BOUNDED SOLUTION OF THIRD ORDER NONLINEAR GENERALIZED DIFFERENCE EQUATIONS

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Abstract. In the paper, we discuss the non linear generalized difference equation

$$\Delta_\ell(a(k)\Delta_\ell(b(k)\Delta_\ell u(k))) = \ell q(k) f(u(k+2\ell)),\ k \in [0,\infty), \quad (1)$$

where $a(k),\ b(k),\ q(k)$ are positive real valued functions, f is real valued functions with $uf(u) > 0$ for all $u \neq 0$. Also, we obtain the sufficient condition for the boundedness of all non oscillatory solutions of (1). Suitable examples are given to illustrate the main results.

Key words: Bounded Solution, Generalized Difference Equation, Nonlinear, Third Order.
Subject Classification: 39A12, 39A70, 47B39, 39B60.

Introduction: The theory of difference equations is based on the operator Δ defined as

$$\Delta u(k) = u(k+1) - u(k),\ k \in N, \quad (2)$$

where N ={0,1,2,3,...}. Eventhough many authors [1, 11] have suggested the definition of Δ as

$$\Delta u(k) = u(k+\ell) - u(k),\ k,\ \ell \in N, \quad (3)$$

no significant progress took place on this line. When we took up the definition of Δ as given in (3), the theory of difference equations are developed in a different direction. We obtained some interesting results in Number Theory. For convenience, we labeled the operator Δ defined by (3) as Δ_ℓ and by defining its inverse Δ_ℓ^{-1}, many interesting results on Number Theory were obtained. By extending the study for complex function and ℓ to be real, some new qualitative properties like rotatory, expanding and shrinking, spiral and weblike were studied for the solutions of difference equations involving Δ_ℓ. The results obtained can be found in [3-7].

In 2009, M. Maria Susai Manuel et.al., extend the theory of generalized difference operator the first kind Δ_ℓ to the generalized difference operator of the third kind $\Delta_{\ell_1,\ell_2,\ell_3}$ for the positive reals ℓ_1, ℓ_2, ℓ_3 and obtain some significant results, relations, the discrete version of Leibnitz theorem, binomial theorem and Newton's formula according to $\Delta_{\ell_1,\ell_2,\ell_3}$. Also, find the formulae for the sum of second partial sum of higher powers of arithmetic progression, the sum of second partial sums of product of consecutive terms of arithmetic progression and sum of all second partial sums of arithmetic-geometric progression are derived by using the solutions of third order generalized difference equation in the field of Numerical Methods [9].

Hence, in this paper, we discuss the sufficient conditions for the boundedness of all non oscillatory solutions of the third order nonlinear difference equation (1).

Throughout this paper, we denote a real number $k\ell + j$ by k.

Main Results:

Lemma 2.1. Any eventually positive solution $u(k)$ of equation (1) belongs to one of the following four classes:

(M_1) $u(k) > 0, \Delta_\ell u(k) > 0, \Delta_\ell(b(k)\Delta_\ell u(k)) > 0;$

(M_2) $u(k) > 0, \Delta_\ell u(k) > 0, \Delta_\ell(b(k)\Delta_\ell u(k)) < 0;$

(M_3) $u(k) > 0, \Delta_\ell u(k) < 0, \Delta_\ell(b(k)\Delta_\ell u(k)) > 0;$

(M_4) $u(k) > 0, \Delta_\ell u(k) < 0, \Delta_\ell(b(k)\Delta_\ell u(k)) < 0;$

for all sufficiently large k.

Proof. Let $u(k)$ be an eventually positive solution of equation (1). Then from (1) we have $\Delta_\ell(b(k)\Delta_\ell u(k)) > 0$ for large k. Hence, we obtain $\Delta_\ell(b(k)\Delta_\ell u(k)),\ \Delta_\ell u(k),\ u(k)$ are eventually one of sign. Thus, we have proved our lemma.

If we assume $\sum_{k=1}^{\infty} \frac{1}{a(k)} = \sum_{k=1}^{\infty} \frac{1}{b(k)} = \infty,$

then, by Kirguadze's lemma any eventually positive solution of equation (1) must be of M_1-type or M_2-type.

Theorem 2.2. Assume f is non decreasing function $f(u)/u$ is non increasing for $u > 0$ and

$$\sum_{k=1}^{\infty} \frac{1}{b(k)} \sum_{r=1}^{\left[\frac{k}{\ell}\right]-1} \frac{1}{a(r)} \sum_{s=1}^{r-1} q(s) < \infty. \quad (4)$$

Then every M_1- type solution of equation (1) is bounded.

