OSCILLATORY BEHAVIOR OF SECOND ORDER SUBLINEAR DIFFERENCE EQUATION

V.CHANDRESEKAR, A.BENEVATHO JAISON

Abstract: In this paper, the authors present a new oscillation criterion for second order sublinear difference equation where is positive function has no restriction on its sign, are such that for.

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Introduction: Difference equations usually described the evaluation of some certain phenomena over time and are also important in describing dynamically discrete system, see [4]. For example, in the numerical integration, the standard approach is to use the difference equations. Similarly, the population dynamics have discrete generations; the size of the (k+1)st generation is a function of the kth generation u(k). This can be expressed as difference equation of the form

$$u(k+1) = f(u(k)),$$

see for example [2]. Further, the concept of difference equations with many examples in applications such as asymptotic behavior of solutions of difference equations were studied extensively by Elaydi [9] where the analytic and geometric approaches were also combined in order to studying difference equations. Further, in [4], both classical and modern treatments of the difference equations were presented in excellent form. For related results on difference equations, see [1-4].

Recently, there has been increasing interest in the study of oscillatory behaviors to the solutions of linear and nonlinear difference equations. The reference [11,12,14,19] concern with the oscillation of non-linear difference equations. The reference [13,15,16,18,20] studied respectively the oscillation of sublinear, super linear and half linear difference equations. Hence, the purpose of this paper is to present two new oscillation criteria for second-order sublinear difference equation

$$\Delta[a(k)\psi(u(k))\Delta u(k)] + q(k)f(u(k)) = 0, \quad (1)$$

where (i) $\alpha \in C^1([k_0,\infty); (0,\infty))$,

(ii) $q \in C([k_0,\infty);R)$, has no restriction on its sign,

(iii) $\psi \in C(R,R), \psi(u) > 0$ for $u \neq 0$ (iv) $f \in C^1(R,R)$ satisfies

$$ku(k) > 0, \ \Delta f(u) \geq 0 \text{ for } x \neq 0. \quad (2)$$

It will be supposed that equation (1) is strongly sublinear in the sense that

$$\sum_{r=1}^{x} \frac{\psi(-x-r)}{f(-x-r)} < \infty \text{ and } \sum_{r=1}^{x} \frac{\psi(-x+r)}{f(-x+r)} < \infty.$$

By the solution of (1), we shall always mean a nontrivial function u(k) that satisfies (1) for all $k \in [0,\infty)$. A solution u(k) of (1) is called oscillatory if for any $K \in [0,\infty)$ there exist an $k \geq K$ such that $u(k)u(k+\ell) \leq 0$. Otherwise it is called nonoscillatory. i.e., a solution of (1) is called nonoscillatory if it is eventually negative or positive.

The problem of finding oscillation criteria for second-order superlinear and sublinear difference equations, respectively which involve the average behavior of the integral of the alternating coefficient, has received the attention of some authors [16,17]. Both oscillation results established here involve the average behavior of the integral of the alternating coefficient q. These criteria are based on the idea of the use of a class of the parameter functions $H(k,s)$. Recently, some authors proceeds further in this direction and extended those results to the more general difference equation with a damped term

$$\Delta[a(k)\psi(u(k))\Delta u(k)] + p(k)\Delta u(k) + q(k)f(u(k)) = 0.$$

Throughout this paper, we define the constant $M_{f,\psi}$ is defined as follows.

$$M_{f,\psi} = \min\left\{ \frac{\inf_{u>0}\left(\Delta f(u)\Phi(u)/\psi(u)\right)}{1+\inf_{u>0}\left(\Delta f(u)\Phi(u)/\psi(u)\right)}, \frac{\inf_{u<0}\left(\Delta f(u)\Phi(u)/\psi(u)\right)}{1+\inf_{u<0}\left(\Delta f(u)\Phi(u)/\psi(u)\right)} \right\},$$

where,

$$\Phi(u(k)) = \begin{cases} \sum_{r=1}^{x} \frac{\psi(x-r)}{f(x-r)}, & u > 0 \\ \sum_{r=1}^{x} \frac{\psi(-x+r)}{f(-x+r)}, & u < 0. \end{cases} \quad \mathbf{D}$$

Definition 1.1. A function $f(n, y_1, \ldots, y_m)$ is said to be strongly sublinear if there exists constants $\beta \in (0,1)$, β is a quotient of odd positive integers and $d>0$ such that $|y|^{-\beta}|f(n,y,...,y)|$ is nonincreasing in $|y|$ for $0<|y|\le d$.

Theorem 1.2.

