OSCILLATION THEOREMS FOR SECOND KIND ADVANCED GENERALIZED FUNCTIONAL DIFFERENCE EQUATIONS

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Abstract. In this paper, we consider the second kind advanced generalized functional difference equation

$$\Delta_m[a(k)\Delta_\ell u(k)]+p(k)u(g(k))=0 \qquad (1)$$

where $a(k)>0,\ \sum_{k=k_0}^{\infty}\frac{1}{a(k)}=\infty,\ p(k)\geq 0,\ p(k)\neq 0,\quad g(k)=k+\ell,\ g(k)$ is a monotone increasing real valued function. Also, we obtain some new oscillation criteria through an appropriate generalized Riccati equation.

Keywords: Functional difference equation, Oscillation, Second kind.
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Introduction: Difference equations usually describe the evolution of some certain phenomena over time and are also important in describing dynamics for fundamentally discrete system, see [1]. For example, in the numerical integration, the standard approach is to use the difference equations. Similarly, the population dynamics have discrete generations; the size of the $(k+1)^{th}$ generation $u(k+1)$ is a function of the k generation $u(k)$. This can be expressed as difference equation of the form $u(k+1)=f(u(k))$, see for example [15]. Further, the concept of difference equations with many examples in application such as asymptotic behavior of solutions of difference equations were studied extensively by Elaydi [15] where the analytic and geometric approaches were also combined in order to studying difference equations. Further, in [15], both classical and modern treatments of the difference equations were presented in excellent form. For related results on difference equations, [2-4, 11-13].

But recently, M. Maria Susai Manuel, et.al., by extending the theory of generalized difference equation of first kind to generalized difference equation of second kind and obtain some significant results, relations, theorems and formulae in Number Theory by using the solutions of second kind difference equation [10].

Notice that the generalized second kind difference equation

$$\Delta_m[a(k)\Delta_\ell u(k)]+p(k)(u(k+\ell))=0 \qquad (2)$$

is a special case of (1). With this background, in this paper, we discuss the oscillatory behavior of solutions of (1) and we obtain some better sufficient conditions for (1) to be oscillatory. Also we establish the relation between the oscillation of (1) and the oscillation of (2) and obtain a comparison theorem between more general equations. Our results extend some results for functional differential equations to generalized functional difference equations. In this paper, the details of the proof of results for non-oscillatory solutions will be carried out only for eventually positive solutions, since the arguments are similar for eventually negative solutions.

By a solution of (1) we shall always mean a nontrivial function $u(k)$ that satisfies (1) for all $k\in[0,\infty)$. A solution $u(k)$ of (1) is called Oscillatory if for any $k\in[0,\infty)$ there exists a real number $k\geq K$ such that $u(k)u(k+\ell)\leq 0$. Otherwise it is called nonoscillatory, is a solution of (1) is called nonoscillatory if it is eventually negative or positive.

Preliminary Lemmas: In this section, we prove some lemmas that will be useful in obtaining our main results.

Lemma 2.1. Suppose $u(k)$ is a onoscillatory solution of (1), then $u(k)\Delta_\ell u(k)>0$ holds for sufficiently large k.

Proof. Since $u(k)$ is a nonoscillatory solution, $g(k)$ is a monotone increasing function, we assume that there exists $k_0\in[0,\infty)$ ($[0,\infty)$ denotes the set of all positive real numbers) such that $u(k)>0, u(g(k))>0$, for all $k\geq k_0$. From (1), we obtain

$$\Delta_m[a(k)\Delta_\ell u(k)]=-p(k)u(g(k))\leq 0.$$

Hence $a(k)\Delta_\ell u(k)$ is a monotone nonincreasing function. If there exists $k_1\geq k_0$, such that $\Delta_\ell u(k)\leq 0$, then from $p(k)\neq 0$, there exists $k_2\geq k_1$, such that $a(k_2)\Delta_\ell u(k_2)<a(k_1)\Delta_\ell u(k_1)\leq 0$. For $k\geq k_2$, we have $a(k)\Delta_\ell u(k)\leq a(k_2)\Delta_\ell u(k_2)$

$$\Delta_\ell u(k) \le \frac{a(k_2)\Delta_\ell u(k_2)}{a(k)}.$$

Summing the above inequality from k_2 to $k-1$, we get

$$u(k) \le u(k_2) + \sum_{t=k_2}^{k-1} \frac{a(k_2)\Delta_\ell u(k_2)}{a(t)}$$

$$(or) \quad u(k) \le u(k_2) + a(k_2)\Delta_\ell u(k_2) \sum_{t=k_2}^{k-1} \frac{1}{a(t)}.$$

