GENERALIZED q-DERIVATIVE OPERATOR OF THE SECOND KIND AND ITS APPLICATIONS

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Abstract. In this paper, the authors define the generalized q-derivative operator of the second kind and its relation with q-shift operator. Also, we present the discrete version of Leibnitz theorem according to the generalized q-derivative operator of the second kind. By defining, its inverse and using stirling numbers of first kind, we establish a formula for sum of all partial sums of higher powers of geometric progression in the field of finite difference method.

Key words: q -Derivative operator, Geometric progression, Partial sums, Second kind.
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Introduction: The theory of q-derivative equations is based on the definition of the q-derivative operator

$$D_q y_k = \frac{y_{kq} - y_k}{(q-1)k}, q \neq 1 \qquad (1)$$

where y_k is sequence of positive integers [4, 10]. The definition of $D_q y_k$ is simply the derivative between two successive values of the sequence y_k and y_{kq}, $k \in N$. In [9], introduced the derivative operator D_{xy} on two variables x and y and turns out to be suitable for dealing with the Cauchy polynomials $p_n(x,y)$. Also, derived a binomial identity which unifies the two identities of Rota and Goldman, as well as the q-vandermonde identity. It is equivalent to the Goldman-Rota binomial identity, and it could be regarded as a homogeneous generalization of the q-Vandermonde identity. Using this operator, the q-Leibnitz formula and generating function of the homogeneous Rogers-Szeg¨o polynomials are derived [10]. But recently V.Chandrasekar and K.Suresh, took up the definition of D_q as given in (1) and extended the theory of q-derivative operator into the generalized q-derivative operator is defined as

$$D_q y_k = \frac{y_{kq} - \ell y_k}{(q-\ell)k}, \; q \neq \ell, \qquad (2)$$

and developed the theory of q-derivative equations in a different direction. For convenience, we labelled the operator D_q defined by (2) as $D_{q(\ell)}$ and by defining its inverse $D^{-1}_{q(\ell)}$ many interesting results in the field numerical methods were obtained [3].

Hence in this paper, we extend the theory of generalized q-derivative operator of first kind $D_{q(\ell)}$ to the generalized q-derivative operator of the second kind $D_{q_1,q_{2(\ell)}}$ and obtain some significant results, relations, theorems and finding the formulae for the sum of all partial sums of higher powers of geometric progression and the sum of all partial sums of product of consecutive terms of geometric progression in the field of numerical methods using the inverse of generalized q-derivative operator of second kind and the Stirling numbers of first kind and second kind respectively.

Prelimanaries: In this section, we define the generalized q-derivative operator of the second kind and obtaining the relation between the shift operator, generalized q-derivative operator and polynomials.

Definition 2.1. Let $u(k)$ be a real valued function on $[0,\infty)$. Then the generalized $q-$derivative operator of second kind is defined as

$$D_{q_1,q_{2(\ell)}} u(k) = \frac{1}{(q_1-\ell)(q_2-\ell)k^2}\Big[u(kq_1q_2) - \ell[u(kq_1)+u(kq_2)] + \ell^2 u(k)\Big], q_i \neq \ell, i = 1,2. \; (3)$$

Definition 2.2. Let $u(k)$ be a real valued function on $[0,\infty)$. Then the $q-$shift operator is defined as

$$E_{q_1q_2} u(k) = u(q_1q_2k). \qquad (4)$$

Lemma 2.3. Relation between generalized $q-$derivative operator of second kind and $q-$shift operator is

$$D_{q_1,q_{2(\ell)}} = \prod_{t=1}^{2} \frac{\left(E_{q_t} - \ell\right)}{(q_t-\ell)k^2}. \qquad (5)$$

Proof. (5) follows by Definitions 2.1 and 2.2.

Remark 2.4. (i) When $q_1 = q_2 = 1$, we get $E_{11}u(k) = u(k)$.

(ii)When $q_1 = q_2 = 1$, we obtain $D_{11(\ell)} = \frac{1}{k^2}$.