Let $H:D=\{(k,s)\,|\,k\ge s\ge k_0\}\to R$ be a continuous together with partial derivative with respect to the second variable, satisfying the conditions

$$H(k,k)=0,\ \text{for}\ k\ge k_0,$$

$$H(k,s)>0,\ \text{for}\ k>s\ge k_0 \tag{3}$$

$$\frac{\partial H(k,s)}{\partial s}=0,\ \text{for}\ k\ge k_0,$$

$$\frac{\partial H(k,s)}{\partial s}\le 0,\ \text{for}\ (k,s)\in D \tag{4}$$

$$\frac{\partial}{\partial s}\left(a(s)\frac{\partial H(k,s)}{\partial s}\right)\ge 0,\ \ \text{for}\ (k,s)\in D \tag{5}$$

If

$$\liminf_{k\to\infty}\frac{\frac{\partial H(k,s)}{\partial s}}{H(k,s)}>-\infty,\ \ \text{for every}\ s\ge k_0. \tag{6}$$

there exists a positive function $\rho\in C^2([k_0,\infty))$, with $\Delta(a(k)\Delta\rho(k))\le 0$ such that for some $\alpha\in[0,M_{f,\psi}]$,

$$\limsup_{k\to\infty}\frac{1}{H(k,k_0)}\int_{k_0}^{k}H(k,s)\rho^{\alpha}(s)q(s)=\infty.$$ Then equation (1) is oscillatory.

In the case where $a(k)\equiv 1$ and $\psi(u)\equiv 1$, (1) reduces to the equation

$$\Delta^2 u+q(k)f(u)=0. \tag{7}$$

Taking $H(k,s)=(k-s)^{(m)}$ for some integer $m\ge 1$, which obviously satisfies conditions (3)-(6), Theorem 1.2 reduces to theorem 1 of continuous analogue in [10] for equation (7).

Theorem 1.3. Suppose that

$$I_f=\min\left\{\begin{array}{l}\dfrac{\inf_{u>0}\Delta f(u)F(u)}{1+\inf_{u>0}\Delta f(u)F(u)},\\[2ex] \dfrac{\inf_{u<0}\Delta f(u)F(u)}{1+\inf_{u<0}\Delta f(u)F(u)}\end{array}\right\}>0,$$

where $F(u(k))=\sum_{r=1}^{k}\frac{1}{f(k-r)},k>0,$

$\sum_{r=1}^{k}\frac{1}{f(-k+r)},k<0.$ Let m be an integer $m\ge 1$ and $\rho\in C^2([t_0,\infty))$ be a positive function such that $(\Delta\rho)^2\le c\rho(\Delta^2-\rho)$, on $N(k_0)$, where c is a positive constant.

Equation (7) is oscillatory if there exists a continuous function on $N(k_0)$ with

$$\sum_{k_0}^{\infty}\frac{\varphi^2{}_+(s)}{s}=\infty, \tag{8}$$

where $\varphi_+(s)=\max\{\varphi(s)\},s\ge k_0$ and $\limsup_{k\to\infty}\sum_{s=K}^{k}(k-s)^{(m)}[\rho(s)]^{I_f}q(s)\ge\varphi(K)$, for every $K\ge k_0$.

In particular, by taking $m=1$ then the previous theorem reduces to theorem 1 of continuous analogue in [12].

In this paper, we proceed further in direction of Theorem 1.2 and present two oscillation theorems which include as particular cases some previous theorems, as Theorem 1.3.

Throughout this paper, we denote a real number $k\ell+j$ by k.

Main Results:

Theorem 2.1 Let (i) $\rho\in C^2([k_0,\infty))$ be a positive function such that

$$a(k)[\Delta\rho(k)]^2\le c\Delta[a(k)\Delta\rho(k)]\rho(k), \tag{9}$$

for any $k\ge k_0$, for some positive constant c,

(ii) $H(k,s)$ be a function of second order differnce on D with respect to the second variable which satisfies conditions (3)-(6)

$$\text{and}\ \ 0<\inf_{s\ge k_0}\left[\liminf_{k\to\infty}\frac{H(k,s)}{H(k,k_0)}\right]\le\infty, \tag{10}$$

$$\sum_{k_0}^{\infty}s\Omega^2(s)<\infty,$$

$$\Omega(s)=\limsup_{k\to\infty}\frac{-\Delta_s H(k,s)}{H(k,s)},\ s\ge k_0 \tag{11}$$

Then equation (1) is oscillatory if there exist a function $\varphi\in C([k_0,\infty))$ such that condition (8) holds and

$$\limsup_{k\to\infty}\frac{1}{H(k,K)}\sum_{K}^{k}H(k,s)\eta^{\beta}(s)q(s)\ge\varphi(K), \tag{12}$$

for every $K\ge k_0$ and some $\alpha\in[0,M_{f,\psi}]$.