From the relation $\sum_{t=k_0}^{\infty} \frac{1}{a(t)} = \infty$

and $a(k_2)\Delta_\ell u(k_2) < 0$, we have $\lim_{k\to\infty} u(k) = -\infty$.

This is a contradiction. Thus, $\Delta_\ell u(k) > 0$, $u(k)\Delta_\ell u(k) > 0$. This completes the proof of Lemma 2.1.

Lemma 2.2. Assume (1) has a nonoscillatory solution $u(k) > 0$. Let $x(k) = \frac{a(k)\Delta_\ell u(k)}{u(k)}$ then $x(k) > 0$ holds for sufficiently large k. In addition, $x(k)$ satisfies

$$\lim_{k\to\infty} x(k) = 0, \tag{3}$$

$$\sum_k^{\infty} p(t) \prod_t^{g(t)-1} \left(1 + \frac{x(\tau)}{a(\tau)}\right) < \infty, \tag{4}$$

$$\sum_k^{\infty} \frac{x(t+m)x(t)}{a(t)} < \infty, \tag{5}$$

$$x(k) = \sum_k^{\infty} p(t) \prod_t^{g(t)-1} \left(1 + \frac{x(\tau)}{a(\tau)}\right) + \sum_k^{\infty} \frac{x(t+m)x(t)}{a(t)} < \infty. \tag{6}$$

Proof. By Lemma 2.1, it is easy to prove that there exists $k_0 \in [0,\infty)$ such that for $k \ge k_0$

$$x(k) = \frac{a(k)\Delta_\ell u(k)}{u(k)} > 0.$$

By definition of $x(k)$, we have

$$\frac{u(k+\ell) - u(k)}{u(k)} = \frac{x(k)}{a(k)}$$

$$\frac{u(k+\ell)}{u(k)} = 1 + \frac{x(k)}{a(k)}.$$

Thus, $$\prod_{k_0}^{k-1} \frac{u(t+\ell)}{u(t)} = \prod_{k_0}^{k-1} \left(1 + \frac{x(k)}{a(k)}\right)$$

$$u(k) = u(k_0) \prod_{k_0}^{k-1} \left(1 + \frac{x(k)}{a(k)}\right).$$

Now,

$$\Delta_m x(k) = \frac{a(k+m)\Delta_\ell u(k+m)}{u(k+m)} - \frac{a(k)\Delta_\ell u(k)}{u(k)}$$

$$= \frac{a(k+m)\Delta_\ell u(k+m)u(k) - a(k)\Delta_\ell u(k)u(k+m)}{u(k+m)u(k)}$$

$$= \frac{u(k+m)\Delta_m(a(k)\Delta_\ell u(k)) - a(k+m)\Delta_\ell u(k+m)\Delta_m u(k)}{u(k+m)u(k)}$$

$$= \frac{(-u(k+m)p(k))u(g(k))}{u(k+m)u(k)} - \frac{a(k+m)\Delta_\ell u(k+m)\Delta_m u(k)}{u(k+m)u(k)}$$

$$= \frac{-p(k)u(g(k))}{u(k)} - \frac{a(k)a(k+m)\Delta_\ell u(k+m)\Delta_m u(k)}{u(k+m)u(k)a(k)}$$

$$= \frac{-p(k)u(g(k))}{u(k)} - \frac{x(k+m)x(k)}{a(k)}.$$

Therefore,

$$\Delta_m x(k) + p(k) \prod_{k_0}^{g(k)-1} \left(1 + \frac{x(k)}{a(k)}\right) + \frac{x(k+m)x(k)}{a(k)} = 0. \tag{7}$$

Summing this from $k_1 \ge k_0$ to $k-1$, we get

$$x(k) - x(k_1) + \sum_{k_1}^{k-1} p(t) \prod_t^{g(t)-1} \left(1 + \frac{x(\tau)}{a(\tau)}\right) + \sum_{k_1}^{k-1} \frac{x(t+m)x(t)}{a(t)} = 0. \tag{8}$$

From (7), we have $\Delta_m x(k) < 0$.

