OSCILLATORY BEHAVIOUR OF SOLUTIONS OF CERTAIN TYPE OF THIRD ORDER GENERALIZED MIXED NEUTRAL DIFFERENCE EQUATION

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Abstract: In this paper, we discuss the oscillatory and asymptotic properties of the mixed type third order generalized neutral difference equation of the form

$$\Delta_\ell\left(a(k)\Delta_\ell^2\left(u(k)+b(k)u(k-\tau_1\ell)+c(k)u(k+\tau_2\ell)\right)\right)+q(k)u^\beta(k+\ell-\sigma_1\ell)$$
$$+p(k)u^\beta(k+\ell+\sigma_2\ell)=0 \qquad (1)$$

where a(k),b(k), c(k), q(k) and p(k) are positive real valued functions, β is a ratio of odd integers, τ_1,τ_2,σ_1 *and* σ_2 are positive integers. We establish some sufficient conditions which ensure that all solutions are either oscillatory or converges to zero. Some examples are inserted to illustrate the main results.

Keywords: Asymptotic, Neutral difference equation, Oscillation, Nonoscillation.
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Introduction: The theory of difference equations based on the operator Δ defined as

$$\Delta u(k)=u(k+1)-u(k),\ k\in N, \qquad (2)$$

where $N=\{1,2,3,....\}$. Even though many authors [2 , 14 - 15] have suggested the definition of Δ as $\Delta u(k)=u(k+\ell)-u(k),\ k\in[0,\infty),\ell\in(0,\infty)$ (3) no significant progress took place on this line. But recently, when we took up the definition of Δ as given in the (3) and developed the theory of difference equation in a different direction and obtained some interesting result in the application of number theory. For convenience we labelled the operator Δ defined by (3) as Δ_ℓ and by defining its inverse Δ_ℓ^{-1} many interesting result on number theory were obtained . By extending the study for sequence of complex number and ℓ to be real, some new qualitative properties like rotator, expanding and shrinking, spiral, and weblike were studied for the solution of difference equations involving Δ_ℓ. The results obtained can be found in [10 - 12].

Recently M.Maria Susai Manuel, et. al., have extended the definition of Δ_ℓ by $\Delta_{\ell_1,\ell_2,\ell_3}$ is defined as

$$\Delta_{\ell_1,\ell_2,\ell_3}u(k)=u(k+\ell_1+\ell_2+\ell_3)-$$
$$\left[u(k+\ell_1+\ell_2)+u(k+\ell_1+\ell_3)+u(k+\ell_2+\ell_3)\right]$$
$$+\left[u(k+\ell_1)+u(k+\ell_2)+u(k+\ell_3)\right]-u(k), \quad (4)$$

for the real valued function $u(k)$ and $\ell_i\in(0,\infty),\ i=1,2,3$ and also derive the sum of second partial sum of higher powers of consecutive terms of arithmetic progression and sum of second partial sum of higher powers of geometric progression by using the solution of the generalised difference equation [13].

In [7], J.R.Graef worked on the oscillation and Non oscillation of the solution of arbitrary order of differential equation and E.Thandapani obtained the discrete analogous of arbitrary order difference equation [8]. In [9], Grace considered the third order mixed type neutral difference equation is of the form

$$\Delta^3(x_n+ax_{n-m}-bx_{n_{+k}})\pm(qx_{n-g}+px_{n+h})=0 \quad (5)$$

and established some sufficient conditions for the oscillation of all solutions of equation (5). Also Agarwal, Grace and Bohner extended the m[th] order neutral type difference equation

$$\Delta^m(x_n+ax_{n-k}+bx_{n_{+\sigma}})+(qx_{n-g}+px_{n+h})=0 \quad (6)$$

and obtained some oscillation theorem for oscillation of all solution of equation (6).

In [8] the author considered $\ell=1$ and $k\in N(a)$ for an integer 'a' but in this paper, the theory is extended for all real $k\in[0,\infty),a\in[0,\infty)$ and for any real ℓ and oscillatory behaviour of solution of certain type of third mixed generalized neutral difference equation (1) is discussed.

Main Results: In this section, we present some new oscillation criteria for equation (1). For the sake of convenience, when we write a functional inequality without specifying its domain of validity we assume that it holds for all sufficiently large k.

We begin with the following lemmas which are crucial in the proof of the main results. For simplicity, we use the following notations:

(i) $z_\ell(k)=u(k)+b(k)u(k-\tau_1\ell)+c(k+\tau_2\ell)$, (ii) $R(k)=Q(k)+P(k)$,

$(iii)Q(k)=\min\left\{q(k),q(k-\tau_1\ell),q(k+\tau_2\ell)\right\}$,

$(iv) P(k) = \min\{p(k), p(k-\tau_1\ell), p(k+\tau_2\ell)\}$,

$(v) \eta(k) = \left(\frac{d}{4}\right)^{\beta-1} \frac{\gamma(k-\sigma_1\ell)^\beta}{2^\beta} R(k)$ for some $\gamma \in (j,\ell)$ *and* $d > 0$.

