}} u(k)$.

Lemma 3.2. [5] If s_r^n and S_r^n are the Stirling numbers of the first and second kinds respectively and $k_r^{(n)} = k(k-\ell)\ldots(k-(n-1)\ell)$,

then $k_\ell^{(n)} = \sum_{r=1}^{n} s_r^n \ell^{n-r}k^r, k^n = \sum_{r=1}^{n} S_r^n \ell^{n-r}k_\ell^{(r)}$. (12)

Definition 3.3. If $0<\ell<k$ and

$$v(k) = \sum_{r=3}^{[k/\ell]} \frac{(r-2)^{(2)}}{2}\alpha^{r-3}u(k-r\ell) \quad (13)$$

satisfies the generalized third kind α-difference equation $v(k+3\ell) - 3\alpha v(k+2\ell) + 3\alpha^2 v(k+\ell)$

$$-\alpha^3 v(k) = u(k), \qquad (14)$$

then (13) is a numerical solution of (14).

Lemma 3.4. If $v(k)$ is any solution of the equation (14), then $w_3(k) = v(k) - \frac{1}{2}\left[\frac{k}{\ell}\right]^{(2)}$

$$\alpha^{\left[\frac{k}{\ell}\right]-2} c_{1j} - \left[\frac{k}{\ell}\right]\alpha^{\left[\frac{k}{\ell}\right]-1} c_{2j} - \alpha^{\left[\frac{k}{\ell}\right]} c_{3j}, \qquad (15)$$

where $c_{1j} = \Delta^{-1}_{\alpha(\ell)} v(j),\ c_{2j} = \Delta^{-1}_{\alpha(\ell,\ell)} v(j),$ $c_{3j} = \Delta^{-1}_{\alpha(\ell,\ell,\ell)} v(j)$ is also a solution of the equation (14). Now $w_3(k)$ is called as complete solution of the equation (14).

Lemma 3.5. Let P_k be any function of $k \in [0,\infty)$, then $v(k) = \frac{\lambda^k}{(\lambda^\ell - \alpha)^3}\sum_{t=0}^{\infty}(-1)^t \frac{(t+2)^{(t)}}{t!}$

$$\left[\frac{\lambda^\ell \Delta_\ell}{\lambda^\ell - \alpha}\right]^t P_k \Bigg|_j^k - \frac{\lambda^j}{(\lambda^\ell - \alpha)}\sum_{t=0}^{\infty}(-1)^t \left[\frac{\lambda^\ell \Delta_\ell}{\lambda^\ell - \alpha}\right]^t P_j$$

$$\left\{\left[\frac{k}{\ell}\right]^{(2)} \alpha^{\left[\frac{k}{\ell}\right]-2}\right\}\Bigg|_j^k - \frac{\lambda^j}{(\lambda^\ell - \alpha)^2}\sum_{t=0}^{\infty}(-1)^t \frac{(t+1)^{(t)}}{t!}$$

$$\left[\frac{\lambda^\ell \Delta_\ell}{\lambda^\ell - \alpha}\right]^t P_j \left\{\left[\frac{k}{\ell}\right]\alpha^{\left[\frac{k}{\ell}\right]-1}\right\}\Bigg|_j^k \qquad (16)$$

is a complete solution of the equation (14), for $u(k) = \lambda^k P_k$.

Proof. From the generalized α-difference operator, we find that

$$\Delta_{\alpha(\ell)}\lambda^k F_k = \lambda^{k+\ell}F_{k+\ell} - \alpha\lambda^k F_k = \lambda^k(\lambda^\ell E^\ell - \alpha)F_k = \lambda^k P_k$$

where $P_k = (\lambda^\ell E^\ell - \alpha)F_k$ (or) $F_k = (\lambda^\ell E^\ell - \alpha)^{-1}P_k$, which yields $\Delta^{-1}_{\alpha(\ell)}\lambda^k P_k = \lambda^k(\lambda^\ell E^\ell - \alpha)^{-1}P_k$

$$= \frac{\lambda^k}{\lambda^\ell - \alpha}\left[1 - \frac{\lambda^\ell}{\lambda^\ell - \alpha}\Delta_\ell + \frac{\lambda^{2\ell}}{(\lambda^\ell - \alpha)^2}\Delta_\ell^2 + \ldots\right]P_k$$