Lemma 2.5. If $q_1, q_2, \ell \in (0,\infty)$, then

$$D_{q_1,q_2(\ell)} = D_{q_1(\ell)}\left[D_{q_2(\ell)}\right] = D_{q_2(\ell)}\left[D_{q_1(\ell)}\right]. \quad (6)$$

Lemma 2.6. If c_1 and c_2 are any two non-zero scalars, $u(k)$ and $v(k)$ are real valued functions on $[0,\infty)$, then

$$D_{q_1,q_2(\ell)}\left[c_1u(k)+c_2v(k)\right] = c_1D_{q_1,q_2(\ell)}u(k)+c_2D_{q_1,q_2(\ell)}v(k).$$

Lemma 2.7. If $u(k)$ and $v(k)$ are real valued functions, then

$$D_{q_1,q_2(\ell)}\left[u(k)v(k)\right] = u(k)D_{q_1,q_2(\ell)}v(k) + \frac{1}{(q_1-\ell)(q_2-\ell)k^2}\Big[v(q_1q_2k)(q_1-1)kD_{q_1(1)}[u(q_2k)] +v(q_1q_2k)(q_2-1)kD_{q_2(1)}[u(k)] - \ell v(q_1k)(q_1-1)k D_{q_1(1)}[u(k)] - \ell v(q_2k)(q_2-1)kD_{q_2(1)}[u(k)]\Big].$$

Lemma 2.8. Let $u(k)$ and $v(k)\neq 0$ be the real valued functions, Then

$$D_{q_1,q_2(\ell)}\left[\frac{u(k)}{v(k)}\right] = \frac{\frac{1}{(q_1-\ell)(q_2-\ell)k^2}}{v(k)v(q_1k)v(q_2k)v(q_1q_2k)}\Big[v(k)v(q_1k)v(q_2k)(q_1-\ell)kD_{q_1(\ell)}[u(q_2k)] - \ell u(q_2k)v(q_1k)v(k)(q_1-\ell)k\,D_{q_1(\ell)}[v(q_2k)] - \ell v(q_1q_2k)v(q_1k)v(q_2k)(q_1-\ell)k\,D_{q_1(\ell)}[u(k)] - \ell u(q_1k)v(q_1q_2k)v(q_2k)(q_1-1)k\,D_{q_1(1)}[v(k)] + \ell(1-\ell)u(q_2k)v(k)v(q_1k)v(q_2k)\Big].$$

3. HIGHER ORDERS OF $D_{q_1,q_2(\ell)}$

In this section, we define the higher orders of $D_{q_1,q_2(\ell)}$ and establish the generalized version of Leibnitz theorem according to $D_{q_1,q_2(\ell)}$.

Definition 3.1. The generalized second order q-derivative operator of second kind denoted by $D^2_{q_1,q_2(\ell)}$ is defined as

$$D^2_{q_1,q_{2(\ell)}} = D_{q_1,q_2(\ell)}\left(D_{q_1,q_2(\ell)}\right).$$

In general, the n^{th} order generalized q-derivative operator denoted by $D^n_{q_1,q_2(\ell)}$ is defined as

$$D^n_{q_1,q_2(\ell)} = D_{q_1,q_2(\ell)}\left(D^{n-1}_{q_1,q_2(\ell)}\right).$$

Remark 3.2. If m and n are any positive integers, then

$$D^m_{q_1,q_2(\ell)}D^n_{q_1,q_2(\ell)} = D^n_{q_1,q_2(\ell)}D^m_{q_1,q_2(\ell)}.$$

Remark 3.3. If c is a constant and $u(k)$ is real valued function, then

$$D^m_{q_1,q_2(\ell)}\left[cu(k)\right] = cD^m_{q_1,q_2(\ell)}\left[u(k)\right].$$

Lemma 3.4. If m and n are any positive integers, then

$$D^m_{q_1,q_2(\ell)}k^n = \prod_{i=0}^{m-1}\left[(n-2i)_{q_1,q_2(\ell)}\right]k^{n-2m}, \quad (7)$$

where $\left[(n-2i)_{q_1,q_2(\ell)}\right] = \prod_{s=1}^{2}\left(\frac{q_s^{n-2i}-\ell}{q_s-\ell}\right)$. (8)

Proof. From (3), we have

$$D_{q_1,q_2(\ell)}k^n = \prod_{t=1}^{2}\left(\frac{q_t^n-\ell}{q_t-\ell}\right)k^{n-2}.$$

Again by applying (3) in the above expression, we obtain

$$D^2_{q_1,q_2(\ell)}k^n = \prod_{t=1}^{2}\left(\frac{q_t^n-\ell}{q_t-\ell}\right)\left(\frac{q_t^{n-2}-\ell}{q_t-\ell}\right)k^{n-4}.$$

Repeating this process we get proof.