Throught this paper, we denote k means $k\ell + j$, $\ell \in (0,\infty)$, $0 \le j = k - \left[\frac{k}{\ell}\right]\ell < \ell.$

Lemma 2.1. Assume $A \ge 0, B \ge 0, \beta \ge 1$. Then $(A+B)^\beta \le 2^{\beta-1}(A^\beta + B^\beta)$.

Proof. The proof of the lemma is simple and so omitted.

Lemma 2.2. Let $u(k)$ be a positive solution of (1). Then there are only two cases for $k \ge k_1 \in [0,\infty)$ sufficiently large

(1) $z_\ell(k) > 0, \Delta_\ell z_\ell(k) > 0, \Delta_\ell^2 z_\ell(k) > 0,$
$\Delta_\ell(a(k)\Delta_\ell^2 z_\ell(k)) \le 0$

(2) $z_\ell(k) > 0, \Delta_\ell z_\ell(k) < 0, \Delta_\ell^2 z_\ell(k) > 0,$
$\Delta_\ell(a(k)\Delta_\ell^2 z_\ell(k)) \le 0$

Proof. Let $u(k)$ be a positive solution of equation (1). Then there exist a real number $k_1 \ge k_0$ such that $u(k) > 0, u(k-\sigma_1\ell) > 0, u(k+\sigma_2\ell) > 0,$ $u(k-\tau_1\ell) > 0$ *and* $u(k+\tau_2\ell) > 0$ for all $k \ge k_1$. Then $z_\ell(k) > 0$ *for all* $k \ge k_1$.

It follows from equation (1) that

$$\Delta_\ell(a(k)\Delta_\ell^2 z_\ell(k)) = -q(k)u^\beta(k+\ell-\sigma_1\ell) -p(k)u^\beta(k+\ell+\sigma_2\ell) < 0, \quad k \ge k_1. \quad (7)$$

Hence, $a(k)\Delta_\ell^2 z_\ell(k)$ is strictly decreasing for all $k \ge k_1$. We claim that $\Delta_\ell^2 z_\ell(k) > 0$ for all $k \ge k_1$. If not, then there is a real $k_2 \ge k_1$ and $M < 0$ such that

$$a(k)\Delta_\ell^2 z_\ell(k) \le a(k_2)\Delta_\ell^2 z_\ell(k) \le M, \quad k \ge k_2.$$

Summing this from k_2 *to* $k-1$, we have

$$\Delta_\ell z_\ell(k) \le \Delta_\ell z_\ell(k) + M\sum_{s=k_2}^{k-1} \frac{1}{a(s)}.$$

Letting $k \to \infty$, then $\Delta_\ell z_\ell(k) \to -\infty$. Thus there exists an integer $k_3 \ge k_2$ such that $\Delta_\ell z_\ell(k) < 0$ for all $k \ge k_1$. This implies that $\Delta_\ell z_\ell(k) \to -\infty$ *as* $k \to \infty$, a contradiction. Hence $\Delta_\ell^2 z_\ell(k) > 0$ for $k \ge k_3$. This completes the proof.

Lemma2.3. Let $z_\ell(k) > 0,$ $\Delta_\ell z_\ell(k) > 0, \Delta_\ell^2 z_\ell(k) > 0, \Delta_\ell^3 z_\ell(k) \le 0$ for all $k \ge K \in [0,\infty)$. Then for any $\gamma \in (j,l)$, and for some real K_1, one has

$$\frac{z_\ell(k+\ell)}{\Delta_\ell z_\ell(k)} \ge \frac{(k-K)}{2} \ge \frac{\gamma k}{2} \quad for\, k \ge K_1 \ge K. \quad (8)$$

Proof. Since $\Delta_\ell z_\ell(k) = \Delta_\ell z_\ell(K) + \sum_{s=k_2}^{k-1} \Delta_\ell^2 z_\ell(s),$

we have $\Delta_\ell z_\ell(k) \ge (k-K)\Delta_\ell^2 z_\ell(k).$

Summing the last inequality from K to $k-1$, we have

$z_\ell(k) \ge z_\ell(K) + (k-K)\Delta_\ell z_\ell(k) - z_\ell(k) + z_\ell(K)$ or $\frac{z_\ell(k+\ell)}{\Delta_\ell z_\ell(k)} \ge \frac{(k-K)}{2} \ge \frac{\gamma k}{2}, for\ k \ge K_1 \ge K.$ The proof is now completed.

