THEORY OF GENERALIZED FUZZY DIFFERENCE EQUATION AND ITS APPLICATIONS

V.CHANDRASEKAR , J.KATHIRAVAN

Abstract: In this paper, we discuss the existence of the oscillatory behavior of the boundedness and the asymptotic behavior of the positive solutions of the generalized fuzzy difference equation. Also, we derive the sum of higher powers of the sequence of positive fuzzy numbers by using the numerical-complete solution of the generalized fuzzy difference equation.

Key words: Closed form solution, Fuzzy numbers, Generalized fuzzy difference equation, Numerical solution.
AMS Subject Classification: 39A12, 39A70, 47B39, 39B60.

Introduction: Difference equations usually describe the evolution of some certain phenomena over time and are also important in describing dynamics for fundamentally discrete system, see [1]. For example, in the numerical integration, the standard approach is to use the difference equations. Similarly, the population dynamics have discrete generations, the size of the $(k+1)^{st}$ generation $u(k+1)$ is a function of the k^{th} generation $u(k)$. This can be expressed as difference equation of the form

$$u(k+1) = f(u(k)),$$

see for example [1]. Further, the concept of difference equations with many examples in applications such as asymptotic behavior of solutions of difference equations were studied extensively by Elaydi [5] where the analytic and geometric approaches were also combined in order to studying difference equations. Further in [8], both classical modern treatments of the difference equations were presented in excellent form. For a detail study of the theory of difference equations and their applications and the references cited therein, see [13, 14, 16].

Each and every practical system is endowed with uncertainties. Heisen-berg's uncertainty principle is for micro level in physical systems. But some form of uncertainty is inherent in biological, natural and social systems. The more complexity of a system the greater is its uncertainty due to fuzziness. That is, each quantity we want to measure becomes fuzzy valued instead of precise valued. Thus, equations become fuzzy equations and differential (difference) equations become fuzzy differential (difference) equations. Fuzzy dynamical system can be visualized as a collection of fuzzy set of crisp dynamical systems representing some uncertain or imprecise system and is more complex than a probabilistic dynamical system or a stochastic process. Fuzzy differential equation attracted the attention of some authors and sufficient theory was developed [4, 6, 7, 12, 13]. Though Lakshmikantham. V and Vatsala A.S [9] introduced the notion of fuzzy difference equation and studied the stability theory required to fuzzy difference equations, theory required to study matrix fuzzy difference equations which occur in system theory, economics, inventory analysis, probability models for learning, approximate solutions of ordinary and partial differential equations is still not available in the literature.

Recently there has been increase in interest in the study of difference equations because they have many applications to economics, biology, etc (see [1, 5, 8]). Moreover in [2] Deeba et.al., studied the fuzzy anolog of difference equation which arises in population genetics. Furthermore in [3] Deeba and Korvin studied the second order difference equation.

$$u(k+1) = u(k) - abu(k-1) + m, \quad k \in \mathrm{N},$$

where $u(k)$ is a function of fuzzy numbers and $a, b, m, u(0), u(1)$ are fuzzy numbers. This equation is linearized model of nonlinear model which determines the carbondioxide (CO_2) level in the blood. Finally in [11] Negoita and Ralescu investigated properties of various types of fuzzy dynamical systems.

With this background, in this paper, we discuss the generalized the fuzzy non-linear difference equation

$$u(k+h) = A + \frac{B}{u(k)}, \; k \in [0,\infty), \; h \in (0,\infty) \quad (1)$$

where $u(k)$ is a function of fuzzy numbers and $A, B, u(0)$ are fuzzy numbers. We note that (1) is a special case of the generalized Riccati difference equation

$$u(k+h) = \frac{Au(k)+B}{Cu(k)+D}, \quad k \in [0,\infty),$$

where $A, B, C, D, u(0)$ are constants.

Also, we investigate the existence of the behavior boundedness and persistence of the positive solution of the generalized fuzzy non-linear difference equations (1). Moreover, we derive the formula for the sum of fuzzy numbers by using the inverse of generalized difference operator.

2. Preliminaries:

Definition 2.1. If $m \in \mathrm{N}(1)$, then the equation of the form

$$f\left(k, v(k), v(k+h), \ldots, v(k+mh)\right) = 0 \qquad (2)$$

is called the generalized linear difference equation.