Proof. Let $u(k)$ be an unbounded solution of equation (1) of M_1-type. By lemma 2.1, we have $u(k)>0$, $\Delta_\ell u(k)>0$ and $\Delta_\ell(b(k)\Delta_\ell u(k))>0$ for $k\in[0,\infty)$, from (1)

$$q(k)=\frac{\Delta_\ell(a(k)\Delta_\ell(b(k)\Delta_\ell u(k)))}{\ell f(u(k+2\ell))}$$
$$=\frac{a(k+\ell)\Delta_\ell(b(k+\ell)\Delta_\ell u(k+\ell))}{\ell f(u(k+2\ell))}-\frac{a(k)\Delta_\ell(b(k)\Delta_\ell u(k))}{\ell f(u(k+2\ell))}$$
$$\geq\frac{a(k+\ell)\Delta_\ell(b(k+\ell)\Delta_\ell u(k+\ell))}{\ell f(u(k+2\ell))}-\frac{a(k)\Delta_\ell(b(k)\Delta_\ell u(k))}{\ell f(u(k+\ell))} \quad (5)$$
$$=\Delta_\ell\left[\frac{a(k)\Delta_\ell(b(k)\Delta_\ell u(k))}{\ell f(u(k+\ell))}\right] \text{ for } k\in[0,\infty).$$

Summing (5) from N to $r-1$, we get

$$\sum_{s=N}^{r-1}q(s)+\frac{a(N)\Delta_\ell(b(N)\Delta_\ell u(N))}{\ell f(u(N+\ell))}\geq\frac{a(r)\Delta_\ell(b(r)\Delta_\ell u(r))}{\ell f(u(r+\ell))}$$

and therefore

$$\frac{1}{a(r)}\sum_{s=N}^{r-1}q(s)+\frac{a(N)\Delta_\ell(b(N)\Delta_\ell u(N))}{a(r)\ell f(u(N+\ell))}$$
$$\geq\frac{\Delta_\ell(b(r)\Delta_\ell u(r))}{\ell f(u(r+\ell))}\geq\frac{b(r+\ell)\Delta_\ell u(r+\ell)}{\ell f(u(r+\ell))}-\frac{b(r)\Delta_\ell u(r)}{\ell f(u(r))}=\Delta_\ell\left[\frac{b(r)\Delta_\ell u(r)}{\ell f(u(r))}\right].$$

Summing once again, from $r=N$ to $\left[\frac{k}{\ell}\right]-1$, we obtain

$$\sum_{r=N}^{\left[\frac{k}{\ell}\right]-1}\frac{1}{a(r)}\sum_{s=N}^{r-1}q(s)+\sum_{r=N}^{\left[\frac{k}{\ell}\right]-1}\frac{a(N)\Delta_\ell(b(N)\Delta_\ell u(N))}{\ell a(r)f(u(r+\ell))}\geq\frac{b(k)\Delta_\ell u(k)}{\ell f(u(k))}-\frac{b(N)\Delta_\ell u(N)}{\ell f(u(N))}.$$

Hence

$$\frac{\Delta_\ell u(k)}{\ell f(u(k))}\leq\frac{1}{b(k)}\sum_{r=N}^{\left[\frac{k}{\ell}\right]-1}\frac{1}{a(r)}\sum_{s=N}^{r-1}q(s)+\sum_{r=N}^{\left[\frac{k}{\ell}\right]-1}\frac{1}{a(r)}\frac{a(N)\Delta_\ell(b(N)\Delta_\ell u(N))}{b(k)\ell f(u(N+\ell))}+\frac{b(N)\Delta_\ell u(N)}{b(k)\ell f(u(N))}. \quad (6)$$

Since $f(u)/u$ is non increasing for $u>0$, from (6) we have

$$\frac{\Delta_\ell u(k)}{\ell u(k)}\leq\frac{f(u(N))}{u(N)}\frac{\Delta_\ell u(k)}{\ell f(u(k))}$$
$$\leq\frac{f(u(N))}{\ell b(k)u(N)}\sum_{r=N}^{\left[\frac{k}{\ell}\right]-1}\frac{1}{a(r)}\sum_{s=N}^{r-1}q(s)+\sum_{r=N}^{\left[\frac{k}{\ell}\right]-1}\frac{f(u(N))}{a(r)u(N)}\frac{a(N)\Delta_\ell(b(N)\Delta_\ell u(N))}{\ell f(u(N+\ell))b(k)}+\frac{b(N)\Delta_\ell u(N)}{\ell u(N)b(k)}. \quad (7)$$

Let $g(t)=u(k)+\frac{(t-k)}{\ell}\Delta_\ell u(k)$ for $k\leq t\leq k+\ell$.