Corollary 3.5. If n and i are any positive integers, then

$$D^n_{q_1,q_2(\ell)}k^{2n} = \prod_{i=0}^{n-1}\left[(n-2i)_{q_1,q_2(\ell)}\right], \quad (9)$$

where $\left[(n-2i)_{q_1,q_2(\ell)}\right]$ is given in (8).

Proof. The proof follows by induction method on n.

Lemma 3.6. If $P_k = a_0k^n + a_1k^{n-1} + \ldots + a_n$ is any n^{th} degree polynomial in k, then

$$D^m_{q_1,q_2(\ell)}P_k = \sum_{j=0}^{n}a_j\prod_{i=0}^{m-1}[(n-2i-j)_{q_1,q_2(\ell)}]k^{n-2m-j}. \quad (10)$$

Proof. The proof follows from (7).

Corollary 3.7. If $P_k = a_0k^{2n} + a_1k^{2n-1} + \ldots + a_{2n}$ is any polynomial in k of degree $2n$, then

$$D^n_{q_1,q_2(\ell)}P_k = \sum_{j=0}^{2n}a_j\prod_{i=0}^{n-1}\left[(2n-2i-j)_{q_1,q_2(\ell)}\right]k^{-j}$$

Proof. The proof follows from (7).

Lemma 3.8. If n is any positive integer, then

$$D^n_{q_1,q_2(\ell)} = \frac{\sum_{s=0}^{n}\sum_{t=0}^{n}(-\ell)^{s+t}\,nC_s\,nC_t\,E_{q_1}^{n-s}E_{q_2}^{n-t}}{\prod_{t=1}^{2}(q_t-\ell)^n\,k^{2n}}. \quad (11)$$

Proof. The proof follows by operating q-derivative operator n-times in (5) and using binomial theorem.

Lemma 3.9. If $u(k)$ is a real valued function, then

$$D^n_{q_1,q_2(\ell)}u(k)=\frac{\sum_{s=0}^{n}\sum_{t=0}^{n}(-\ell)^{s+t}\,nC_s\,nC_t\,u(q_1^{n-s}q_2^{n-t}k)}{\prod_{t=1}^{2}(q_t-\ell)^n k^{2n}}.$$

Proof. The proof follows by operating $u(k)$ on both sides in (11).

The following is the Leibnitz theorem according to $D_{q_1,q_2(\ell)}$.

Theorem 3.10. If $u(k)$ and $v(k)$ are any two real valued functions, then

$$D^n_{q_1,q_2(\ell)}\left[u(k)v(k)\right]=\sum_{r_1=0}^{n}\sum_{r_2=0}^{n}\Big[nC_{r_1}nC_{r_2} \quad (12)$$

$$D^{r_1}_{q_1(\ell)}\left[D^{r_2}_{q_2(\ell)}u(k)\right]D^{n-r_1}_{q_1(\ell)}\left[D^{n-r_2}_{q_2(\ell)}v\left(q^{r_1+r_2}k\right)\right]\Big]$$

Proof. From (6), we have

$$D^n_{q_1,q_2(\ell)}=D^n_{q_1(\ell)}D^n_{q_2(\ell)}\ . \quad (13)$$

From the Leibnitz theorem of the generalized q-derivative operator of first kind , we obtain

$$D^n_{q_2(\ell)}[u(k)v(k)]=\sum_{t-0}^{n}nC_t(D^t_{q_2(\ell)}u(k))(D^{n-t}_{q_2(\ell)}v(q^tk)). \quad (14)$$

(12) follows from (13) and (14).