Proof. Let $u(k)$ be a nonoscillatory solution of the difference equation (1) and let $K_0\ge k_0$ be such that

$u(k) \neq 0$ for all $K \geq k_0$. Furthermore, we define

$$w(k) = \rho^{\alpha}(k)\phi(u(k)),\ k \geq K_0. \qquad (13)$$

Then, we obtain for $k \geq K_0$

$$\Delta w(k) = \rho^{\alpha}(k)\frac{\psi(k)}{f(k)} + w(k+1) - \phi(u(k))\rho^{\alpha}(k)$$

(14)

which implies

$$\Delta^2 w(k) = \rho^{\alpha}(k+1)\Delta\frac{\psi(k)}{f(k)} - \rho^{\alpha}(k)\left[\frac{\psi(k)}{f(k)} + \phi(u(k+1))\right] + w(k+2).$$

Let $\frac{\psi(k)}{f(k)} + \phi(u(k+1)) \geq 1$, for $k \neq 0$.

Then, we have

$$\Delta^2 w(k) \leq \rho^{\alpha}(k+1)\Delta\frac{\psi(k)}{f(k)} - \rho^{\alpha}(k) + w(k+2).$$

Thus, for every k, K and $k \geq K \geq K_0$, we get

$$-\sum_{K}^{k} H(k,s)\Delta\left(a(s)\Delta w(s)\right)$$
$$= [H(k,K)a(k)\Delta w(k)] - a(k)\Delta_s H(k,K)\, w(k+1) - \sum_{K}^{k} w(s+2)\Delta_s\left[a(s)\Delta_s H(k,s)\right]_K^k$$
$$= [H(k,K)a(k)\Delta w(k)] + a(k)\Delta_s H(k,K) w(k+1) - \sum_{K}^{k} w(s+2)\Delta_s\left[a(s)\Delta_s H(k,s)\right]_K^k$$

for

$$-\sum_{K}^{k} q\rho^{\alpha}(k+1) \qquad (15)$$

every $k \geq K \geq K_0$, we have

$$-\frac{1}{H(k,K)}\sum_{K}^{k} H(k,s)\Delta\left(a(s)\Delta w(s)\right) = \frac{1}{H(k,K)}[H(k,K)a(k)\Delta w(k)] + \frac{1}{H(k,K)}a(k)\Delta_s H(k,K) w(k+1) - \frac{1}{H(k,K)}\sum_{K}^{k} w(s+2)\Delta_s\left[a(s)\Delta_s H(k,s)\right]_K^k - \frac{1}{H(k,K)}\sum_{K}^{k} q\rho^{\alpha}(k+1) \qquad (16)$$

Thus by condition (12), we get for $K \geq K_0$

$$a(k)\Delta w(k) - a(k)w(k+1)\liminf_{k\to\infty}\frac{\Delta_s H(k,s)}{H(k,K)} \geq \liminf_{k\to\infty}\frac{1}{H(k,K)}\sum_{K}^{k} w(s+2)\Delta_s[a(s)\Delta_s H(k,s)] + \phi(k) + \liminf_{k\to\infty}\frac{1}{H(k,K)}\sum_{K}^{k} H(k,s)a(s)\rho^{\alpha}(k) + \liminf_{k\to\infty}\frac{1}{H(k,K)}\sum_{K}^{k} H(k,s)a(s)w(s+2).$$

and, we conclude that for every $T \geq T_o$

$$\liminf_{k\to\infty}\frac{1}{H(k,K)}\sum_{K}^{k} w(s+2)\Delta_s[a(s)\Delta_s H(k,s)] < \infty \qquad (17)$$

$$\liminf_{k\to\infty}\frac{1}{H(k,K)}\sum_{K}^{k} H(k,s)a(s)\rho^{a}(k) < \infty \quad (18)$$

$$\liminf_{k\to\infty}\frac{1}{H(k,K)}\sum_{K}^{k} (k,s)a(s)w(s+2) < \infty \quad (19)$$

$$\varphi(k) \leq a(k)\Delta w(k) + a(k)w(k+1)\Omega(k). \quad (20)$$

Hence,

$$\liminf_{k\to\infty}\frac{1}{H(k,K)}\sum_{K}^{k} w(s+2)\Delta_s[a(s)\Delta_s H(k,s)] \geq \liminf_{k\to\infty}\frac{1}{H(k,K)}\sum_{K}^{k} w(s+2)a(s)\Delta_s^2 H(k,s) + \liminf_{k\to\infty}\frac{1}{H(k,K)}\sum_{K}^{k} w(s+2)\Delta_s H(k,s)\Delta_s a(s),$$

so that, by (19), we have

$$\liminf_{k\to\infty}\sum_{K}^{k}\frac{w(s+2)a(s+1)\Delta_s^2 H(k,s)}{H(k,K)} < \infty. \quad (21)$$