Then $g'(t)=\frac{\Delta_\ell u(k)}{\ell}$ and $g(t)\geq u(k)$ for $k\leq t\leq k+\ell$. Hence

$$\frac{\Delta_\ell u(k)}{\ell u(k)}=\int_k^{k+\ell}\frac{g'(t)}{u(k)}dt\geq\int_k^{k+\ell}\frac{g'(t)}{g(t)}dt$$
$$\frac{\Delta_\ell u(k)}{\ell u(k)}=\ln[u(k)+\Delta_\ell u(k)]-\ln[u(k)]$$
$$=\ln[u(k+\ell)]-\ln[u(k)]. \quad (8)$$

Now, summing both sides of (8) from $k=N$ to $n-1$, we obtain

$$\sum_{k=N}^{n-1}\frac{\Delta_\ell u(k)}{\ell u(k)}\geq\sum_{k=N}^{n-1}[\ln u(k+\ell)-\ln u(k)]$$
$$=\ln u(n)-\ln u(N).$$

Hence, by (7) we get

$$\ln u(n)-\ln u(N)$$
$$\leq\frac{f(u(N))}{\ell u(N)}\sum_{k=N}^{n-1}\frac{1}{b(k)}\sum_{r=N}^{\left[\frac{k}{\ell}\right]-1}\frac{1}{a(r)}\sum_{s=N}^{r-1}q(s)+\frac{f(u(N))}{u(N)}\frac{a(N)\Delta_\ell(b(N)\Delta_\ell u(N))}{\ell f(u(N+\ell))}\times$$
$$\left(\sum_{k=N}^{n-1}\frac{1}{b(k)}\sum_{r=N}^{\left[\frac{k}{\ell}\right]-1}\frac{1}{a(r)}\right)+\frac{b(N)\Delta_\ell u(N)}{\ell u(N)}\sum_{k=N}^{n-1}\frac{1}{b(k)}.$$

From (4) there follows the convergence of the series

$$\sum_{k=1}^{\infty}\frac{1}{b(k)}\sum_{r=1}^{\left[\frac{k}{\ell}\right]-1}\frac{1}{a(r)} \text{ and } \sum_{k=1}^{\infty}\frac{1}{b(k)},$$

and therefore $\ln[u(k)]$ is bounded. Which is a contradiction and hence the proof.

Theorem 2.3. Assume the condition (1) holds. Then every M_2 -type solution of equation (1) is bounded.

Proof . If $u(k)$ is a solution of the equation (1) of M_2 -type then there exists K such that $u(k)>0$, $\Delta_\ell u(k)>0$ and

$\Delta_\ell(b(k)\Delta_\ell u(k))<0$ for all $k \ge K$. Summing both sides of the inequality from $k=K$ to $s-1$, we obtain

$$\Delta_\ell u(s)<\frac{b(K)\Delta_\ell u(K)}{b(s)} \text{ for } s \ge K .$$

Summing once again the last inequality from $s=K$ to $s=k-1$, we get

$$u(k)<b(K)\Delta_\ell u(K)\sum_{s=K}^{k-1}\frac{1}{b(s)}+u(K).$$

From condition (1) which implies $\sum_{k=1}^{\infty}\frac{1}{b(k)}<\infty$ and $b(K)\Delta_\ell(u(K))>0$. So the solution $u(k)$ must be bounded. This completes our proof.

Consequently by lemma 2.1, Theorem 2.2, Theorem 2.3, following theorem is obtained.

Theorem 2.4. Let f be nondecreasing, $f(u)/u$ is nonincreasing for $u>0$ and

$$\sum_{k=1}^{\infty}\frac{1}{b(k)}\sum_{r=1}^{\left[\frac{k}{\ell}\right]-1}\frac{1}{a(r)}\sum_{s=1}^{r-1}q(s)<\infty .$$

Then every eventually positive solution of equation (1) is bounded.