and taking the limits from j to k, we obtain

$$\Delta^{-1}_{\alpha(\ell)}\lambda^k P_k\,\Big|_j^k = \frac{\lambda^k}{\lambda^\ell - \alpha}\sum_{t=0}^{\infty}(-1)^t\left[\frac{\lambda^\ell}{\lambda^\ell - \alpha}\right]^t P_k\Bigg|_j^k$$

which is a complete solution of the generalized first kind α - difference equation $v(k+\ell) - \alpha v(k) = \lambda^k P_k$. Now taking $\Delta^{-1}_{\alpha(\ell)}$ in the above expression, we get

$$\Delta^{-1}_{\alpha(\ell,\ell)}\lambda^k P_k =$$

$$\frac{\lambda^k}{(\lambda^\ell - \alpha)^2}\sum_{t=0}^{\infty}(-1)^t \tfrac{(t+1)}{t!}\left[\frac{\lambda^\ell}{\lambda^\ell - \alpha}\right]^t P_k\Bigg|_j^k - \frac{\lambda^\ell}{\lambda^\ell - \alpha}$$

$$\sum_{t=0}^{\infty}(-1)^t\left[\frac{\lambda^\ell}{\lambda^\ell - \alpha}\right]^t P_j\left\{\left[\frac{k}{\ell}\right]\alpha^{\left[\frac{k}{\ell}\right]-1}\right\}\Bigg|_j^k. \qquad (17)$$

Now (16) follows by taking $\Delta^{-1}_{\alpha(\ell)}$ on (17) and applying the limit from j to k.

Lemma3.6. If $k \geq 3\ell$, then $v(k) = \sum_{r=3}^{\left[k/\ell\right]}\frac{(r-2)^{(2)}}{2}\alpha^{r-3}\lambda^{k-r\ell}P_{k-r\ell}$ is a numerical solution of (14) for $u(k) = \lambda^k P_k$. *Proof.* From the inverse of generalized α - difference operator of first kind and taking the limits from j to k, we find

$$\Delta^{-1}_{\alpha(\ell)}\lambda^k P_k\Big|_j^k = \sum_{r=1}^{\left[k/\ell\right]}\alpha^{r-1}\lambda^{k-r\ell}P_{k-r\ell}$$

which is a numerical solution of the generalized first kind α -difference equation $v(k+\ell) - \alpha v(k) = \lambda^k P_k$. Again by taking $\Delta^{-1}_{\alpha(\ell)}$ on the above expression and applying the limit from j to k, we get

$$\Delta^{-1}_{\alpha(\ell,\ell)}\lambda^k P_k\Big|_j^k = \sum_{r=1}^{\left[k/\ell\right]}(r-1)\alpha^{r-2}\lambda^{k-r\ell}P_{k-r\ell}. \qquad (18)$$

Again and again by taking $\Delta^{-1}_{\alpha(\ell)}$ on (18), applying the limits from j to k and using Lemma 3.3, we get proof.

Theorem 3.7. Let P_k be any function of $k \in [0,\infty)$, then

$$\sum_{r=3}^{\left[k/\ell\right]}\frac{(r-2)^{(2)}}{2}\alpha^{r-3}\lambda^{k-r\ell}P_{k-r\ell} = \frac{\lambda^k}{(\lambda^\ell - \alpha)^3}\sum_{t=0}^{\infty}(-1)^t$$

$$P_j\left\{\left[\frac{k}{\ell}\right]^{(2)}\alpha^{\left[\frac{k}{\ell}\right]-2}\right\}\Bigg|_j^k - \frac{\lambda^j}{(\lambda^\ell - \alpha)^2}\sum_{t=0}^{\infty}(-1)^t$$

$$\frac{(t+1)^{(t)}}{t!}\left[\frac{\lambda^\ell \Delta_\ell}{\lambda^\ell - \alpha}\right]^t P_j\left\{\left[\frac{k}{\ell}\right]\alpha^{\left[\frac{k}{\ell}\right]-1}\right\}\Bigg|_j^k. \qquad (19)$$

Proof. The proof follows by (17) and (18).

The following theorem is the formula for sum of second partial sums on product of consecutive terms of arithmetic-geometric progression.

Theorem 3.8. Consider all the terms in Theorem 3.7.