Using $x(k) > 0$,we get $0 \le \lim_{k\to\infty} x(k) = r < \infty$.

From (8),we have $\sum_k^{\infty} p(t) \prod_t^{g(t)-1} \left(1 + \frac{x(\tau)}{a(\tau)}\right) < \infty$

and $\sum_k^{\infty} \frac{x(t+m)x(t)}{a(t)} < \infty$.

If $r > 0$, then there exists $k_2 \ge k_1$ such that

$$x(k+m)x(k) \ge \left(\frac{r}{2}\right)^2 = \frac{r^2}{4}$$

holds for $k \ge k_2$. Therefore,

$$\sum_{k_1}^{k-1} \frac{x(t+m)x(t)}{a(t)} \geq \sum_{k_2}^{k-1} \frac{r^2}{4a(t)} \to \infty \text{ as } k \to \infty.$$

This contradicts the fact that

$$\sum_{k_1}^{\infty} \frac{x(t+m)x(t)}{a(t)} < \infty.$$

Thus, $\lim_{k\to\infty} x(k) = 0$ and

$$x(k_1) = \sum_{k_1}^{\infty} p(t) \prod_{t}^{g(t)-1} \left(1 + \frac{x(\tau)}{a(\tau)}\right) + \sum_{k_1}^{\infty} \frac{x(t+m)x(t)}{a(t)}.$$

This completes the proof of Lemma 2.2. Define a function $\alpha_i(k), i = 0,1,2,..$ as follows

$$\alpha_0(k) = \sum_{k}^{\infty} p(t)$$

$$\alpha_{i+1}(k) = \sum_{k}^{\infty} p(t) \prod_{t}^{g(t)-1} \left(1 + \frac{\alpha_i(\tau)}{a(\tau)}\right) + \sum_{k}^{\infty} \frac{\alpha_i(t+m)\alpha_i(t)}{a(t)}, \; for \; i = 0,1,2,... \quad (9)$$

The $\alpha_i(k)$ are defined so long as they are finite. Then, by induction, we obtain that $\alpha_{i+1}(k) \geq \alpha_i(k) > 0$ and $\lim_{i\to\infty} \alpha_i(k) = 0$ if they are defined, where $i = 0,1,2,...$

Lemma 2.3. Equation (1) has a nonoscillatory solutions if and only if there exists a positive real valued function $u(k)$ such that $\lim_{k\to\infty} u(k) = 0, \Delta_m x(k) < 0,$ and

$$x(k) \geq \sum_{k}^{\infty} p(t) \prod_{t}^{g(t)-1} \left(1 + \frac{x(\tau)}{a(\tau)}\right) + \sum_{k}^{\infty} \frac{x(t+m)x(t)}{a(t)}. \quad (10)$$

Proof. If equation (1) has a nonoscillatory solution $u(k)$, define $x(k) = \frac{a(k)\Delta_\ell u(k)}{u(k)}$. Then, by Lemma 2.2, we have $x(k) \to 0$ as $k \to \infty$, $\Delta_m x(k) < 0$ and (10) holds.