Lemma 2.4. Let $u(k)$ be a positive solution of (1) and the corresponding $z_\ell(k)$ satisfy Lemma 2.2 of (2). If

$$\sum_{k=k_0}^{\infty}\sum_{s=k}^{\infty}\left[\frac{1}{a(s)}\sum_{t=s}^{\infty}(q(t)+p(t))\right] = \infty \quad (9)$$

holds, then $\lim_{k\to\infty} u(k) = 0.$

Proof. Let $u(k)$ be a positive solution of equation (1). Since $z_\ell(k) > 0$ and $\Delta_\ell z_\ell(k) < 0,$ *then* $\lim_{k\to\infty} z_\ell(k) = L \ge 0$ exists. We shall prove that L = 0. Assume that $L > 0$. Then for any $\in > 0$, we have $L + \in > z_\ell(k)$ eventually. Choose $0 < \in < \frac{L(1-b-c)}{b+c}$. From (1), we have

$$\begin{aligned} u(k) &= z_\ell(k) - b(k)u(k-\tau_1\ell) - c(k)u(k+\tau_2\ell) \\ &> L - (b+c)z_\ell(k-\tau_1\ell) \\ &> L - (b+c)(L+\in) = \gamma(L+\in) > \gamma z_\ell(k), \end{aligned}$$

where $\gamma = \frac{L-(b+c)(L+\in)}{(L+\in)} > 0.$ Using the above inequality, in (7), we arrive

$$\Delta_\ell(a(k)\Delta_\ell^2 z_\ell(k)) \le -q(k)\gamma^\beta z_\ell^{\ \beta}(k+\ell-\sigma_1\ell) - p(k)\gamma^\beta z_\ell^{\ \beta}(k+\ell+\sigma_2\ell) \le -\gamma^\beta (q(k)+p(k)) z_\ell^{\ \beta}(k+\ell+\tau_1\ell).$$

Summing this inequality from k to ∞, and using $z_\ell(k) \ge L$, we obtain

$$\Delta_\ell^2 z_\ell(k) \ge (\gamma L)^\beta \left[\frac{1}{a(k)}\sum_{s=k}^{\infty}(p(s)+q(s))\right].$$

Summing again from k to ∞, we obtain

$$-\Delta_\ell z_\ell(k) \ge (\gamma L)^\beta \sum_{s=k}^{\infty}\frac{1}{a(s)}\sum_{t=s}^{\infty}(p(t)+q(t)).$$

Summing from k_1 to ∞, we obtain $z_\ell(k_1) \ge (\gamma L)^\beta \sum_{k=k_1}^{\infty}\sum_{s=k}^{\infty}\left[\frac{1}{a(s)}\sum_{t=s}^{\infty}(p(t)+q(t))\right]$. This contradicts (9). Then L = 0, moreover the inequality $0 \le u(K) \le z_\ell(k)$ implies that $\lim_{k\to\infty} u(k) = 0$ and the proof is completed.

Next, we establish some oscillation results which ensures that every solution of difference equation (1) oscillates or converges to zero.

Theorem 2.5. Assume that condition (9) holds, $\sigma_1 \ge \tau_1$ *and* $\beta \ge 1$. If there exists a positive real valued function $\rho(k)$ and a real number $k_1 \in [0,\infty)$ with

$$\limsup_{k\to\infty}\sum_{s=k_1}^{k-1}\left[\rho(s\ell+j)\eta(s\ell+j) - \frac{\left(1+b^\beta+\frac{c^\beta}{2^{\beta-1}}\right)}{4}\frac{a((s\ell+j)-\sigma_1\ell)(\Delta_\ell\rho(s\ell+j))^2}{\rho(s\ell+j)}\right] = \infty \qquad (10)$$

holds, then every solution $u(k)$ of equation (1) oscillates or $\lim_{k\to\infty} u(k) = 0$.

Proof. Let $u(k)$ be a nonoscillatory solution of equation (1). Without loss of generality, we may assume that there exists a real number $K \ge k_0$ such that $u(k) > 0,\ u(k-\sigma_1\ell) > 0,\ u(k+\sigma_2\ell) > 0,$ $u(k-\tau_1\ell) > 0,\ u(k+\tau_2\ell) > 0$ for all $k \ge K$. Then we have $z_\ell(k) > 0$ for all $k \ge K$. From equation (1) for all $k \ge K$, we have

$$\begin{aligned}
&\Delta_\ell(a(k)\Delta_\ell^2 z_\ell(k)) + q(k)u^\beta(k+\ell-\sigma_1\ell) \\
&+p(k)u^\beta(k+\ell+\sigma_2\ell) \\
&+b^\beta\Delta_\ell(a(k-\tau_1\ell)\Delta_\ell^2 z_\ell(k-\tau_1\ell) \\
&+b^\beta q(k-\tau_1\ell)u^\beta(k+\ell-\tau_1\ell-\sigma_1\ell) \\
&+b^\beta p(k-\tau_1\ell)u^\beta(k+\ell-\tau_1\ell+\sigma_1\ell) \\
&+\frac{c^\beta}{2^{\beta-1}}\Delta_\ell(a(k+\tau_2\ell)\Delta_\ell^2 z_\ell(k+\tau_2\ell)) \\
&+\frac{c^\beta}{2^{\beta-1}}q(k+\tau_2\ell)u^\beta(k+\ell+\tau_2\ell-\sigma_2\ell) \\
&+\frac{c^\beta}{2^{\beta-1}}p(k+\tau_2\ell)u^\beta(k+\ell+\tau_2\ell+\sigma_2\ell) = 0. \qquad (11)
\end{aligned}$$