Definition 2.2. [15] If A is a function from $[0,\infty)$ into the interval $[0,1]$, then A is called a fuzzy set.

Definition 2.3. Let $v(k)$ be a function of fuzzy numbers defined on $[0,\infty)$. Then the generalized nonlinear fuzzy difference equation is defined as $v(k+h) = f(v(k), w, q),\ h \in (0,\infty), \qquad (3)$

where w, q and $v(0)$ are fuzzy numbers and $f : [0,\infty)\times[0,\infty)\times[0,\infty) \to [0,\infty)$, $[0,\infty)$ is the set of all positive real numbers.

Definition 2.4. Let $v(k)$ be a function of fuzzy numbers defined on $[0,\infty)$. Then the first order non-linear generalized fuzzy difference equation is defined as

$$v(k+h) - v(k) = u(k),\ h \in [0,\infty). \qquad (4)$$

Definition 2.5. A function is of the form

$$v(k) = \sum_{r=1}^{\left[\frac{k}{h}\right]} u(k - rh) \qquad (5)$$

satisfies (4), we call it as numerical solution of that equation. A function $v(k) = v_1(k)$, other than numerical solution, satisfying the equation (4), is called closed form solution of that difference equation and it is denoted as $\Delta_h^{-1} u(k)$.

Note that numerical solution $v(k)$ and closed form solution $v_1(k)$ of the equation (4) need not be equal always. We observed that, it is also possible to find a particular closed form solution coinciding with numerical solutions of (4).

Consider the following example.

Example 2.6. The functions $c_{1(h)}(k) = \dfrac{k_h^{(4)}}{4h}$ and $u_{1(h)}(k) = \sum_{r=1}^{\left[\frac{k}{h}\right]} (k - rh)_h^{(3)}$ are solutions of (4), when $u(k) = k_h^{(3)}$. If k is an integer multiple of h, then $c_{1(h)}(k) = u_{1(h)}(k)$, otherwise $c_{1(h)}(k) \neq u_{1(h)}(k)$.

The example 2.6 shows that, equation (4) has two type of solutions; One is closed form solution $c_{1(h)}(k)$ and another one is numerical solution given as

$$u_{1(h)}(k) = \sum_{r=1}^{\left[\frac{k}{h}\right]} u(k - rh) \qquad (6)$$

Let $0 < h < k$ and k is not an integer multiple of h. If for given equation (4), then there exists a closed form solution $c_{1(h)}(k)$ such that

$$u_{1(h)}(k) = c_{1(h)}(k) - c_{1(h)}(j), \qquad (7)$$

then $c_{1(h)}(k)$ is called complete solution of equation (4) and the relation (7) can be called as numerical-complete solution of equation (4).

Definition 2.7. If $u(k)$ and $v(k)$ are positive real valued function, then we say that $u(k), v(k)$ oscillates about $(u, v),\ u, v \in [0,\infty)$ if for every $k_0 \in K$ there exist $k_1, k_2 \in K, k_1, k_2 \geq k_0$ such that

$(u(k_1) - u)(u(k_2) - u) \leq 0$ and

$(v(k_1) - v)(v(k_2) - v) \leq 0$

$(u(k_1) - u)(v(k_1) - v) \geq 0$ and

$(u(k_2) - u)(v(k_2) - v) \geq 0$.

Definition 2.8. Let $u(k)$ be a function of positive fuzzy numbers and let u be a positive fuzzy number. We say that $u(k)$ oscillates about u if for every $k_0 \in K$ there exists $k_1, k_2 \in K,\ k_1, k_2 \geq k_0$ such that

$$MIN\left\{u(k_2), u\right\} = u(k_2) \ \ and \ \ MIN\left\{u(k_1), u\right\} = u$$

(or)

$$MIN\left\{u(k_2), u\right\} = u \ \& \ MIN\left\{u(k_1), u\right\} = u(k_1). (8)$$

Definition 2.9. Let A be a fuzzy set . Then the generalized a-cuts is defined as, $\left[A\right]_a = \left\{t \in [0,\infty) : A(t) \geq a\right\},\ a \in (j, h]$.