Because of (19) and (21) there exist an integer valued function $\tau(k), k \in N(0)$ in $N(K_0)$ with $\lim_{k\to\infty}\tau(k) = \infty$ and such that

$$\lim_{k\to\infty}\sum_{K_0}^{\tau(k)}\frac{H(\tau(k),s)a(s)w(s+2)}{H(\tau(k),K_0)} < \infty \qquad (22)$$

$$\lim_{k\to\infty}\sum_{K_0}^{\tau(k)}\frac{w(s+2)a(s+1)\Delta_s^2 H(\tau(k),s)}{H(\tau(k),K_0)} < \infty \quad (23)$$

Now, we shall establish that

$$\limsup_{k\to\infty}\frac{a(k+1)\, w(k+2)}{k} < \infty. \qquad (24)$$

Let us consider an arbitrary positive constant μ, by condition (10), and a constant ρ with

$$\inf_{s\geq k_0}\left[\liminf_{k\to\infty}\frac{H(k,s)}{H(k,k_0)}\right]>\rho>0. \qquad (25)$$

Suppose that (24) fails. Then there exists a $K_1 \geq K_0$ such that

$$\frac{a(k+1)\,\mathrm{w}(k+2)}{k} \geq \frac{\mu}{\rho} \text{ for all } k \geq K_1$$

Thus, we obtain for $k \geq K_1$

$$\frac{1}{H(k,K_0)}\sum_{K_0}^{k}\mathrm{w}(s+2)a(s+1)\Delta_s^2 H(k,s)$$

$$\geq\frac{\mu}{\rho H(k,K_0)}\sum_{K_0}^{k}s\Delta_s^2 H(k,s)$$

$$\geq\frac{-\mu}{\rho H(k,K_0)}\left[K_1\Delta_s H(\mathrm{k},K_1)-H(\mathrm{k},K_1+1)\right]$$

$$\geq\frac{\mu H(\mathrm{k},K_1+1)}{\rho H(k,\mathrm{k}_0)}.$$

Since (25) ensures that

$$\liminf_{k\to\infty}\frac{H(\mathrm{k},K_1+1)}{H(k,\mathrm{k}_0)}>\rho$$

there exists a $K_2 \geq K_1+1$ such that

$$\frac{H(\mathrm{k},K_1+1)}{H(k,\mathrm{k}_0)}\geq\rho \text{ for every } k \geq K_2 \qquad (26)$$

Thus,

$$\frac{1}{H(k,K_0)}\sum_{K_0}^{k}\mathrm{w}(s+2)a(s+1)\Delta_s^2 H(k,s)\geq\mu$$

for all $k \geq K_2$ and consequently, for all sufficiently large k,

$$\frac{\sum_{K_0}^{\tau(k)}\mathrm{w}(s+2)a(s+1)\Delta_s^2 H(\tau(k),s)}{H(\tau(k),K_0)}\geq\mu.$$

Since $\mu > 0$ is arbitrary, we prove that

$$\lim_{k\to\infty}\frac{\sum_{K_0}^{\tau(k)}\mathrm{w}(s+2)a(s+1)\Delta_s^2 H(k,s)}{H(\tau(k),K_0)}=\infty,$$

which contradicts (23). So we proved (24).

Next, we shall prove that

$$\sum_{K_0}^{\infty}a(s)\,\mathrm{w}(s+2)<\infty. \qquad (27)$$

Suppose on the contrary, that there exists a $K_1 \geq K_0$ such that

$$\sum_{K_0}^{k}a(s)\,\mathrm{w}(s+2)\geq\frac{\mu}{\rho}, \quad \text{for every } k \geq K_1,$$

where μ is arbitrary positive constant. Then, for all $k>K_1$ we have

$$\frac{1}{H(k,K_0)}\sum_{K_0}^{k}H(k,s)a(s)w(s+2)$$

$$=\frac{1}{H(k,K_0)}\sum_{K_0}^{k}\left(-\Delta_s H(k,s)\right)\left(\sum_{k_0}^{s}w(\tau+2)a(\tau)\right)$$

$$\geq\frac{\mu\sum_{K_1}^{k}\left(-\Delta_s H(k,s)\right)}{\rho H(k,K_0)}\geq\frac{\mu H(k,K_1+1)}{\rho H(k,K_0)}.$$