If the assumption that, $f(u)$ is non decreasing and $f(u)/u$ is non increasing for $u>0$, is replaced by the assumption that $f(u)$ is non increasing and $f(u)/u$ is non decreasing for $u<0$, then we conclude that every eventually negative solution of equation (1) is bounded.

Example 2.5. Consider the equation

$$\Delta_\ell((k+\ell)^2\Delta_\ell(k^2\Delta_\ell u(k)))$$
$$=\frac{(u(k+2\ell))^{1/3}}{\ell(k+\ell)^{1/3}(k+2\ell)^{2/3}(k+3\ell)},\ k\in[2\ell,\infty).(9)$$

All conditions of Theorem 2.4. are satisfied. Hence every eventually positive solution of (9) is bounded. One such solution is

$$u(k)=1-\frac{1}{\left[\frac{k}{\ell}\right]}.$$

Theorem 2.6. Suppose f is non increasing for $u>0$ and (1) holds. Then every M_1 -type solution of equation (1) is bounded.

Proof. Let $u(k)$ be a solution of M_1 -type equation (1). Then there exists such a real number K that $u(k)>0$, $\Delta_\ell u(k)>0$ and $\Delta_\ell(b(k)\Delta_\ell u(k))>0$ for $k \ge K$. Since $f(u(k+\ell)) \le f(u(k))$ for $k \ge K$, we have

$$\Delta_\ell(a(k)\Delta_\ell(b(k)\Delta_\ell u(k)))$$
$$=\ell q(k)f(u(k+2\ell))\le \ell q(k)f(u(K)),\ k\ge K.$$

Summing the above inequality from K to $r-1$,

$$a(r)\Delta_\ell(b(r)\Delta_\ell u(r))-a(K)\Delta_\ell(b(K)\Delta_\ell u(K))$$
$$\le \ell f(u(K))\sum_{s=K}^{r-1}q(s).$$

Hence,

$$\Delta_\ell(b(r)\Delta_\ell u(r))\le\frac{1}{a(r)}a(K)\Delta_\ell(b(K)\Delta_\ell u(K))$$
$$+\frac{\ell}{a(r)}f(u(K))\sum_{s=K}^{r-1}q(s). \quad (10)$$

Summing (10) from K to $\left[\frac{k}{\ell}\right]-1$, we get

$$\Delta_\ell u(k)\le\frac{b(K)\Delta_\ell u(K)}{b(k)}+\frac{a(K)\Delta_\ell(b(K)\Delta_\ell u(K))}{b(k)}$$
$$\times\sum_{r=K}^{\left[\frac{k}{\ell}\right]-1}\frac{1}{a(r)}+\frac{\ell f(u(K))}{b(k)}\sum_{r=K}^{\left[\frac{k}{\ell}\right]-1}\frac{1}{a(r)}\sum_{s=K}^{r-1}q(s).$$

Finally, we get

$$u(k)\le u(K)+b(K)\Delta_\ell u(K)\sum_{k=K}^{k-1}\frac{1}{b(k)}$$
$$+a(K)\Delta_\ell(b(K)\Delta_\ell u(K))\sum_{k=K}^{k-1}\frac{1}{b(k)}\sum_{r=K}^{\left[\frac{k}{\ell}\right]-1}\frac{1}{a(r)}$$
$$+\ell f(u(K))\sum_{k=K}^{k-1}\frac{1}{b(k)}\sum_{r=K}^{\left[\frac{k}{\ell}\right]-1}\frac{1}{a(r)}\sum_{s=K}^{r-1}q(s).$$

From (4), we have $\sum_{k=1}^{\infty}\frac{1}{b(k)}\sum_{r=1}^{\left[\frac{k}{\ell}\right]-1}\frac{1}{a(r)}<\infty$

and $\sum_{k=1}^{\infty}\frac{1}{b(k)}<\infty$, and which yields $u(k)$ is bounded. Consequently by lemma 2.1, Theorem 2.3 and Theorem 2.6, we get the following theorem.

Theorem 2.7. Let f be non increasing function for

$u \ge 0$ and $\sum_{k=1}^{\infty}\frac{1}{b(k)}\sum_{r=1}^{\left[\frac{k}{\ell}\right]-1}\frac{1}{a(r)}\sum_{s=1}^{r-1}q(s)<\infty.$

Then every eventually positive solution of equation (1) is bounded.