Definition 2.10. Let $\overline{H}$ be a closure of the set H. Then D is called generalized fuzzy number if the following conditions hold:

(i) D is a normal, (ii) D is a convex fuzzy set, (iii) D is upper semicontinuous, (iv) The support of D,

supp $\overline{\bigcup_{a \in (j,h]} [D]_a} = \overline{\{t : D(t) > 0\}}$ is compact.

Remark 2.11. A fuzzy number D is called positive if supp $(D) \subset (0,\infty)$.

Definition 2.12. Suppose $D \in F(R)$, and D satisfies the following conditions hold:

(i) D is a normal, (ii) D is a convex fuzzy set, (iii) D is upper semicontinuous, (iv) $[D]_a$ is bounded, $a \in (j,h]$.

Then D is called a noncompact fuzzy number. We use $\tilde{E}$ to denote the noncompact fuzzy number space.

Definition 2.13. If $u(k)$ is a function of positive fuzzy numbers which satisfies (3), then $u(k)$ is called positive solution of (3).

Definition 2.14. A fuzzy number u is called an equilibrium for the equation (3) if $u = f(u,w,q)$.

Properties 2.15. Fuzzy arithmetic based on the following two properties of fuzzy numbers.

(i) Each fuzzy set and thus also each fuzzy number, can fully and uniquely be represented by generalized a-cuts.

(ii) The generalized a-cuts of each fuzzy number are closed intervals of real numbers for all $a \in (j,h]$.

Remark 2.16. [15] The four arithmetic operations on closed intervals are defined as follows:

(i) Addition : $[a,b]+[d,e]=[a+d,b+e]$ (ii) Subtraction : $[a,b]-[d,e]=[a-e,b-d]$ (iii) Multiplication : $[a,b][d,e]$

$$=[\min(ad,ae,bd,be),\max(ad,ae,bd,be)]$$

(iv) Division : $[a,b]\big/[d,e]=[a,b].\left[\frac{1}{e},\frac{1}{d}\right]$

$$=\left[\min\left(\frac{a}{d},\frac{a}{e},\frac{b}{d},\frac{b}{e}\right),\max\left(\frac{a}{d},\frac{a}{e},\frac{b}{d},\frac{b}{e}\right)\right]$$

provided $0 \notin [d,e]$.

The above four operations are also called interval valued arithmetic operations.

3. Main Results:

Lemma 3.1. Consider the system of generalized difference equation

$$u(k+h)=\alpha+\frac{\beta}{v(k)},\ and\ \ v(k+h)=\gamma+\frac{\delta}{u(k)},$$

$k \in [0,\infty)$, $h \in (0,\infty)$, (9)

where $\alpha,\beta,\gamma,\delta,u(j),v(j)$ are positive reals. Then

(i) Equation

$$\gamma\lambda^2+\lambda(\delta-\alpha\gamma-\beta)-\alpha\delta=0$$

$$(resp.\alpha\mu^2+\mu(\beta-\alpha\gamma-\delta)-\gamma\beta=0) \qquad (10)$$

has two real distinct roots λ_1,λ_2 (resp. μ_1,μ_2) where one of them is positive and other is negative.

(ii) The solution of $u(k),v(k)$ of (9) with initial interval value $u(j)$, $v(j)$ has the form

$$u(2k+h)=\frac{\lambda_2 K_1^k u(h)-\lambda_1}{K_1^k u(h)-1},$$

$$v(2k+h)=\frac{\mu_2 K_2^k v(h)-\mu_1}{K_2^k v(h)-1},$$

$$u(2k)=\frac{\lambda_2 K_1^k u(j)-\lambda_1}{K_1^k u(j)-1}, \qquad (11)$$

$$v(2k)=\frac{\mu_2 K_2^k v(j)-\mu_1}{K_2^k v(j)-1},\ k\in[0,\infty)$$

where $K_1=\frac{\gamma\lambda_2+\delta}{\gamma\lambda_1+\delta}$, $K_2=\frac{\alpha\mu_2+\beta}{\alpha\mu_1+\beta}$,

$$u(k)=\frac{u(k)-\lambda_1}{u(k)-\lambda_2},v(k)=\frac{v(k)-\mu_1}{v(k)-\mu_2},k\in[0,\infty). \quad (12)$$

(iii) For any positive solution of (9),we have $\lim_{k\to\infty}u(k)=\max\{\lambda_1,\lambda_2\}$, $\lim_{k\to\infty}v(k)=\max\{\mu_1,\mu_2\}$

Proof. Since $\alpha,\beta,\gamma,\delta$ are positive constants. Hence the proof of (i) is obvious.