Then,

$$\sum_{r=3}^{[k/\ell]}\frac{(r-2)^{(2)}}{2}\alpha^{r-3}\lambda^{k-r\ell}(k-r\ell)_\ell^{(n)}=\frac{\lambda^k}{(\lambda^\ell-\alpha)^3}\sum_{t=0}^{\infty}(-1)^t\frac{(t+2)^{(t)}}{t!}n^{(t)}\ell^t\left[\frac{\lambda^\ell\Delta_\ell}{\lambda^\ell-\alpha}\right]^t k_\ell^{(n-t)}\Bigg|_j^k$$
$$-\frac{\lambda^j}{(\lambda^\ell-\alpha)}\sum_{t=0}^{\infty}(-1)^t n^{(t)}\ell^t\left[\frac{\lambda^\ell\Delta_\ell}{\lambda^\ell-\alpha}\right]j_\ell^{(n-t)}\left\{\left[\frac{k}{\ell}\right]^{(2)}\alpha^{\left[\frac{k}{\ell}\right]-2}\right\}\Bigg|_j^k$$
$$-\frac{\lambda^j}{(\lambda^\ell-\alpha)^2}\sum_{t=0}^{\infty}(-1)^t\frac{(t+1)^{(t)}}{2t!}n^{(t)}\ell^t\left[\frac{\lambda^\ell\Delta_\ell}{\lambda^\ell-\alpha}\right]^t j_\ell^{(n-t)}\left\{\left[\frac{k}{\ell}\right]\alpha^{\left[\frac{k}{\ell}\right]-1}\right\}\Bigg|_j^k. \qquad (20)$$

Proof. The proof follows by taking $P_k=k_\ell^{(n)}$ in (19).
Following example illustrates Theorem 3.8.

Example 3.9. Taking n = 3 in (20), we get

$$\left[\frac{\lambda^\ell\Delta_\ell}{\lambda^\ell-\alpha}\right]j_\ell^{(n-t)}\left\{\left[\frac{k}{\ell}\right]^{(2)}\alpha^{\left[\frac{k}{\ell}\right]-2}\right\}\Bigg|_j^k-\frac{\lambda^j}{(\lambda^\ell-\alpha)^2}$$
$$\sum_{t=0}^{\infty}(-1)^t\frac{(t+1)^{(t)}}{2t!}n^{(t)}\ell^t\left[\frac{\lambda^\ell\Delta_\ell}{\lambda^\ell-\alpha}\right]^t j_\ell^{(n-t)}\left\{\left[\frac{k}{\ell}\right]\alpha^{\left[\frac{k}{\ell}\right]-1}\right\}\Bigg|_j^k-\left[\frac{\lambda^\ell\Delta_\ell}{\lambda^\ell-\alpha}\right]^t$$
$$j_\ell^{(3-t)}\left\{\left[\frac{k}{\ell}\right]^{(2)}\alpha^{\left[\frac{k}{\ell}\right]-2}\right\}\Bigg|_j^k-\frac{\lambda^j}{(\lambda^\ell-\alpha)^2}\sum_{t=0}^{3}\frac{(t+1)^{(t)}}{2}$$
$$\frac{(-1)^t3^{(t)}\ell^t}{t!}\left[\frac{\lambda^\ell\Delta_\ell}{\lambda^\ell-\alpha}\right]^t j_\ell^{(3-t)}\left\{\left[\frac{k}{\ell}\right]^{(2)}\alpha^{\left[\frac{k}{\ell}\right]-2}\right\}\Bigg|_j^k.(21)$$

In Particular, $k=31,\ell=3,\alpha=5,\lambda=3$ and $j=1$ in (21), we get

$$\sum_{r=3}^{10}(r-2)^{(2)}5^{r-3}3^{31-3r}(31-3r)_3^{(3)}=\sum_{t=0}^{3}(-1)^t\frac{(t+2)^{(t)}(3)^{(t)}}{t!}$$
$$\frac{3^t3^{31+3t}}{(3^3-5)^{t+3}}(31)_1^{31}-\sum_{t=0}^{3}(-1)^t\frac{(t+1)^{(t)}(3)^{(t)}3^t3^{1+3t}}{2t!(3^3-5)^{t+2}}(1)_3^{3-t}$$
$$\left\{\left[\frac{k}{3}\right]5^{\left[\frac{k}{3}\right]-1}\right\}\Bigg|_1^{31}-\sum_{t=0}^{3}(-1)^t\frac{(3)^{(t)}3^t3^{1+3t}}{(3^3-5)^{t+1}}(1)_3^{3-t}$$
$$\left\{\left[\frac{k}{3}\right]^{(2)}5^{\left[\frac{k}{3}\right]-2}\right\}\Bigg|_1^{31}=135814023031416.$$

Following is the formula for sum of second partial sums of higher powers of arithmetic-geometric progressions.