Conversely, if there exists a function $x(k)$ such that (10) is satisfied, $x(k) \to 0$ as $k \to \infty$, and $\Delta_m x(k) < 0,$ then $x(k) > 0$. Moreover,

$$x(k) \geq \sum_{k}^{\infty} p(t) = \alpha_0(k),$$

$$x(k) \geq \sum_{k}^{\infty} p(t) \prod_{t}^{g(t)-1} \left(1 + \frac{\alpha_0(\tau)}{a(\tau)}\right) + \sum_{k}^{\infty} \frac{\alpha_0(t+m)\alpha_0(t)}{a(t)} = \alpha_1(k),$$

and by induction, we can easily obtain that

$x(k) \geq \alpha_i(k),$ for $i = 1,2,...$

Hence, there exists a finite limit

$$\lim_{i\to\infty} \alpha_i(k) = \alpha(k) < \infty, \quad k \in [0,\infty).$$

By Levi monotone convergence theorem and (9), let $i \to \infty,$ we have

$$\alpha(k) \geq \sum_{k}^{\infty} p(t) \prod_{t}^{g(t)-1} \left(1 + \frac{\alpha(\tau)}{a(\tau)}\right) + \sum_{k}^{\infty} \frac{\alpha(t+m)\alpha(t)}{a(t)}.$$

Let $u(k) = \prod_{k_0}^{k-1} \left(1 + \frac{\alpha(t)}{a(t)}\right) > 0.$

Then, $\Delta_\ell u(k) = \left(\frac{\alpha(k)u(k)}{a(k)}\right)$ and

$$\Delta_m \left(a(k)\Delta_\ell u(k)\right) = \alpha(k+m)\Delta_m u(k) + u(k)\Delta_m \alpha(k).$$

So $\Delta_m \left(a(k)\Delta_\ell u(k)\right) + p(k)u(g(k))$

$$= \alpha(k+m)\Delta_m u(k) + u(k)\Delta_m \alpha(k) + p(k)u(g(k))$$

$$= \alpha(k+m)\Delta_m u(k) + u(k) \left[-p(k) \prod_{k}^{g(k)-1} \left(1 + \frac{\alpha(t)}{a(t)}\right) - \frac{\alpha(k+m)\alpha(k)}{a(k)}\right] + p(k)u(g(k))$$

$$= \alpha(k+m)\Delta_m u(k) - u(k)\frac{\alpha(k+m)\alpha(k)}{a(k)} = 0.$$

Hence, $u(k)$ is a positive nonoscillatory solution of (1). Hence the proof.

Main Results:

Theorem 3.1. Suppose (1) satisfies $\sum_{k_0}^{\infty} p(\mathrm{t}) = \infty$ then (1) is oscillatory.

Proof. If not there exists a nonoscillatory solution of (1) without loss of generality, we assume $u(k) > 0$ for $k \geq k_0$. By Lemma 2.1 we have $\Delta_\ell u(k) > 0.$ Hence $u(k)$ is a monotone increasing function. Summing both sides of (1) from k_0 to $k-1,$ we get

$$a(k)\Delta_\ell u(k) - a(k_0)\Delta_\ell u(k_0) + \sum_{k_0}^{k-1} p(\mathrm{k})x(g(\mathrm{k})) = 0$$

or

$$\sum_{k_0}^{k-1} p(\mathrm{k})x(g(\mathrm{k})) = a(k_0)\Delta_\ell u(k_0) - a(k)\Delta_\ell u(k).$$

Since $g(\mathrm{k}_0) \le g(\mathrm{k}),\ u(g(\mathrm{k}_0)) \le u(g(\mathrm{k}))$,
it follows that

$$\sum_{k_0}^{k-1} p(\mathrm{k})u(g(\mathrm{k}_0)) \le \sum_{k_0}^{k-1} p(\mathrm{k})u(g(\mathrm{k}))$$
$$= a(k_0)\Delta_\ell u(k_0) - a(k)\Delta_\ell u(k)$$
$$< a(k_0)\Delta_\ell u(k_0)$$

which contradicts the fact that $\sum_{k_0}^{\infty} p(\mathrm{t}) = \infty.$

Thus, (1) is oscillatory. This completes the proof of Theorem 3.1.

Example 3.2. Consider the second kind generalized difference equation

$$\Delta_{\ell,m} u(k) + \frac{1}{k} u(\mathrm{g}(k)) = 0, \qquad (11)$$

where $\mathrm{g}(k) \ge k + m, k \ge 0$, $\mathrm{g}(k)$ is a monotone increasing positive real valued function.