(ii) From equation (9), we have

$$u(2k+3h)=\frac{(\alpha\gamma+\beta)u(2k+h)+\alpha\delta}{\gamma u(2k+h)+\delta},$$

$$v(2k+3h)=\frac{(\alpha\gamma+\delta)v(2k+h)+\gamma\beta}{\alpha v(2k+h)+\beta},$$

$$u(2k+2h)=\frac{(\alpha\gamma+\beta)u(2k)+\alpha\delta}{\gamma u(2k)+\delta}, \qquad (13)$$

$$v(2k+2h)=\frac{(\alpha\gamma+\delta)v(2k)+\gamma\beta}{\alpha v(2k)+\beta}.$$

(11) and (12) follows from (13).

(iii) Suppose that

$$\max\{\lambda_1,\lambda_2\}=\lambda_1>0,\ \max\{\mu_1,\mu_2\}=\mu_2>0. (14)$$

The other cases can be discuss similarly.

First we prove that

$$|K_1|<1. \qquad (15)$$

Let us assume the contrary that $|K_1|\geq 1$. Then

$$-|\gamma\lambda_2+\delta|\leq\gamma\lambda_1+\delta\leq|\gamma\lambda_2+\delta|. \qquad (16)$$

From (10) and (14), we have $\lambda_2<0$. Then since γ,δ are positive numbers and (16) holds it follows that

$$\gamma\lambda_2+\delta\leq-\gamma\lambda_1-\delta. \qquad (17)$$

Then from (10) and (17), we get $\frac{\alpha\gamma+\beta-\delta}{\gamma}=\lambda_1+\lambda_2\leq-\frac{2\delta}{\gamma}$, which is a contradiction, since α,β,γ and δ are positive numbers. Therefore (15) is true. Similarly, (14) holds, we can prove that

$$\left|K_2\right| > 1. \quad (18)$$

Hence, from (11), (15) and (18) the proof of (iii) follows immediately. This completes the proof of lemma.

Lemma 3.2. Let f be a continuous function from $[0,\infty)\times[0,\infty)\times[0,\infty)\to[0,\infty)$ and A,B,C are fuzzy numbers. Then

$$[f(A,B,C)]_a = f([A]_a,[B]_a,[C]_a),\ a\in(j,h].$$

Theorem 3.3. Let $D\in E$, and $[D]_a=[D_{-(a)},D_{+(a)}],\ a\in(j,h]$. Then $D_{-(a)}$ and $D_{+(a)}$ can be regarded as functions on $(j,h]$ which satisfy

(i) $D_{-(a)}$ is nondecreasing and left continuous, (ii) $D_{+(a)}$ is nonincreasing and left continuous, (iii) $D_{-(1)}\le D_{+(1)}$.

Proposition 3.4. Let A and B be the positive fuzzy numbers of (1). Then for every positive fuzzy number $u(j)$ there exists a unique positive solution $u(k)$ of (1) with initial interval value $u(j),\ j\in[0,h)$.

Proof. Suppose that there exists a function of fuzzy numbers $u(k)$ satisfying (1) with initial interval value $u(j)$. Consider the generalized a-cuts for $a\in(j,h]$,

$$[u(k)]_a=[L_{k,a},R_{k,a}],\quad k\in[0,\infty),$$
$$[A]_a=[A_{\ell,a},A_{r,a}],\quad [B]_a=[B_{\ell,a},B_{r,a}]. \quad (19)$$

Then from (1), (19) and lemma 3.2, it follows that

$$[u(k+h)]_a=[L_{k+h,a},R_{k+h,a}] = \left[A+\frac{B}{u(k)}\right]_a$$
$$=\left[A_{l,a}+\frac{B_{\ell,a}}{R_{k,a}},A_{r,a}+\frac{B_{r,a}}{L_{k,a}}\right]$$

from which for $k\in[0,\infty)$ we take,

$$L_{k+h,a}=A_{\ell,a}+\frac{B_{\ell,a}}{R_{k,a}},$$

Then

$$R_{k+h,a}=A_{r,a}+\frac{B_{r,a}}{L_{k,a}},\ a\in(j,h]. \quad (20)$$

from Lemma 3.1, eqution (20) has a unique solution $(L_{k,a},R_{k,a})$ with initial value $(L_{0,a},R_{0,a})$ which is given by (11) and (12) where

$$\alpha=A_{\ell,a},\quad \beta=B_{\ell,a},\quad \gamma=A_{r,a},$$
$$\delta=B_{r,a},\quad u(k)=L_{k,a},\quad v(k)=R_{k,a}.$$