Using Lemma 2.2 in (11), we have

$$\Delta_\ell(a(k)\Delta_\ell^2 z_\ell(k)) + b^\beta\Delta_\ell(a(k-\tau_1\ell)\Delta_\ell^2 z_\ell(k-\tau_1\ell) + \frac{c^\beta\Delta_\ell(a(k+\tau_2\ell)\Delta_\ell^2 z_\ell(k+\tau_2\ell))}{2^{\beta-1}} + \frac{Q(k)}{4^{\beta-1}}$$

$$z_\ell^{\ \beta}(k+\ell-\sigma_1\ell) + \frac{P(k)z_\ell^{\ \beta}(k+\ell+\sigma_2\ell)}{4^{\beta-1}} \le 0.(12)$$

By Lemma 2.2, there are two cases for $z_\ell(k)$. First assume that case (1) holds for all $k \ge K_1 \ge K$. It follows from $\Delta_\ell z_\ell(k) > 0$ that $z_\ell(k+\sigma_2\ell) \ge z_\ell(k-\sigma_1\ell)$. Thus, by (12), we obtain

$$\Delta_\ell(a(k)\Delta_\ell^2 z_\ell(k)) + b^\beta\Delta_\ell(a(k-\tau_1\ell)\Delta_\ell^2 z_\ell(k-\tau_1\ell) + \frac{c^\beta}{2^{\beta-1}}\Delta_\ell(a(k+\tau_2\ell)\Delta_\ell^2 z_\ell(k+\tau_2\ell)) + \frac{R(k)}{4^{\beta-1}}z_\ell^{\ \beta}(k+\ell-\sigma_1\ell) \le 0. \qquad (13)$$

Define

$$w_1(k) = \rho(k)\frac{a(k)\Delta_\ell^2 z_\ell(k)}{\Delta_\ell z_\ell(k-\sigma_1\ell)},\ k \ge K_1. \qquad (14)$$

Then $w_1(k) > 0$ *for* $k \ge K_1$. From (14), we have

$$\Delta_\ell w_1(k) = \frac{w_1(k+\ell)\Delta_\ell\rho(k)}{\rho(k+\ell)} + \frac{\rho(k)\Delta_\ell(a(k)\Delta_\ell^2 z_\ell(k))}{\Delta_\ell z_\ell(k-\sigma_1\ell)} - w_1(k+\ell)\left(\frac{\Delta_\ell^2 z_\ell(k-\sigma_1\ell)}{\Delta_\ell z_\ell(k-\sigma_1\ell)}\right).$$

By (7), we have

$$a(k-\sigma_1\ell)\Delta_\ell^2 z(k-\sigma_1\ell) \ge a(k+\ell)\Delta_\ell^2 z(k+\ell).$$

Thus from (14), we obtain

$$\Delta_\ell w_1(k) \le \frac{w_1(k+\ell)\Delta_\ell \rho(k)}{\rho(k+\ell)} + \frac{\rho(k)\Delta_\ell(a(k)\Delta_\ell^2 z_\ell(k))}{\Delta_\ell z_\ell(k-\sigma_1\ell)} - \frac{\rho(k) w_2^2(k+\ell)}{\rho^2(k+\ell)a(k-\sigma_1\ell)}. \qquad (15)$$

Next , we define for $k \ge K_1$,

$$w_2(k) = \rho(k)\frac{a(k-\tau_1\ell)\Delta_\ell^2 z_\ell(k-\tau_1\ell)}{\Delta_\ell z_\ell(k-\sigma_1\ell)}. \qquad (16)$$

Then $w_2(k) > 0 \; for \, k \ge K_1$. Then from (16), we obtain

$$\Delta_\ell w_2(k) = \frac{\Delta_\ell \rho(k)}{\rho(k+\ell)} w_2(k+\ell) + \rho(k)\frac{\Delta_\ell(a(k-\tau_1\ell)\Delta_\ell^2 z_\ell(k-\tau_1\ell))}{\Delta_\ell z_\ell(k-\sigma_1\ell)} - w_2(k+\ell)\left(\frac{\Delta_\ell^2 z_\ell(k-\sigma_1\ell)}{\Delta_\ell z_\ell(k-\sigma_1\ell)}\right).$$