Here $a(k) = 1 > 0,\ p(\mathrm{k}) = \frac{1}{k} > 0, \sum_{k_0}^{\infty} \frac{1}{a(\mathrm{k})} = \infty,$

$$\sum_{k_0}^{\infty} p(t) = \sum_{k_0}^{\infty} \frac{1}{t} = \infty.$$

So the equation (11) is oscillatory.

Remark 3.3. We can prove Theorem 3.1 by using Lemma 2.2.

As in proof of Lemma 2.2 and Lemma 2.3, we get the following theorem.

Theorem 3.4. Equation (1) has nonoscillatory solutions if and only if equation (6) has an eventually positive real valued function $x(k)$ satisfying $\Delta_\ell x(k) < 0$ and $x(k) \to 0$ as $k \to \infty$.

Theorem 3.5. Equation (1) has nonoscillatory solutions if and only if every $\alpha_i(k)$ in (9) is defined and $\lim_{i\to\infty} \alpha_i(k) = \alpha(\mathrm{k}) < \infty$.

Proof. Suppose $u(\mathrm{k})$ is a nonoscillatory solution of (1). Without loss of generality, we assume that $u(\mathrm{k})$ is eventually positive.

Let $x(\mathrm{k}) = \left(\frac{a(\mathrm{k})\Delta_\ell u(\mathrm{k})}{u(\mathrm{k})} \right)$. By Lemma 2.3, there exists a decreasing function $x(\mathrm{k})$ such that $x(\mathrm{k}) \to 0$ as $k \to \infty$ and

$$x(k) \ge \sum_{k}^{\infty} p(t) \prod_{t}^{g(t)-1} \left(1 + \frac{x(\tau)}{a(\tau)}\right) + \sum_{k}^{\infty} \frac{x(t+m)x(t)}{a(t)}.$$

Consequently, by the proof of Lemma 2.3, we have

$$x(k) \ge \sum_{k}^{\infty} p(t) = \alpha_0(k),$$

$$x(k) \ge \sum_{k}^{\infty} p(t) \prod_{t}^{g(t)-1} \left(1 + \frac{\alpha_0(\tau)}{a(\tau)}\right) + \sum_{k}^{\infty} \frac{\alpha_0(t+m)\alpha_0(t)}{a(t)} = \alpha_1(k)$$

and by induction, we can easily obtain that

$$x(k) \ge \alpha_{i+h}(k) = \sum_{k}^{\infty} p(t) \prod_{t}^{g(t)-1} \left(1 + \frac{\alpha_i(\tau)}{a(\tau)}\right) + \sum_{k}^{\infty} \frac{\alpha_i(t+m)\alpha_i(t)}{a(t)}.$$

Hence, (9) is bounded, so

$$\lim_{i\to\infty} \alpha_i(k) = \alpha(\mathrm{k}) < \infty.$$

Conversely, if $\lim_{i\to\infty} \alpha_i(k) = \alpha(\mathrm{k}) < \infty$,

then from $\alpha_{i+h}(n) \ge \alpha_i(n)$, we get

$$\alpha(k) \ge \alpha_i(k), \mathrm{i} = 1, 2, \ldots$$

Using Levi monotone convergence theorem, we have

$$\alpha(k) = \sum_{k}^{\infty} p(t) \prod_{t}^{g(t)-1} \left(1 + \frac{\alpha(\tau)}{a(\tau)}\right) + \sum_{k}^{\infty} \frac{\alpha(t+\ell)\alpha(t)}{a(t)}$$

$$\Delta_m \alpha(k) = -p(k) \prod_{k}^{g(k)-1} \left(1 + \frac{\alpha(t)}{a(t)}\right) - \frac{\alpha(k+m)\alpha(k)}{a(k)} < 0$$

$$\lim_{k\to\infty} \Delta_m \alpha(k) = 0.$$

By Lemma 2.3, we obtain (1) has nonoscillatory solutions. This completes the proof of Theorem3.3.