By (26), we get,

$$\frac{1}{H(k,K_0)}\sum_{K_0}^{k}H(k,s)a(s)w(s+2)\geq\mu,$$

for all $k>K_2$, so that for all sufficiently large k, we have

$$\frac{\sum_{K_0}^{\tau(k)}H(\tau(k),s)a(s)w(s+2)}{H(\tau(k),K_0)}\geq\mu,$$

which contradicts (22), and hence, prove (27). By similar arguments, using (17), we can prove

$$\sum_{K_0}^{\infty}w(s)a(s)\rho^{a}(k)<\infty. \qquad (28)$$

Further, $\Delta\rho(k)$ is nonincreasing function, so that we have for $k>K_0$

$$\rho(k)-\rho(K_0)=\sum_{K_0}^{k}\Delta\rho(s)>(k-K_0)\Delta\rho(k)$$

which ensures that

$$\limsup_{k\to\infty}\frac{k\Delta\rho(k)}{\rho(k)}<\infty. \qquad (29)$$

Using (9), we derive, for every $k\geq K_0$,

$$\sum_{K_0}^{k}w(s+2)a(s)=\sum_{K_0}^{k}\frac{\left[\Delta w(s+1)\right]^2}{w(s+2)}a(s)$$

$$-\frac{w^2(s+1)}{w(s+2)}a(s)+2w(s+1)a(s)$$

$$\geq\sum_{K_0}^{k}\frac{\left[\Delta w(s+1)\right]^2}{w(s+2)}a(s)+(c+2)w(s+1)a(s).$$

Therefore,

$$\sum_{K_0}^{\infty}\frac{\left[\Delta w(s+1)\right]^2}{w(s+2)}a(s)$$
$$\le\sum_{K_0}^{k}w(s+2)a(s)-\sum_{K_0}^{k}(c+2)w(s+1)a(s),$$

which, since of (24), (27)-(29), ensures that

$$\sum_{K_0}^{\infty}\frac{\left[\Delta w(s+1)\right]^2}{w(s+2)}a(s)<\infty. \qquad (30)$$

Finally, by using (20), we have

$$\sum_{K_0}^{\infty}\frac{\left[\varphi_+(s+1)\right]^2}{s}$$
$$\le\sum_{K_0}^{\infty}a^2(s)\frac{\left[\Delta w(k+1)+w(k+2)\Omega(k)\right]^2}{s}$$
$$\le M\sum_{K_0}^{\infty}a(s)\frac{\left[\Delta w(k+1)+w(k+2)\Omega(k)\right]^2}{w(s+2)}$$
$$\le 2M\sum_{K_0}^{\infty}a(s)\frac{\Delta^2 w(s+1)}{w(s+2)}$$
$$+2M\sum_{K_0}^{\infty}a(s)w(s+2)\Omega^2(\mathrm{s})$$
$$\le 2M\sum_{K_0}^{\infty}a(s)\frac{\Delta^2 w(s+1)}{w(s+2)}+2M^2\sum_{K_0}^{\infty}s\Omega^2(s),$$

where $M=\sup_{k=K_0}a(k)w(k+2)/k$ and by (24), M is finite. Thus, because of (30) and (11), we come to contradiction with assumption (8).

Example 2.2. Consider the equation

$$\Delta\left[\frac{u^2(k)}{k}\Delta\,\mathrm{u(k)}\right]+k^{\lambda}\cos k[|\,\mathrm{u}\,|^{(\alpha+2)}\,\mathrm{sgn(k)}$$
$$+u^{(3)}(k)]=0,\qquad \alpha\in\{0,1\}, \qquad (31)$$

Then, for all $u\neq 0$ $uf(u)>0$,

$$\Delta f(u)=(\alpha+2)|\,\mathrm{u}\,|^{(\alpha+1)}+3\mathrm{u}^{(2)}>0$$

Further, for every $u\neq 0$

$$\Phi(\mathrm{u})=\sum_{r=1}^{k}\frac{1}{u^{(\alpha)}(k-r)+u(k-r)}$$
$$\le\sum_{r=1}^{k}\frac{1}{u^{(\alpha+1)}(k-r)}=\frac{1}{(1-\alpha)\mathrm{u}^{(\alpha-1)}}$$

and consequently,

$$\inf_{u>0}\phi(\mathrm{u})\frac{\Delta f(\mathrm{u})}{\psi(\mathrm{u})}=\inf_{u<0}\phi(\mathrm{u})\frac{\Delta f(\mathrm{u})}{\psi(\mathrm{u})}$$
$$\le\inf_{u>0}\frac{[(\alpha+2)|\,\mathrm{u}\,|^{(\alpha+1)}+3\mathrm{u}^{(2)}]}{u^{(2)}(1-\alpha)(\mathrm{u}-2)^{(\alpha-1)}}$$
$$\le\inf_{u>0}\frac{(\alpha+2)(\mathrm{u}-2)^{(\alpha-1)}+3}{(1-\alpha)(\mathrm{u}-2)^{(\alpha-1)}}<\frac{(\alpha+2)}{(1-\alpha)}.$$