Note that $\sigma_1 > \tau_1$ for any $\ell \in [0,\infty)$. By (7) we find

$$a(k-\sigma_1\ell)\Delta_\ell^2 z_\ell(k-\sigma_1\ell) \ge a(k+\ell-\tau_1\ell)\Delta_\ell^2 z_\ell(k+\ell-\tau_1\ell).$$

Hence by (16), we obtain

$$\Delta_\ell w_2(k) \le \frac{\Delta_\ell \rho(k)}{\rho(k+\ell)} w_2(k+\ell) + \rho(k)\frac{\Delta_\ell(a(k-\tau_1\ell)\Delta_\ell^2 z_\ell(k-\tau_1\ell))}{\Delta_\ell z_\ell(k-\sigma_1\ell)} - \rho(k)\frac{w_2^2(k+\ell)}{\rho^2(k+\ell)a(k-\sigma_1\ell)}. \qquad (17)$$

Similarly, we define for $k \ge K_1$

$$w_3(k) = \frac{\rho(k)a(k+\tau_2\ell)\Delta_\ell^2 z_\ell(k+\tau_2\ell)}{\Delta_\ell z_\ell(k-\sigma_1\ell)}. \qquad (18)$$

Then $w_3(k) > 0$. From (18), we have

$$\Delta_\ell w_3(k) = \frac{\Delta_\ell \rho(k)}{\rho(k+\ell)} w_3(k+\ell) + \rho(k)\frac{\Delta_\ell(a(k+\tau_2\ell)\Delta_\ell^2 z_\ell(k+\tau_2\ell))}{\Delta_\ell z_\ell(k-\sigma_1\ell)} - w_3(k+\ell)\left(\frac{\Delta_\ell^2 z_\ell(k-\sigma_1\ell)}{\Delta_\ell z_\ell(k-\sigma_1\ell)}\right).$$

Using the equation (7), we obtain

$$a(k-\sigma_1\ell)\Delta_\ell^2 z_\ell(k-\sigma_1\ell) \ge a(k+\ell+\tau_2\ell)\Delta_\ell^2 z_\ell(k+\ell+\tau_2\ell).$$

Then by (18), we obtain

$$\Delta_\ell w_3(k) \le \frac{\Delta_\ell \rho(k)}{\rho(k+\ell)} w_3(k+\ell) + \rho(k)\frac{\Delta_\ell(a(k+\tau_2\ell)\Delta_\ell^2 z_\ell(k+\tau_2\ell))}{\Delta_\ell z_\ell(k-\sigma_1\ell)} - \rho(k)\frac{w_3^2(k+\ell)}{\rho^2(k+\ell)a(k-\sigma_1\ell)}. \qquad (19)$$

From (15), (17) and (19), we have

$$\begin{aligned}\Delta_\ell w_1(k) + b^\beta \Delta_\ell w_2(k) + \frac{c^\beta}{2^{\beta-1}}\Delta_\ell w_3(k) &\le -\rho(k)\frac{R(k)}{4^{\beta-1}}\frac{z_\ell^{\beta}(k+\ell-\sigma_1\ell)}{\Delta_\ell z_\ell(k-\sigma_1\ell)} + \frac{w_1(k+\ell)\Delta_\ell\rho(k)}{\rho(k+\ell)} - \frac{\rho(k)w_1^2(k+\ell)}{\rho^2(k+\ell)a(k-\sigma_1\ell)} \\ &+ b^\beta\left(\frac{w_2(k+\ell)\Delta_\ell\rho(k)}{\rho(k+\ell)} - \frac{\rho(k)w_2^2(k+\ell)}{\rho^2(k+\ell)a(k-\sigma_1\ell)}\right) \\ &+ \frac{c^\beta}{2^{\beta-1}}\left[\frac{w_3(k+\ell)\Delta_\ell\rho(k)}{\rho(k+\ell)} - \frac{\rho(k)w_3^2(k+\ell)}{\rho^2(k+\ell)a(k-\sigma_1\ell)}\right]. \end{aligned} \qquad (20)$$