Restating Theorem 3.5 as sufficient conditions for (1) to be oscillatory, we have the following theorem.

Theorem 3.6. If one of the following conditions holds, then (1) is oscillatory.

(1) There exists $i_0 \ge 1$ such that $\alpha_{i_0}(k)$ in (9) is not defined.

(2) Every $\alpha_i(k)$ in (9) is defined, but for any $k_1 \in [0, \infty)$, there exists $k_2 \in [0, \infty), k_2 \ge k_1$, such that $\lim_{i\to\infty} \alpha_i(k_2) = \infty.$

Example 3.7. Consider the second kind generalized difference equation

$$\Delta_m \left(\frac{1}{k^3} \Delta_\ell u(k) \right) + \frac{1}{k^2} u(g(k)) = 0, \ell \in N(1),$$

$k \in N(0)$, where $g(k) \ge k + \ell$, $g(k)$ is a monotone increasing integer function. Clearly,

$$a(k)=\frac{1}{k^3}>0,\sum_{k_0}^{\infty}\frac{1}{a(t)}=\sum_{k_0}^{\infty}t^3=\infty$$

$$p(k)=\left(\frac{1}{k^2}\right)\geq 0,\ \text{and}\ p(k)\neq 0$$

$$\alpha_0(k)=\sum_k^{\infty}p(t)=\sum_k^{\infty}\frac{1}{t^2}<\infty,$$

$$\alpha_1(k)\geq\sum_k^{\infty}p(t)\prod_t^{g(t)-1}\left(1+\frac{\alpha_0(\tau)}{a(\tau)}\right)+\sum_k^{\infty}\frac{\alpha_0(t+\ell)\alpha_0(t)}{a(t)}$$

$$\geq\sum_k^{\infty}p(t)\prod_t^{g(t)-1}\left(1+\frac{\alpha_0(\tau)}{a(\tau)}\right)$$

$$=\sum_k^{\infty}\frac{1}{t^2}\prod_t^{g(t)-1}\left(1+\frac{\sum_\tau^{\infty}\frac{1}{\sigma^2}}{\frac{1}{\tau^3}}\right)$$

$$=\sum_k^{\infty}\frac{1}{t^2}\prod_t^{g(t)-1}\left(1+\tau^3\sum_\tau^{\infty}\frac{1}{\sigma^2}\right)$$

$$\geq\sum_k^{\infty}\frac{1}{t^2}\left(1+t^3\sum_\tau^{\infty}\frac{1}{\sigma^2}\right)$$

$$\geq\sum_k^{\infty}\frac{1}{t^2}\left(1+t^3\frac{1}{t^2}\right)=\sum_k^{\infty}\frac{1}{t^2}(1+t)\geq\sum_k^{\infty}\frac{1}{t},$$

i.e., $\alpha_1(k)$ is not defined. By Theorem 3.6, equation (1) is oscillatory.

Remark. 3.8. In Theorem 3.6 &Theorem 3.7, by taking $\ell=m=1$, we obtained the result to discrete equation in [16] .

Theorem 3.9. If (1) has nonoscillatory solutions, then (2) has nonoscillatory solutions. If (2) is oscillatory, then (1) is oscillatory.

Proof. If (1) has nonoscillatory solutions, by Lemma 2.3, there exists a positive real function $x(k)$ such that (10) is valid. From $g(k)\geq k+m$, we have

$$x(k)\geq\sum_k^{\infty}p(t)+\sum_k^{\infty}\frac{x(t+m)x(t)}{a(t)}.$$

By Lemma 2.3, equation (2) has nonoscillatory solutions. This completes the proof of Theorem 3.5.

Using a similar method, we can get the more general comparison theorem.

Let $\Delta_m(b(k)\Delta_\ell u(k))+q(k)u(b(k))=0,$ (12) where

$$b(k)>0,\quad \sum_{t=k_0}^{\infty}\frac{1}{b(t)}=\infty, q(k)\geq 0,\quad q(k)\neq 0,$$

$b(k)\geq k+m,\ k\geq 0,\ b(k)$ is a monotone increasing function.