On the other hand, for $u\neq 0$, we derive

$$\Phi(\mathrm{u})\frac{\Delta f(\mathrm{u})}{\psi(\mathrm{u})}=\sum_{r=1}^{k}\frac{1}{u^{(\alpha)}(k-r)+u(k-r)}$$
$$\ge\sum_{r=1}^{k}\frac{1}{2u^{(\alpha)}(k-r)}=\frac{1}{2(1-\alpha)},$$

if $|\,k\,|\le 1$ and

$$\Phi(\mathrm{u})\frac{\Delta f(\mathrm{u})}{\psi(\mathrm{u})}=\sum_{r=1}^{k}\frac{1}{u^{(\alpha)}(k-r)+u(k-r)}$$
$$\ge\sum_{r=1}^{k}\frac{1}{u^{(\alpha)}(k-r)+u(k-r)}$$
$$\ge\sum_{r=1}^{k}\frac{1}{2u^{(\alpha)}(k-r)}=\frac{(\alpha+2)}{2(1-\alpha)}$$

if $|\,k\,|\ge 1$ and therefore, for $u\neq 0$ we have

$$\Phi(\mathrm{u})\frac{\Delta f(\mathrm{u})}{\psi(\mathrm{u})}$$
$$\ge\frac{[(\alpha+2)|\,\mathrm{u}\,|^{(\alpha+1)}+3\mathrm{u}^{(2)}]}{2(1-\alpha)(\mathrm{u}-2)^{(\alpha-1)}u^{(2)}(k)}>\frac{(\alpha+2)}{2(1-\alpha)}$$

for $|\,k\,|\le 1$ and

$$\Phi(\mathrm{u})\frac{\Delta f(\mathrm{u})}{\psi(\mathrm{u})}\ge\frac{[(\alpha+2)|\,\mathrm{u}\,|^{\alpha+1}+3\mathrm{u}^{(2)}]}{2(1-\alpha)(\mathrm{u}-2)^{(\alpha-1)}u^{(2)}(k)}$$
$$>\frac{3}{2(1-\alpha)}>\frac{(\alpha+2)}{2(1-\alpha)}$$

for $|\,k\,|\ge 1$. Hence,

$$\inf_{u>0}\phi(\mathrm{u})\frac{\Delta f(\mathrm{u})}{\psi(\mathrm{u})}=\inf_{u<0}\phi(\mathrm{u})\frac{\Delta f(\mathrm{u})}{\psi(\mathrm{u})}=\frac{(\alpha+2)}{2(1-\alpha)}.$$

Accordingly,

$$M_{f,\psi}\ge\frac{(\alpha+2)/2(1-\alpha)}{1+(\alpha+2)/(1-\alpha)}=\frac{\alpha+2}{6},$$

so that we can take $\beta=(\alpha+2)/6$.

Now, we can choose $\rho(\mathrm{k})=\mathrm{k}^{\mu}$, where we take μ such that $0<\mu<\min\left\{1,\dfrac{6\lambda}{\alpha+2}\right\}.$

Then, we have that $0<\mu<1$, so that the function ρ is concave and moreover condition (9) is satisfied for arbitrary constant c such that $c\ge\mu/(1-\upsilon-\mu)$.

Further, by taking $H(\mathrm{k,s})=(\mathrm{k-s})^{(2)}$ for $k\ge s\ge k_0$, for every $K\ge k_0$, we obtain

$$\limsup_{k\to\infty}\frac{1}{k^2}\sum_K^k(\mathrm{k-s})^{(2)}\,\mathrm{s}^{(\delta)}\cos\mathrm{s}$$
$$\ge -K^{(\delta)}\sin\mathrm{K}+\mathrm{T}^{(\delta)},\quad \delta=\lambda+\mu\frac{\alpha+2}{6}.$$