On the other hand, using $a(k)$ nondecreasing and $\Delta_\ell^2 z_\ell(k) > 0 \; for \, k \ge K_1$. We have $\Delta_\ell^3 z_\ell(k) \le 0 \; for \, k \ge K_1$. Then by Lemma 2.3 we find for any $\gamma \in (j,\ell)$, and for k sufficiently large

$$\frac{z_\ell(k+\ell-\sigma_1\ell)}{\Delta_\ell z_\ell(k-\sigma_1\ell)} \ge \frac{\gamma(k-\sigma_1\ell)}{2} \qquad (21)$$

due to (8). Since $z_\ell(k) > 0, \Delta_\ell z_\ell(k) > 0$ and $\Delta_\ell^2 z_\ell(k) > 0 \; for \, k \ge K_1$, we have

$$z_\ell(k) = z_\ell(K_1) + \sum_{s=K_1}^{k-1}\Delta_\ell z_\ell(s) \ge (k-K_1)\Delta_\ell z_\ell(K_1) \ge \frac{dk}{2} \qquad (22)$$

for some d > 0 and k sufficiently large. From (21), (22) and $\beta > 1$ we have $\frac{z_\ell^{\beta}(k+\ell-\sigma_1\ell)}{\Delta_\ell z_\ell(k-\sigma_1\ell)} \ge \frac{d^{\beta-1}\gamma(k-\sigma_1\ell)}{2^\beta}$. Combining the last inequality with (20), and then apply the completing the square in the right hand side of the resulting inequality, we obtain

$$\Delta_\ell w_1(k)+b^\beta\Delta_\ell w_2(k)+\frac{c^\beta}{2^{\beta-1}}\Delta_\ell w_3(k)\le-\rho(k)\eta(k)+\frac{(1+b^\beta+\frac{c^\beta}{2^{\beta-1}})a(k-\sigma_1\ell)\left(\Delta_\ell\rho(k)\right)^2}{4\rho(k)}. \quad (23)$$

Summing the last inequality from $K_2\ge K_1$ to $k-1$, we obtain

$$\sum_{s=K_2}^{k-1}\left[\rho(s\ell+j)\eta(s\ell+j)-\frac{\left(1+b^\beta+\frac{c^\beta}{2^{\beta-1}}\right)}{4}\frac{a((s\ell+j)-\sigma_1\ell)\left(\Delta_\ell\rho(s\ell+j)\right)^2}{\rho(s\ell+j)}\right]\le w_1(K_2)+b^\beta w_2(K_2)+\frac{c^\beta}{2^{\beta-1}}w_3(K_2).$$

Taking limsup in the last inequality, we get a contradiction to (10).

Assume that Lemma 2.2 of (2) holds. Then by Lemma 2.4, we can obtain $\lim_{k\to\infty}u(k)=0$. This completes the proof.

Let $\rho(k)=k$ and $\beta=1$. Then, we obtain following corollary by Theorem 2.5.

Corollary 2.6. Assume that condition (9) holds and $\sigma_1\ge\tau_1$. If there is a real number $K_1\in[0,\infty)$ with

$$\lim_{k\to\infty}\sup\sum_{s=K_1}^{k-1}\left[(s\ell+j)\eta(s\ell+j)-\frac{(1+b+c)a((s\ell+j)-\sigma_1\ell)}{4(s\ell+j)}\right]=\infty$$

holds, then every solution $u(k)$ of equation (1) oscillates or $\lim_{k\to\infty}u(k)=0$.

Theorem 2.7. Assume that condition (9) holds, $\sigma_1\le\tau_1$ *and* $\beta\ge1$. If there exists a positive real valued function $\rho(k)$, and a real number $K_1\in[0,\infty)$ with

$$\lim_{k\to\infty}\sup\sum_{s=K_2}^{k-1}\left[\rho(s\ell+j)\eta(s\ell+j)-\frac{\left(1+b^\beta+\frac{c^\beta}{2^{\beta-1}}\right)}{4}\frac{a((s\ell+j)-\sigma_1\ell)\left(\Delta_\ell\rho(s\ell+j)\right)^2}{\rho(s\ell+j)}\right]=\infty \quad (24)$$

holds, then every solution $u(k)$ equation (1) oscillates or $\lim_{k\to\infty}u(k)=0$.

Proof. Proceeding as in the proof of Theorem 2.5, we obtain (12). By Lemma 2.2, there are two cases for $z_\ell(k)$. Assume that case (1) holds for all $k\ge K_1\ge K$. Then, we obtain (13). Let us assume the following transformations:

$$w_1(k)=\rho(k)\frac{a(k)\Delta_\ell^2z_\ell(k)}{\Delta_\ell z_\ell(k-\tau_1\ell)},\ k\ge K_1,$$

$$w_2(k)=\rho(k)\frac{a(k-\tau_1\ell)\Delta_\ell^2z_\ell(k-\tau_1\ell)}{\Delta_\ell z_\ell(k-\tau_1\ell)},\ k\ge K_1,$$

$$w_3(k)=\rho(k)\frac{a(k+\tau_2\ell)\Delta_\ell^2z_\ell(k+\tau_2\ell)}{\Delta_\ell z_\ell(k-\tau_1\ell)},\ k\ge K_1.$$