Theorem 3.10. Consider all the terms in Theorem 3.7. Then,

$$\sum_{r=3}^{[k/\ell]}\frac{(r-2)^{(2)}}{2}\alpha^{r-3}\lambda^{k-r\ell}(k-r\ell)^n=\sum_{t=0}^{n}\frac{(-1)^t(t+2)^{(2)}n^{(t)}}{2(\lambda^\ell-\alpha)^{t+3}}$$
$$\ell^t\lambda^{k+t\ell}\sum_{i=1}^{n-t}S_i^{n-t}\ell^{n-t-i}k_\ell^{(i)}\Bigg|_j^k-\sum_{t=0}^{n}(-1)^t\frac{(t+1)}{t!}$$
$$\frac{n^{(t)}\ell^t\lambda^{k+t\ell}}{(\lambda^\ell-\alpha)^{t+2}}\sum_{i=1}^{n-t}S_i^{n-t}\ell^{n-t-i}j_\ell^{(i)}\left\{\left[\frac{k}{\ell}\right]\alpha^{\left[\frac{k}{\ell}\right]-1}\right\}\Bigg|_j^k-\sum_{t=0}^{n}(-1)^t$$
$$\frac{n^{(t)}\ell^t\lambda^{k+t\ell}}{(\lambda^\ell-\alpha)^{t+1}}\sum_{i=1}^{n-t}S_i^{n-t}\ell^{n-t-i}j_\ell^{(i)}\left\{\left[\frac{k}{\ell}\right]^{(2)}\alpha^{\left[\frac{k}{\ell}\right]-2}\right\}\Bigg|_j^k. \qquad (22)$$

Proof. Taking $P_k=k^n$ in (19) and using (12), we get proof.

Corollary 3.11. If $\alpha\neq a^\ell, k\in[\ell,\infty)$, then

$$\sum_{r=3}^{[k/\ell]}\frac{(r-2)^{(2)}}{2}\alpha^{r-3}a^{k-r\ell}(k-r\ell)=\frac{ka^k}{(a^\ell-\alpha)^3}-$$
$$\frac{3\ell a^{k+\ell}}{(a^\ell-\alpha)^4}-\frac{1}{2}\left\{\frac{ja^j}{(a^\ell-\alpha)}-\frac{\ell a^{j+\ell}}{(a^\ell-\alpha)^2}\right\}\left\{\left[\frac{k}{\ell}\right]^{(2)}\alpha^{\left[\frac{k}{\ell}\right]-2}\right\}\Bigg|_j^k$$
$$-\left\{\frac{ja^j}{(a^\ell-\alpha)^2}-\frac{2\ell a^{j+\ell}}{(a^\ell-\alpha)^3}\right\}\left\{\left[\frac{k}{\ell}\right]\alpha^{\left[\frac{k}{\ell}\right]-1}\right\}\Bigg|_j^k. \qquad (23)$$

Proof. The proof follows by taking n=1, λ=a in (22).
Following example illustrates Corollary3.11.

Example 3.12. Taking $k=28,\ell=3,a=7$ and $j=1$ in (23), we obtain

$$\sum_{r=3}^{[k/3]}\frac{(r-2)^{(2)}}{2}5^{r-3}7^{k-r\ell}(k-r\ell)=\frac{k7^k}{(338)^3}-$$
$$\frac{3.3.7^{k+3}}{(338)^4}\Bigg|_1^{28}-\frac{1}{2}\left\{\frac{7}{(338)}-\frac{3.7^4}{(338)^2}\right\}\left\{\left[\frac{k}{3}\right]^{(2)}\right.$$
$$\left.5^{\left[\frac{k}{3}\right]-2}\right\}\Bigg|_1^{28}-\left\{\frac{7}{(338)^2}-\frac{2.7^4}{(338)^3}\right\}\left\{\left[\frac{k}{3}\right]5^{\left[\frac{k}{3}\right]-1}\right\}\Bigg|_j^k$$
$$=224747431048034482.$$

Following theorem illustrates Theorem 3.8.