Lemma 3.11. If n is positive integer, then

$$E^n_{q_1q_2}=\sum_{s=0}^{n}\sum_{t=0}^{n}\Big[nC_s\,nC_t\,(q_1-\ell)^s(q_2-\ell)^t$$

$$k^{s+t}\ell^{2n-(s+t)}D^s_{q_1(\ell)}D^t_{q_2(\ell)}\Big]. \quad (15)$$

Proof. From the shift operator, we have

$$E^n_{q_1q_2}=E^n_{q_1}E^n_{q_2} \quad (16)$$

from the relation $D_{q(\ell)}=\dfrac{(E_q-\ell)}{(q-\ell)k}$, we find

$$E^n_{q_2}=\left(\ell+(q_2-\ell)k\,D_{q_2(\ell)}\right)^n$$

$$=\sum_{t=0}^{n}nc_t\,\ell^{n-t}(q_2-\ell)^t k^t D^t_{q_2(\ell)}. \quad (17)$$

The proof follows from (16) and (17).

Theorem 3.12. If m and n are any two positive integers, then

$$(q_1q_2)^{nm}=\sum_{s=0}^{n}\sum_{t=0}^{n}\Big[nC_s\,nC_t\,\ell^{2n-(s+t)}$$

$$\prod_{i_1=0}^{s-1}\prod_{i_2=0}^{t-1}\left(q_1^{m-t-i_1}-\ell\right)\left(q_2^{m-i_2}-\ell\right)\Big].$$

Proof. The proof follows by operating $u(k)=k^m$ on both sides in (15).

4. Relation Between Polynomial Factorial And Q-Derivative Operator Of Second Kind: In this section, we discuss about the relation between the q-derivative operators of second kind and generalized polynomial factorial by using the Stirling numbers of first kind.

Definition 4.1. [5] If n is positive integer and ℓ is positive real, then the generalized polynomial factorial is defined as

$$k^{(n)}_\ell=k(k-\ell)(k-2\ell)\ldots(k-(n-1)\ell). \quad (18)$$

Lemma 4.2. [5] If s^n_r 's are the Stirling numbers of first kind, then

$$k^{(n)}_\ell=\sum_{r=1}^{n}s^n_r\ell^{n-r}k^r\ . \quad (19)$$

Lemma 4.3. Let *m* and *n* be the positive integers.

$$D^m_{q_1,q_2(\ell)}k^{(n)}_\ell$$

Then

$$=\sum_{r=1}^{n}s^n_r\ell^{n-r}\prod_{i=0}^{m-1}\left[(r-2i)_{q_1,q_2(\ell)}\right]k^{r-2m}. \quad (20)$$

where $\left[(r-2i)_{q_1,q_2(\ell)}\right]$ is given in (8).

Proof. The proof follows from (7) and (19).

Corollary 4.4. If s^n_r's are the Stirling numbers of first kind, then $D_{q_1,q_2(\ell)}k^{(n)}_\ell=\sum_{r=1}^{n}s^n_r\ell^{n-r}\left[r_{q_1,q_2(\ell)}\right]k^{r-2}$. (21)

Proof. Proof follows by taking $m=1$ in (20).

5. Generalized Inverse Of Q-Derivative Operator Of Second Kind And Its Applications: In this section, we derive the sum of all partial sums of higher powers of geometric progression and sum of all partial sums of product of m consecutive terms of geometric progression by using the inverse of generalized q-derivative operator of first kind and second kind respectively.

Definition5.1. The inverse of generalized q-derivative operator of second kind denoted by $D^{-1}_{q_1,q_2(\ell)}$ is defined as if $D_{q_1,q_2(\ell)}v(k)=u(k)$ then $v(k)=D^{-1}_{q_1,q_2(\ell)}u(k)$

and the nth order inverse operator denoted by $D^{-n}_{q_1,q_2(\ell)}$ is defined as

$$D^{-n}_{q_1,q_2(\ell)}=D^{-1}_{q_1,q_2(\ell)}\left(D^{-(n-1)}_{q_1,q_2(\ell)}\right).$$

Theorem 5.2. If *m* is positive integer, then

$$D^{-1}_{q_1,q_2(\ell)}k^m = \frac{k^{m+2}}{\left[(m+2)_{q_1,q_2(\ell)}\right]}. \qquad (22)$$

Proof. The proof follows by taking $m=1$ and n by m in (7) and definition 5.1.