and as in proof of Theorem 2.5, we get

$$\Delta_\ell w_1(k)+b^\beta\Delta_\ell w_2(k)+\frac{c^\beta}{2^{\beta-1}}\Delta_\ell w_3(k)\le-\rho(k)\frac{R(k)}{4^{\beta-1}}\frac{z_\ell^{\beta}(k+\ell-\sigma_1\ell)}{\Delta_\ell z_\ell(k-\tau_1\ell)}+\frac{w_1(k+\ell)\Delta_\ell\rho(k)}{\rho(k+\ell)}-\frac{\rho(k)w_1^2(k+\ell)}{\rho^2(k+\ell)a(k-\tau_1\ell)}+b^\beta\left(\frac{w_2(k+\ell)\Delta_\ell\rho(k)}{\rho(k+\ell)}-\frac{\rho(k)w_2^2(k+\ell)}{\rho^2(k+\ell)a(k-\tau_1\ell)}\right)+\frac{c^\beta}{2^{\beta-1}}\left[\frac{w_3(k+\ell)\Delta_\ell\rho(k)}{\rho(k+\ell)}-\frac{\rho(k)w_3^2(k+\ell)}{\rho^2(k+\ell)a(k-\tau_1\ell)}\right]. \quad (25)$$

On the other hand, we have by Lemma 2.3, for any $\gamma\in(j,\ell)$, we find

$$\frac{z_\ell(k+\ell-\sigma_1\ell)}{\Delta_\ell z_\ell(k-\tau_1\ell)}=\frac{z_\ell(k+\ell-\sigma_1\ell)}{\Delta_\ell z_\ell(k-\sigma_1\ell)}.\frac{\Delta_\ell z_\ell(k-\sigma_1\ell)}{\Delta_\ell z_\ell(k-\tau_1\ell)}\ge\frac{\gamma(k-\sigma_1\ell)}{2}.$$

due to $\tau_1\ge\sigma_1$ and $\Delta_\ell^2z_\ell(k)$ for all $k\ge K_2$. Combining the above inequality with (22) and (25), and then apply the completing the square in the righthand side of the resulting inequality, we obtain

$$\Delta_\ell w_1(k)+b^\beta\Delta_\ell w_2(k)+\frac{c^\beta}{2^{\beta-1}}\Delta_\ell w_3(k)\le-\rho(k)\eta(k)+\frac{(1+b^\beta+\frac{c^\beta}{2^{\beta-1}})a(k-\tau_1\ell)\left(\Delta_\ell\rho(k)\right)^2}{4\rho(k)}.$$

Summing this from K_2 to $k-1$, we get

$$\sum_{s=K_2}^{k-1}\left[\rho(s\ell+j)\eta(s\ell+j)-\frac{\left(1+b^{\beta}+\frac{c^{\beta}}{2^{\beta-1}}\right)}{4}\right.$$
$$\left.\frac{a((s\ell+j)-\tau_1\ell)\left(\Delta_\ell\rho(s\ell+j)\right)^2}{\rho(s\ell+j)}\right]$$
$$\le w_1(K_2)+b^{\beta}w_2(K_2)+\frac{c^{\beta}}{2^{\beta-1}}w_3(K_2).$$

Taking limsup on both sides of the last inequality, we get contradiction with (24).

Assume that case (2) holds. Then by Lemma 2.4, we can obtain $\lim_{k\to\infty}u(k)=0$. The proof is now completed.

Let $\rho(k)=k$ and $\beta=1$. Then, we obtain following corollary by Theorem 2.7.

Corollary 2.8. Assume that condition (9) holds and $\tau_1\ge\sigma_1$ for any $\ell\in(0,\infty)$. If

$$\limsup_{k\to\infty}\sum_{s=K}^{k-1}\left[(s+j\ell)\eta(s+j\ell)\right.$$
$$\left.-\frac{(1+b+c)}{4(s+j\ell)}a((s+j\ell)-\tau_1\ell)\right]=\infty$$ holds

for all sufficiently large K, then every solution $u(k)$ of equation (1) oscillates or $\lim_{k\to\infty}u(k)=0$

3 EXAMPLES

Consider the generalised mixed type neutral difference equation

$$\Delta_\ell^3\left(u(k)+au(k-\tau\ell)+bu(k+\sigma\ell)\right)$$
$$+qu(k-g\ell)+pu(k+h\ell)=0 \qquad (26)$$

where a,b,q and p are positive real constants, τ,σ,g *and* h are positive integers.

Theorem 3.1 If $g>\tau$ and

$$q\frac{(g+\tau)\ell+3}{(g\ell-\tau\ell)^{[(g-\tau)\ell]}}>27(\ell+a+b),$$

Then, every solution of equation (26) is oscillatory.

Next, we present two examples to illustrate the main results.