Example 2.8. Consider the equation

$$\Delta_\ell((k+2\ell)^2\Delta_\ell(k^2\Delta_\ell u(k)))$$
$$=\frac{1}{\ell(k+\ell)(k+2\ell)^2u(k+2\ell)},\ k\in[0,\infty),\ (11)$$

where $j=k-\left[\frac{k}{\ell}\right]\ell$. Here all conditions of Theorem 2.7 are satisfied. Hence every eventually positive solution of (11) is bounded. One such solution is $u(k)=\frac{1}{\left[\frac{k}{\ell}\right]}$.

If the assumption that $f(u)$ is nonincreasing for $u>0$ is replaced by the assumption that $f(u)$ is nondecreasing for $u<0$, then we conclude that every eventually negative solution of (1) is bounded.

Theorem 2.9. If f is non decreasing and

$$\sum_{k=1}^{\infty}\frac{1}{b(k)}\sum_{r=1}^{\left[\frac{k}{\ell}\right]-1}\frac{1}{a(r)}\sum_{s=1}^{r-1}q(s)=\infty, \quad (12)$$

then every M_1 - type solution of equation (1) is unbounded.

Proof. Let $u(k)$ be a M_1-type solution of equation (1). Then there exists $K\in[0,\infty)$ such that $\Delta_\ell(b(k)\Delta_\ell u(k))>0,\ u(k)>0$ and $\Delta_\ell u(k)>0$ for all $k\ge K$. From (1), we have

$$\Delta_\ell(a(k)\Delta_\ell(b(k)\Delta_\ell u(k)))=\ell q(k)f(u(k+2\ell))$$
$$\ge \ell q(K)f(u(K)),\quad k\ge K.\quad (13)$$

Similarly, as in proof of this Theorem 2.6 summing three times of (13), we get

$$u(k)\ge u(K)+b(K)\Delta_\ell u(K)\sum_{k=K}^{n-1}\frac{1}{b(k)}$$
$$+a(K)\Delta_\ell(b(K)\Delta_\ell u(K))\sum_{k=K}^{n-1}\frac{1}{b(k)}\sum_{r=K}^{\left[\frac{k}{\ell}\right]-1}\frac{1}{a(r)}$$
$$+\ell f(u(K))\sum_{k=K}^{n-1}\frac{1}{b(k)}\sum_{r=K}^{\left[\frac{k}{\ell}\right]-1}\frac{1}{a(r)}\sum_{s=K}^{r-1}q(s),$$

and using (12), which gives $u(k)\to\infty$ as $k\to\infty$.

Example 2.10. Consider the equation

$$\Delta_\ell(2^{\left[\frac{k}{\ell}\right]}\Delta_\ell(2^{\left[\frac{k}{\ell}\right]}\Delta_\ell u(k)))$$
$$=\frac{10}{3}4^{\left[\frac{k}{\ell}\right]}u(k+2\ell),\ k\in[0,\infty).\quad (14)$$

Now, satisfies conditions of theorem 2.9 and hence any M_1-type solution of equation (14) is unbounded. One such solution is $u(k)=\left\{(3/2)^{\left[\frac{k}{\ell}\right]}\right\}$. For M_2-type solution we need a stronger condition than (12).

Theorem 2.11. Let f be non decreasing function,

$$\sum_{k=1}^{\infty}\frac{1}{b(k)}\sum_{r=1}^{\left[\frac{k}{\ell}\right]-1}\frac{1}{a(r)}\sum_{s=1}^{r-1}q(s)=\infty \text{ and}$$
$$\sum_{k=1}^{\infty}\frac{1}{b(k)}\sum_{r=1}^{\left[\frac{k}{\ell}\right]-1}\frac{1}{a(r)}<\infty. \quad (15)$$

Then every M_2-type solution of equation (1) is unbounded. The proof is similar to the proof of Theorem 2.9. If the condition (15) is not satisfied, then the above result may fail. It is shown in Example 2.12.

Example 2.12. Consider the equation

$$\Delta_\ell((k+\ell)\Delta_\ell((k+\ell)\Delta_\ell u(k)))$$
$$=\frac{k+2\ell}{\ell k(k+\ell)}u(k+2\ell),\ k\in[0,\infty).\quad (16)$$

All conditions of Theorem 2.11 are satisfied except condition (15), namely

$$\sum_{k=1}^{\infty}\frac{1}{b(k)}\sum_{r=1}^{\left[\frac{k}{\ell}\right]-1}\frac{1}{a(r)}=\sum_{k=1}^{\infty}\frac{1}{k+\ell}\sum_{r=1}^{\left[\frac{k}{\ell}\right]-1}\frac{1}{a(r+\ell)}=\infty.$$

So, we cannot say every M_2-type solution is unbounded. In fact, equation (16) got the bounded solution $u(k)=1-\frac{1}{\left[\frac{k}{\ell}\right]}$.