Theorem 5.3. Let $k_\ell^{(m)}$ be the generalized polynomial factorial. Then,

$$D^{-1}_{q_1,q_2(\ell)}k_\ell^{(m)} = \sum_{r=1}^{m} s_r^m \ell^{m-r} \frac{k^{r+2}}{\left[(r+2)_{q_1,q_2(\ell)}\right]}. \quad (23)$$

Proof. From (19), we have

$$D^{-1}_{q_1,q_2(\ell)}k_\ell^{(m)} = \sum_{r=1}^{m} s_r^m \ell^{m-r} D^{-1}_{q_1,q_2(\ell)}k^r. \qquad (24)$$

(23) follows from (22) and (24).

Theorem 5.4. Let $k\in[0,\infty)$. Then,

$$D^{-1}_{q_1,q_2(\ell)}u(k)$$

$$=\prod_{i=1}^{2}(q_i-\ell)k^2\sum_{r_1=1}^{\infty}\sum_{r_2=1}^{\infty}\frac{\ell^{r_1+r_2-2}}{q_1^{2r_1}q_2^{2r_2}}u\left(\frac{k}{q_1^{r_1}q_2^{r_2}}\right). \quad (25)$$

Proof. From (5) and Definition 5.1, we have

$$v(k)=\prod_{t=1}^{2}\frac{(q_t-\ell)}{\left(E_{q_t}-\ell\right)}k^2u(k).$$

Using $v(k)=D^{-1}_{q_1,q_2(\ell)}u(k)$, we obtain

$$D^{-1}_{q_1,q_2(\ell)}u(k)=\prod_{t=1}^{2}(q_t-\ell)\frac{1}{\left(E_{q_1}-\ell\right)} \qquad (25)$$

$$\frac{1}{E_{q_2}}\left(1-\frac{\ell}{E_{q_2}}\right)^{-1}k^2u(k). \qquad (26)$$

follows from (26) & binomial theorem.

Following corollary illustrates Theorem 5.4.

Corollary 5.5. If n is positive integer, then

$$\sum_{r_1=1}^{\infty}\sum_{r_2=1}^{\infty}\frac{\ell^{r_1+r_2-2}}{\left(q_1^{r_1}q_2^{r_2}\right)^{n+2}}=\frac{1}{\prod_{t=1}^{2}\left(q_t^{n+2}-\ell\right)}. \qquad (27)$$

Proof. The proof follows by substituting $u(k)=k^n$ in (25).

Example 5.6. Putting $n=2$ in (27), we get

$$\sum_{r_1=1}^{\infty}\sum_{r_2=1}^{\infty}\frac{\ell^{r_1+r_2-2}}{\left(q_1^{r_1}q_2^{r_2}\right)^{4}}=\frac{1}{\prod_{t=1}^{2}\left(q_t^{4}-\ell\right)}.$$

In particular, taking $q_1=57$, $q_2=19$ and $\ell=3$, we find

$$\frac{1}{(57)^4(19)^4}+\frac{9}{(57)^8(19)^8}+\frac{81}{(57)^{12}(19)^{12}}+\cdots\infty$$

$$=7.269361969\times10^{-13}.$$

Corollary 5.7. If $k_\ell^{(m)}$ is the generalized polynomial factorial, then

$$\sum_{r_1=1}^{\infty}\sum_{r_2=1}^{\infty}\frac{\ell^{r_1+r_2-2}}{q_1^{2r_1}q_2^{2r_2}}\left(\frac{k}{q_1^{r_1}q_2^{r_2}}\right)_\ell^{(m)}=\sum_{r=1}^{m}\frac{s_r^m\ell^{m-r}k^r}{\prod_{t=1}^{2}(q_t^{r+2}-\ell)}. (28)$$

Proof. The proof follows by taking $u(k)=k_\ell^{(m)}$ in (25) and (19).