Notice that according to selection of the number μ is $\delta<0$. Then there exists a $k_1\ge k_0$ such that for $K\ge k_1$,

$$\limsup_{k\to\infty}\frac{1}{k^2}\sum_K^k(\mathrm{k-s})^{(2)}\,\mathrm{s}^{(\delta)}\cos\mathrm{s}\ge -K^{\delta}\sin\mathrm{K}-\varepsilon.$$

Now, set $\psi(\mathrm{K})=-\mathrm{K}^{(\delta)}\sin K-\varepsilon$ and consider an integer N such that $2N\pi+5\pi/4\ge\max\{\mathrm{k}_1,(1+\sqrt{2}\varepsilon)^{1/\delta}\}$. Then, for all integers $n\ge N$, we have $\varphi(\mathrm{K})\ge\frac{1}{\sqrt{2}}$, for every

$$K\in\left[2n\pi+\frac{5\pi}{4},2n\pi+\frac{7\pi}{4}\right],$$

which implies

$$\sum_{k_0}^{\infty}\frac{\varphi_+^2(s)}{s}\ge\sum_{n=N}^{\infty}\frac{1}{2}\sum_{2n\pi+\frac{5\pi}{4}}^{2n\pi+\frac{7\pi}{4}}\frac{1}{s}$$
$$\ge\frac{1}{2}\sum_{n=N}^{\infty}\ln\left(1+\frac{\pi/2}{2n\pi+\frac{5\pi}{4}}\right)=\infty.$$

Accordingly, all conditions of Theorem 2.1 are satisfied, and hence, equation (31) is oscillatory.

Theorem 2.2. Let (i) $\rho\in C^2([k_0,\infty))$ be a positive function such that $\Delta\rho(k)>0$ and

$\Delta\left[a(k)\Delta\rho(k)\right]\le 0$, for every $\mathrm{k}\ge\mathrm{k}_o$, (32)

(ii) $H(k,s)$ be a second order difference function on D with respect to the second variable which satisfies conditions (10), (11). Then equation (1) is oscillatory if there exists a function $\phi\in N(k_0)$ such that (12) holds for every $K\ge k_0$ and some $\beta\in\left[0,M_{f,\psi}\right]$ and

$$\lim_{k\to\infty}sup\left[\sum_{K_0}^{k}\left(\frac{\Delta\rho(s)}{\rho(s)}\right)^2 s\right]^{-1}\sum_{K_0}^{k}\frac{\varphi_+^2(s)}{s}=\infty.\quad(33)$$

Proof. Let $u(k)$ be a solution of the difference equation (1) on $N(K_0)$, $K_0\ge k_0$ with $u(k)\ne 0$ for all $k\ge K_0$. Let $w(k)$ be define by (1). Then, as an in the proof of the theorem 2.1, we derive (20), (24), (27), (29). Furthermore, for every $k\ge K_0$, we obtain

$$\sum_{K_0}^{k}a(s)\frac{\left[\Delta w(s+1)\right]^2}{w(s+2)}\le\sum_{K_0}^{K}a(s)w(s+2)$$
$$-\sum_{K}^{k}(c+2)a(s)w(s+1)=M\sum_{K}^{k}s\left(1-(c+2)\right)$$

where $M=sup_{k\ge K_0}\left(a(k)w(k+2)/k\right)$.

Accordingly, by taking into account (24), (27) - (29), we conclude that there exists a positive constant k such

$$\sum_{K_0}^{k}a(s)\frac{\left[\Delta w(s+1)\right]^2}{w(s+2)}\le k\sum_{k_0}^{k}s\ ,k\ge K_0.\quad(34)$$

By (20) and (34), for $k\ge K_0$ we have

$$\sum_{\mathrm{k}_0}^{\infty}\frac{\left[\varphi_+(s+2)\right]^2}{s}\le\sum_{\mathrm{k}_0}^{K_0}\frac{\varphi_+(s+2)}{s}+\sum_{K_0}^{k}\frac{\varphi_+(s+2)}{s}$$
$$\le\sum_{\mathrm{k}_0}^{K_0}\frac{\varphi_+(s+2)}{s}$$
$$+\sum_{K_0}^{k}a^2(s)\frac{\left[\Delta w(s+1)+w(s+2)\Omega(s)\right]^2}{s}$$
$$\le\sum_{\mathrm{k}_0}^{K_0}\frac{\varphi_+(s+2)}{s}$$
$$+2M\sum_{K_0}^{k}a(s)\frac{\left[\Delta^2 w(s+1)\right]^2}{w(s+2)}+2M^2\sum_{K_0}^{k}\mathrm{s}\Omega^2(s),$$

which for all $\mathrm{k}\ge K_0$ implies

$$\left\{\sum_{K}^{k}s\right\}^{-1}\sum_{k_0}^{k}\frac{\left[\varphi_+(s+2)\right]^2}{s}\le\left\{\sum_{K}^{K_0}s\right\}^{-1}\sum_{k_0}^{K_0}\frac{\varphi_+(s+2)}{s}$$
$$+2M\sum_{K_0}^{k}a(s)\frac{\left[\Delta^2 w(s+1)\right]^2}{w(s+2)}+2M^2\sum_{K_0}^{k}\mathrm{s}\Omega^2(s),$$

and therefore, by (11) and (13) gives a desired contradiction.