Example 3.2 Consider the equation

$$\Delta_\ell^3\left(u(k)+\frac{u(k-\ell)+u(k+\ell)}{3}\right)+\frac{4^{\left[\frac{k}{\ell}\right]}u^3(k-2\ell)}{640}$$
$$+\left(\frac{31}{30}\right)4^{\left[\frac{k}{\ell}\right]}u^3(k+\ell)=0,\ k\in[0,\infty). \qquad (27)$$

Let $a(k)=1, b(k)=c(k)=\frac{1}{3}, q(k)=\frac{4^{\left[\frac{k}{\ell}\right]}}{640}$,

$p(k)=\left(\frac{31}{30}\right)4^{\frac{k}{\ell}}$, $\tau_1\ell=\tau_2\ell=1$, and $\sigma_1\ell=\sigma_2\ell=0$.

Take $\rho(k)=1$. Then condition (9) holds. On the other hand condition (10) also holds. Therefore by Theorem 2.5, every solution $u(k)$ of (24) oscillates or $\lim_{k\to\infty}u(k)=0$. Hence, we have $u(k)=2^{-\left[\frac{k}{\ell}\right]}$ is a solution of (27).

Example 3.3 Consider the equation

$$\Delta_\ell\left(k\Delta_\ell^2\left(u(k)+bu(k-\tau_1\ell)+cu(k+\tau_2\ell)\right)\right)+$$
$$\frac{c}{k}u(k+\ell-\sigma_1\ell)+\frac{d}{k}u(k+\ell-\sigma_1\ell)=0,(28)$$

for $k\in[0,\infty)$, where c, d are positive constants, $j=k-\left[\frac{k}{\ell}\right]\ell$, $0\le b(k)\le b$, $0\le c(k)\le c$, $b+c<1$. Let $a(k)=k$, $b(k)=c(k)=\frac{1}{3}$, $q(k)=\frac{c}{k}$, $p(k)=\frac{d}{k}$. It is easy to verify that all conditions of Corollary 2.8 hold. Then, from Corollary 2.8, every solution $u(k)$ of equation (28) oscillates or $\lim_{k\to\infty}u(k)=0$.

References:

1. Agarwal R. P, Grace S. R.,*"The Oscillation of Certain Difference Equations,"* Math. Comp. Modelling, 30 (1999), 53–66.
2. Agarwal R P, *Difference Equations and Inequalities,* Marcel Dekkar, New York, 2000.
3. Agarwal R. P, Grace S. R.,*" Oscillation of Certain Third Order Difference Equations,"* Comp. Math. Appl., 42 (2001), 379–384.
4. Agarwal R. P, Grace S. R, Bohner E. A., "*On the Oscillation of Higher Order Neutral Difference Equations of Mixed Type,*" Dynam. Syst. Appl., 11 (2002), 459 – 470.
5. Agarwal R. P, Grace S. R, Wong P. J .Y., *"On the Oscillation of Third Order Nonlinear Difference*

Equations," J. Appl. Math. Comput., 32 (2010), 189.
6. Błażej Szmanda, *"Nonoscillation, Oscillation and Growth of Solutions of Nonlinear Difference Equations Second Order,"* Journal of Mathematical Analysis and Applications, 109 (1) (1985), 22 - 30.
7. J. R. Graef, *"Oscillation, Nonoscillation, and Growth of Solutions of Nonlinear Functional Differential Equations of Arbitrary Order,"* Journal of Mathematical Analysis and Applications, 60 (2) (1977), 398 - 409.
8. Graef J, Thandapani E., *"Oscillatory and Asymptotic Behavior of Solution of Third Order Delay Difference Equation,"* Funk Ekvac, 42 (1999), 355–369.
9. Grace S R. *"Oscillation of Certain Neutral Difference Equations of Mixed Type,"* J. Math. Anal. Appl., 224 (1998), 241–254.
10. M. Maria Susai Manuel, G. Britto Antony Xavier and E. Thandapani, *"Theory of Generalized Difference Operator and Its Applications,"* Far East Journal of Mathematical Sciences, 20 (2) (2006), 163 – 171.
11. M. Maria Susai Manuel, G. Britto Antony Xavier and E. Thandapani, *"Qualitative Properties of Solutions of Certain Class of Difference Equations,"* Far East Journal of Mathematical Sciences, 23 (3) (2006), 295-304.
12. M. Maria Susai Manuel, A. George Maria Selvam and G. Britto Antony Xavier, *"Rotary and Boundedness of Solutions of Certain Class of Difference Equations,"* International Journal of Pure and Applied Mathematics, 33(3) (2006), 333-343.
13. M. Maria Susai Manuel, G.Britto Antony Xavier and V.Chandrasekar, *"On Generalised Difference Operator Of Third Kind and its Application in Number Theory,"* International Journal of Pure and Applied Maths., 53 (1) (2009), 69-81.
14. Ronald E. Mickens, *Difference Equations,* Van Nostrand Reinhold Company, New York, 1990.
15. Saber N.Elaydi, *An Introduction to Difference Equation*, Third edition, Springer, USA, 2000.

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