Theorem 3.13. Let $k\in[0,\infty)$, m is a positive integer and $\alpha\neq1$.
Then,

$$\sum_{r=3}^{[k/\ell]}\frac{(r-2)^{(2)}}{2}\alpha^{r-3}(k-r\ell)_\ell^{(m)}=\sum_{i=0}^{m}\frac{(-1)^i(i+2)^{(2)}}{2(1-\alpha)^{i+3}}m^{(i)}\ell^i$$
$$k_\ell^{(m-i)}\Big|_j^k-\sum_{i=0}^{m}(-1)^i\frac{(i+1)m^{(i)}\ell^ij_\ell^{(m-i)}}{(1-\alpha)^{i+2}}\left\{\left[\frac{k}{\ell}\right]\alpha^{\left[\frac{k}{\ell}\right]-1}\right\}\Bigg|_j^k$$

$$-\sum_{i=0}^{m}(-1)^i \frac{m^{(i)}\ell^i j_\ell^{(m-i)}}{(1-\alpha)^{i+1}}\left\{\left[\frac{k}{\ell}\right]^{(2)}\alpha^{\left[\frac{k}{\ell}\right]-2}\right\}\Bigg|_j^k. \quad (24)$$

Proof. Taking λ = 1 in (20) we get proof.

Corollary 3.14. Let $k\in[0,\infty)$ and m = 3. Then,

$$\sum_{r=3}^{\left[k/\ell\right]}\frac{(r-2)^{(2)}}{2}\alpha^{r-3}(k-r\ell)_\ell^{(3)}=\sum_{i=0}^{3}\frac{(-1)^i}{2}\frac{(i+2)^{(2)}3^{(i)}\ell^i k_\ell^{(3-i)}}{(1-\alpha)^{i+3}}\Bigg\}\Bigg|_j^k-\sum_{i=0}^{3}(-1)^i\frac{(i+1)3^{(i)}\ell^i j_\ell^{(3-i)}}{(1-\alpha)^{i+2}}$$

$$\left\{\left[\frac{k}{\ell}\right]\alpha^{\left[\frac{k}{\ell}\right]-1}\right\}\Bigg|_j^k-\sum_{i=0}^{3}(-1)^i\frac{3^{(i)}\ell^i j_\ell^{(3-i)}}{(1-\alpha)^{i+1}}\left\{\left[\frac{k}{\ell}\right]^{(2)}\alpha^{\left[\frac{k}{\ell}\right]-2}\right\}\Bigg|_j^k. \quad (25)$$

Following is an illustration of Corollary3.14.

Example 3.15. By taking $k=161,\ell=4,\ \alpha=2$ and $j=1$ in (25), we obtain

$$\sum_{r=3}^{[40]}\frac{(r-2)^{(2)}}{2}2^{r-3}(161-4r)_4^{(3)}=\sum_{i=0}^{3}(-1)^i(i+2)^{(2)}3^{(i)}\frac{\ell^i k_4^{(3-i)}}{2(-1)^{i+3}}\Bigg\}\Bigg|_1^{161}-\sum_{i=0}^{3}(-1)^i\frac{(i+1)3^{(i)}4^i 1_\ell^{(3-i)}}{(-1)^{i+2}}$$

$$\left\{\left[\frac{k}{4}\right]2^{\left[\frac{k}{4}\right]-1}\right\}\Bigg|_1^{161}-\sum_{i=0}^{3}(-1)^i\frac{3^{(i)}4^i 1_4^{(3-i)}}{(-1)^{i+1}}\left\{\left[\frac{k}{4}\right]^{(2)}\alpha^{\left[\frac{k}{4}\right]-2}\right\}\Bigg|_1^{161}=57800226755682615.$$