Theorem3.10. Assume $b(k)\geq a(k),\ \ q(k)\leq p(k),$ $b(k)\leq g(k)$ for $k\geq k_0$. If (1) has nonoscillatory solutions, then (12) has nonoscillatory solutions. If (12) is oscillatory, then (1) is oscillatory.

Proof. Since equation (1) has nonoscillatory solutions, it follows from Lemma 2.3 that there exists a positive real valued function $x(k)$ such that (10) is valid.

$$x(k)\geq\sum_k^{\infty}p(t)\prod_t^{g(t)-1}\left(1+\frac{x(t)}{a(t)}\right)+\sum_k^{\infty}\frac{x(t+m)x(t)}{a(t)}$$

$$\geq\sum_k^{\infty}q(t)\prod_t^{b(t)-1}\left(1+\frac{x(t)}{b(t)}\right)+\sum_k^{\infty}\frac{x(t+\ell)x(t)}{b(t)}.$$

By Lemma 2.3, we get (12) has nonoscillatory solutions. This completes the proof of Theorem 3.10.

References:

1. R.P Agarwal, Difference Equations and Inequalities, Marcel Dekker, New York, 2000.
2. S.Z. Chen and L. Erbe, "Oscillation Results for Second Order Scalar and Matrix Difference Equations", Comp. Math. Applic., 28 (l-3) (1994), 55-89.
3. L.H. Erbe and B.G. Zhang, "Oscillation of Discrete Analogues of Delay Equations", Diff. and Integral Equations 2 (1989), 300-309.
4. J. Hooker and W. Patula, "A Second Order nonlinear Difference Equations: Oscillation and Asymptotic Behavior", J. Math. Anal. Appl., 91 (1983), 9-29.
5. M.Maria Susai Manuel, G.Britto Antony Xavier and E.Thandapani, "Theory of Generalized Difference Operator and Its Applications", Far East J. of Mathematical Sciences, 20 (2) (2006), 163 - 171.
6. M.Maria Susai Manuel, G.Britto Antony Xavier and E.Thandapani, "Qualitative Properties of Solutions of Certain Class of Difference Equations", Far East J. of Mathematical Sciences, 23(3) (2006), 295-304.
7. M.Maria Susai Manuel, G.Britto Antony Xavier and E.Thandapani, "Generalized Bernoulli Polynomials Through Weighted Pochhammer Symbols", Far East J. of Applied Mathematics, 26(3) (2007), 321-333.

8. M.Maria Susai Manuel, A.George Maria Selvam and G.Britto Antony Xavier, "Rotatory and Boundedness of Solutions of Certain Class of Difference Equations", International J. of Pure and Applied Mathematics, 33(3) (2006), 333-343.
9. M.Maria Susai Manuel and G.Britto Antony Xavier, "Recessive, Dominant and Spiral Behaviours of Solutions of Certain Class of Generalized Difference Equations", International J. of Differential Eqns. and Appl., 10(4) (2007), 423-433.
10. M. Maria Susai Manuel, G. Britto Antony Xavier and V. Chandrasekar, "Generalized Difference Operator of the Second Kind and its Application to Number Theory", International Journal of Pure and Applied Maths., 47(1) (2008), 127-140.
11. W.T. Patula, Growth, "Oscillation Properties of Second Order Linear Difference Equations", SIAM. J. Math. Anal., 10 (l) (1979), 55-61.
12. W.T. Patula, Growth, "Oscillation and Comparison Theorem for Second Order Linear Difference Equations", SIAM. J. Math. Anal. 10 (6) (1979), 1272-1279.
13. Ch.G. Philos, "Oscillation in Certain Difference Equations", Utilitas Math. 39 (1991), 157- 172S.
14. Ronald E.Mickens, Difference Equations, Van Nostrand Reinhold Company, New York, 1990.
15. Saber N. Elaydi, An Introduction to Difference Equations, Third edition, Springer, USA, 2000.
16. J. Yen, "The Oscillation Properties for the Solutions of the Second Order Differential Equation with integral small Coefficient", Acta Math. Sinica., 30 (2) (1987), 208-215.

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