Example 5.8. In (28), putting $m=2$, we find

$$\sum_{r_1=1}^{\infty}\sum_{r_2=1}^{\infty}\frac{\ell^{r_1+r_2-2}}{q_1^{2r_1}q_2^{2r_2}}\left(\frac{k}{q_1^{r_1}q_2^{r_2}}\right)_\ell^{(2)}$$

$$=\frac{-\ell k}{\prod_{t=1}^{2}\left(q_t^3-\ell\right)}+\frac{k^2}{\prod_{t=1}^{2}\left(q_t^4-\ell\right)}.$$

In particular, by taking $k=197$, $q_1=3$, $q_2=4$ and $\ell=2$, we obtain

$$\sum_{r_1=1}^{\infty}\sum_{r_2=1}^{\infty}\frac{2^{r_1+r_2-2}}{3^{2r_1}4^{2r_2}}\left(\frac{197}{3^{r_1}4^{r_2}}\right)_2^{(2)}=\frac{(-2)(197)}{\left(3^3-2\right)\left(4^3-2\right)}$$

$$+\frac{(197)^2}{\left(3^4-2\right)\left(4^4-2\right)}=1.679874029.$$

Theorem 5.9. Let $u(k)$ be real valued function on $[0,\infty)$. Then,

$$D^{-1}_{q,q(\ell)}u(k)=(q-\ell)^2k^2\sum_{r=2}^{\infty}\frac{(r-1)\ell^{r-2}}{q^{2r}}u\left(\frac{k}{q^r}\right). (29)$$

Proof. Taking $q_1=q_2=q$ in (25), we find

$$D^{-1}_{q,q(\ell)}u(k)=(q-\ell)^2k^2\sum_{r_1=1}^{\infty}\sum_{r_2=1}^{\infty}\frac{\ell^{r_1+r_2-2}}{q^{2r_1+2r_2}}u\left(\frac{k}{q^{r_1+r_2}}\right). (30)$$

(29) follows by simplifying simple algebraic derivations of (30).

Corollary 5.10. If n is positive integer, then

$$\sum_{r=2}^{\infty}(r-1)\frac{\ell^{r-2}}{q^{(n+2)r}}=\left(\frac{1}{q^{n+2}-\ell}\right)^2. \qquad (31)$$

Proof. The proof follows by substituting $u(k)=k^n$ in (29).

Example 5.11. Taking $n=2$ in (31), we get

$$\sum_{r=2}^{\infty}(r-1)\frac{\ell^{r-2}}{q^{4r}}=\left(\frac{1}{q^4-\ell}\right)^2.$$

When $q=61$ *and* $\ell=4$, we find

$$\frac{1}{(61)^8}+\frac{8}{(61)^{12}}+\frac{48}{(61)^{16}}+\cdots\infty .$$
$$= 5.216288112\times 10^{-15} .$$

Corollary 5.12. If $k_\ell^{(m)}$ is the generalized polynomial factorial, then

$$\sum_{r=2}^{\infty}(r-1)\frac{\ell^{r-2}}{q^{2r}}\left(\frac{k}{q^r}\right)_\ell^{(m)} = \sum_{r=1}^{m}\frac{s_r^m\ell^{m-r}k^r}{\left(q^{r+2}-\ell\right)^2}. \quad (32)$$

Proof. The proof follows by taking $u(k)=k_\ell^{(m)}$ in (29) and (19).

Example 5.13. Substituting $m=2$ in (32), we find

$$\sum_{r=2}^{\infty}(r-1)\frac{\ell^{r-2}}{q^{2r}}\left(\frac{k}{q^r}\right)_\ell^{(2)} = \frac{-\ell k}{\left(q^3-\ell\right)^2}+\frac{k^2}{\left(q^4-\ell\right)^2}.$$

In particular, by taking $k=187$, $q=3$ and $\ell=2$, , we obtain

$$\sum_{r=2}^{\infty}(r-1)\frac{2^{r-2}}{3^{2r}}\left(\frac{187}{3^r}\right)_2^{(2)} = \frac{(-2)(187)}{\left(3^3-2\right)^2} + \frac{(187)^2}{\left(3^4-2\right)^2} = 6.201508476.$$

Conclusion: In this paper, we derive the infinite series on the sum of all partial sums of higher powers of geometric progression and sum of all partial sums of product of m consecutive terms of geometric progression in the field of finite difference method. It is very useful to find the sum of certain year of the generation of population of animal propagation problems in Mathematical Biology .

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