Example 2.2. Consider the equation

$$\Delta\left(\frac{u^2(\mathrm{k})\Delta u(k)}{k}\right)+k\left(k+\frac{\ln^2 k}{2}\right)^{-\lambda} \qquad (35)$$
$$\sin k\left[|\,\mathrm{u(k)}\,|^{(\alpha+2)}\,\mathrm{sgn}\,u+u^{(3)}(k)\right]=0$$

for $k \ge k_0 > e,$ where $\alpha \in \{0,1\}$ and $\lambda \le (\alpha+2)/6$. Then, we can take $H(\mathrm{k,s})$ as in Example 2.1 and $\beta=\lambda$ since $\lambda \le M_{f,\psi}$. We define $\rho(k)=k+\ln^2 k/2$ and observe that (32) is fulfilled, while condition (9) is not. Moreover, for any $k \ge K \ge k_0$, we have

$$\sum_K^k H(\mathrm{k,s})\rho^{(\beta)}(\mathrm{s})\mathrm{q(s)} = \sum_K^k (\mathrm{k-s})^{(2)} s\sin s$$
$$=(\mathrm{k-K})^{(2)} K\cos K - k^{(2)}\sin K$$
$$+2k(2\,\mathrm{KsinK}+\mathrm{cosk}+2\,\mathrm{cosK})$$
$$-6\,\mathrm{KcosK}-6\,\mathrm{sink}+6\,\mathrm{sinK}-3\,\mathrm{K}^{(2)}\sin K$$

and consequently,

$$\limsup_{k\to\infty}\frac{1}{H(\mathrm{k,K})}\sum_K^k H(\mathrm{k,s})\rho^{(\beta)}(\mathrm{s})q(\mathrm{s})$$
$$\ge K^{(\beta)}\,\mathrm{cosK}-\mathrm{sinK} \ge \mathrm{KcosK}-2,$$

Thus, (12) holds with $\varphi(\mathrm{K})=\mathrm{KcosK}-2,\ K \ge k_0.$ We consider a number k_1 such that $k_1 \ge \max\{\mathrm{k}_0, 4\sqrt{2}\}$. Next, we choose an integer N such that $2N\pi-\pi/4 \ge k_1$, so, that for every integer $n \ge N$, we obtain $\varphi(\mathrm{K}) \ge \frac{K}{2\sqrt{2}}$, for $K \in \left[2n\pi-\frac{\pi}{4}, 2n\pi+\frac{\pi}{4}\right].$

Then, for $n \ge N,$ we get

$$\sum_{k_0}^{2n\pi+\frac{7\pi}{4}}\frac{\varphi_+^2(s)}{s} \ge \sum_{2n\pi-\frac{\pi}{4}}^{2n\pi+\frac{\pi}{4}}\frac{\varphi_+^2(s)}{s} \ge \frac{1}{s}\sum_{2n\pi-\frac{\pi}{4}}^{2n\pi+\frac{\pi}{4}} s = \frac{\pi^2 n}{8}$$

and therefore,

$$\limsup_{k\to\infty}\frac{1}{\ln k}\sum_{k_0}^{k}\frac{\varphi_+^2(s)}{s}$$
$$\ge \limsup_{k\to\infty}\frac{1}{\ln(2\,n\,\pi+\pi/4)}\sum_{2n\pi-\frac{\pi}{4}}^{2n\pi+\frac{\pi}{4}}\frac{\varphi_+^2(s)}{s} \ge \frac{1}{s}\sum_{2n\pi-\frac{\pi}{4}}^{2n\pi+\frac{\pi}{4}} s=\infty.$$

condition (33) is satisfied if the following condition hold $\limsup_{k\to\infty}\frac{1}{\log k}\sum_{k_0}^{k}\frac{\varphi_+^2(s)}{s}=\infty.$

Consequently, equation (35) is oscillatory by the application of Theorem 2.2.

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Department of mathematics,SKP Engineering College,
Tiruvannamalai-606 611,Tamilnadu, S.India.
drchanmaths@gmail.com,benejaison@gmail.com