Theorem 3.16. Consider all the terms in Theorem 3.7. Then

$$\sum_{r=3}^{\left[k/\ell\right]}\frac{(r-2)^{(2)}}{2}\alpha^{r-3}(k-r\ell)^n=\sum_{t=0}^{n}(-1)^t\frac{(t+2)^{(2)}}{2}\frac{n^{(t)}\ell^t}{(1-\alpha)^{t+3}}\sum_{i=1}^{n-t}S_i^{n-t}\ell^{n-t-i}k_\ell^{(i)}\Bigg|_j^k-\sum_{t=0}^{n}(-1)^t\frac{(t+1)^{(2)}}{(1-\alpha)^{t+2}}$$

$$\frac{n^{(t)}\ell^t}{t!}\sum_{i=1}^{n-t}S_i^{n-t}\ell^{n-t-i}j_\ell^{(i)}\left\{\left[\frac{k}{\ell}\right]\alpha^{\left[\frac{k}{\ell}\right]-1}\right\}\Bigg|_j^k-\sum_{t=0}^{n}\frac{n^{(t)}\ell^t}{(1-\alpha)^{t+1}}$$

$$\sum_{i=1}^{n-t}S_i^{n-t}\ell^{n-t-i}j_\ell^{(i)}\left\{\left[\frac{k}{\ell}\right]^{(2)}\alpha^{\left[\frac{k}{\ell}\right]-2}\right\}\Bigg|_j^k. \quad (26)$$

Proof. Taking $P_k=k^n$ in (22), we get proof.

Following corollary illustrates Theorem3.16.

Corollary 3.17. Let $k\in[0,\infty),\alpha\neq 1$. Then

$$\sum_{r=3}^{\left[k/\ell\right]}\frac{(r-2)^{(2)}}{2}\alpha^{r-3}(k-r\ell)^3=\sum_{t=0}^{3}(-1)^t\frac{3^{(t)}\ell^t}{(1-\alpha)^{t+3}}\sum_{i=1}^{3-t}S_i^{3-t}\ell^{n-t-i}k_\ell^{(i)}\Bigg|_j^k-\sum_{t=0}^{3}\frac{(-1)^t(t+2)^{(2)}}{2}$$

$$\frac{(t+1)^{(2)}3^{(t)}\ell^t}{t!(1-\alpha)^{t+2}}\sum_{i=1}^{3-t}S_i^{3-t}\ell^{3-i}j_\ell^{(i)}\left\{\left[\frac{k}{\ell}\right]\alpha^{\left[\frac{k}{\ell}\right]-1}\right\}\Bigg|_j^k\sum_{t=0}^{3}(-1)^t$$

$$\frac{3^{(t)}\ell^t}{(1-\alpha)^{t+1}}\sum_{i=1}^{3-t}S_i^{3-t}\ell^{3-i}j_\ell^{(i)}\left\{\left[\frac{k}{\ell}\right]^{(2)}\alpha^{\left[\frac{k}{\ell}\right]-2}\right\}\Bigg|_j^k. \quad (27)$$

Proof. Proof follows by taking n = 3 in (26).

Following is an illustration of Corollary3.17.

Example 3.18. Taking $k=211,\ell=7,\alpha=5$ and j = 1 in (27), we obtain

$$\sum_{r=3}^{\left[k/7\right]}\frac{(r-2)^{(2)}}{2}5^{r-3}(k-7r)^3=\sum_{t=0}^{3}(-1)^t(t+2)^{(2)}\sum_{t=0}^{3}(-1)^t\frac{3^{(t)}7^t}{(-4)^{t+3}}\sum_{i=1}^{3-t}S_i^{3-t}7^{3-i}k_3^{(i)}\Bigg|_1^{211}-\sum_{t=0}^{3}(-1)^t\frac{(t+1)3^{(t)}7^t}{(-4)^{t+2}}$$

$$\sum_{i=1}^{3-t}S_i^{3-t}7^{3-i}1_7^{(i)}\left\{\left[\frac{k}{7}\right]\alpha^{\left[\frac{k}{7}\right]-1}\right\}\Bigg|_j^k-$$

$$\sum_{t=0}^{3}(-1)^t\frac{3^{(t)}7^t}{(-4)^{t+1}}\sum_{i=1}^{3-t}S_i^{3-t}7^{3-i}1_7^{(i)}\left\{\left[\frac{k}{7}\right]^{(2)}\alpha^{\left[\frac{k}{7}\right]-2}\right\}\Bigg|_1^{211}$$

$$=98199107014806941135770 5.$$

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