Theorem 2.13. If f is non decreasing and

$$\sum_{k=1}^{\infty}\frac{1}{a(k)}=\sum_{k=1}^{\infty}\frac{1}{b(k)}=\infty$$

$$\sum_{k=1}^{\infty} q(k)\sum_{r=1}^{k}\frac{1}{a(r)}\sum_{s=1}^{r}\frac{1}{b(s)}=\infty,$$

then every non oscillatory solution of equation (1) is unbounded.

Example 2.14. Consider the equation

$$\Delta_{\ell}\left(\frac{1}{6^{\left[\frac{k}{\ell}\right]}}\Delta_{\ell}\left(\frac{1}{3^{\left[\frac{k}{\ell}\right]}}\Delta_{\ell}u(k)\right)\right)$$

$$=\frac{17\times 2^{-\left[\frac{k}{\ell}\right]}\times 3^{-2\left[\frac{k}{\ell}\right]-3}}{\ell(k+5\ell)}u(k+2\ell),\ k\in[0,\infty).\ (17)$$

All conditions of Theorem 2.13 are satisfied. Hence, every non oscillatory solution of (17) is unbounded. One such solutions is $u(k)=k+3\ell$.

Theorem 2.15. Equation (1) cannot have a quickly oscillatory solution.

Proof. Let $z(k)>0$ for all $k\in K$ and $u(k)=(-1)^{\left[\frac{k}{\ell}\right]}z(k)$ is a solution of (1). Then

$$\Delta_{\ell}(a(k)\Delta_{\ell}(b(k)\Delta_{\ell}u(k)))=(-1)^{\left[\frac{k}{\ell}\right]+1}[a(k+\ell)b(k+2\ell)z(k+3\ell)+(a(k+\ell)b(k+2\ell)+a(k+\ell)b(k+\ell))z(k+2\ell)+(a(k+\ell)b(k+\ell)+a(k)b(k+\ell)+a(k)b(k))z(k+\ell)+a(k)b(k)z(k)].$$

Therefore equation (1) can be written as

$$(-1)^{\left[\frac{k}{\ell}\right]+1}[a(k+\ell)b(k+2\ell)z(k+3\ell)+(a(k+\ell)b(k+2\ell)+a(k+\ell)b(k+\ell))z(k+2\ell)+(a(k+\ell)b(k+\ell)+a(k)b(k+\ell)+a(k)b(k))z(k+\ell)+a(k)b(k)z(k)]=\ell q(k)f((-1)^{\left[\frac{k}{\ell}\right]+2}z(k+2\ell)).$$

By taking ℓ is even and $k\in$ {multiple of ℓ} we have,

$$-[a(k+\ell)b(k+2\ell)z(k+3\ell)+(a(k+\ell)b(k+2\ell)+a(k+\ell)b(k+\ell))z(k+2\ell)+(a(k+\ell)b(k+\ell)+a(k)b(k+\ell)+a(k)b(k))z(k+\ell)+a(k)b(k)z(k)]$$

$$=\ell q(k)f((-1)^{\left[\frac{k}{\ell}\right]+2}z(k+2\ell)),$$

where

$$-[a(k+\ell)b(k+2\ell)z(k+3\ell)+(a(k+\ell)b(k+2\ell)+a(k+\ell)b(k+\ell))z(k+2\ell)+(a(k+\ell)b(k+\ell)+a(k)b(k+\ell)+a(k)b(k))z(k+\ell)+a(k)b(k)z(k)]<0$$

and by the assumption $uf(u)>0$,

$$\ell q(k)f(z(k+2\ell))>0.$$

On the other hand, by taking ℓ is odd and $k\in$ {multiple of ℓ}, we find

$$[a(k+\ell)b(k+2\ell)z(k+3\ell)+(a(k+\ell)b(k+2\ell)+a(k+\ell)b(k+\ell))z(k+2\ell)+(a(k+\ell)b(k+\ell)+a(k)b(k+\ell)+a(k)b(k))z(k+\ell)+a(k)b(k)z(k)]=\ell q(k)f(-z(k+2\ell)).$$

The left side of the above equation is always positive, and the right side is always negative. This contradiction proves our theorem.

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