Conversely, we prove that $[L_{k,a},R_{k,a}]$, where $[L_{k,a},R_{k,a}]$, is the solution of the system (20) with initial interval value $(L_{0,a},R_{0,a})$, determines the solution $u(k)$ of (1) with initial interval value $u(j)$ such that

$$[u(k)]_a=[L_{k,a},R_{k,a}],\ a\in(j,h],\ k\in[0,\infty).(21)$$

From Theorem 3.3 and $A,B,\ u(0)$ are fuzzy numbers for $a_1,a_2\in(j,h],\ \ a_1\le a_2$, we have

$$0<A_{\ell,a_1}\le A_{\ell,a_2}\le A_{r,a_2}\le A_{r,a_1},$$
$$0<B_{\ell,a_1}\le B_{\ell,a_2}\le B_{r,a_2}\le B_{r,a_1}, \quad (22)$$
$$0<L_{0,a_1}\le L_{0,a_2}\le R_{0,a_2}\le R_{0,a_1}.$$

By induction method, we prove

$$0<L_{k,a_1}\le L_{k,a_2}\le R_{k,a_2}\le R_{k,a_1},k\in[0,\infty). \quad (23)$$

From (22) we have that (23) hold for $k=0$, suppose that (23) are true for $k=n$ then from (20) and (22), it follows that

$$L_{n+h,a_1}=A_{\ell,a_1}+\frac{B_{\ell,a_1}}{R_{n,a_1}}\le A_{\ell,a_2}+\frac{B_{\ell,a_2}}{R_{n,a_2}}=L_{n+h,a_2}$$
$$=A_{\ell,a_2}+\frac{B_{\ell,a_2}}{R_{n,a_2}}\le A_{r,a_2}+\frac{B_{r,a_2}}{L_{n,a_2}}=R_{n+h,a_2}$$
$$=A_{r,a_2}+\frac{B_{r,a_2}}{L_{n,a_2}}\le A_{r,a_1}+\frac{B_{r,a_1}}{L_{n,a_1}}=R_{n+h,a_1}.$$

Therefore (23) are satisfied. Moreover from (20) we have,

$$L_{1,a}=A_{\ell,a}+\frac{B_{\ell,a}}{R_{0,a}},R_{1,a}=A_{r,a}+\frac{B_{r,a}}{L_{0,a}}. \quad (24)$$

Since $A,B,u(j)$ are positive fuzzy numbers from Theorem 3.3, we have $A_{\ell,a},A_{r,a},B_{\ell,a},B_{r,a},L_{0,a},R_{0,a}$ are left continuous. So from (24) we have that $L_{1,a},R_{1,a}$ are also left continuous. By using induction method, we can easily prove that $L_{k,a},R_{k,a},\ k\in[h,\infty)$ are left continuous.

Now, we prove that the support of $u(k)$, supp $u(k)=\overline{\bigcup_{a\in(j,h]}[L_{k,a},R_{k,a}]}$ is compact. It is sufficient to prove that $\bigcup_{a\in(j,h]}[L_{k,a},R_{k,a}]$ is bounded. Let $k=1$.

Since $A,B,u(j)$ are positive fuzzy numbers, there exists constants $M_i>0,N_i>0,i=0,1,2$ such that for all $a \in (j,h]$, .

$[A_{\ell,a}, A_{r,a}] \subset [M_1, N_1], [B_{\ell,a}, B_{r,a}] \subset [M_2, N_2],$

$$[L_{0,a}, R_{0,a}] \subset [M_0, N_0]. \quad (25)$$

Therefore, from (24) and (25), we can easily prove that

$$[L_{1,a}, R_{1,a}] \subset \left[M_1 + \frac{M_2}{N_0}, N_1 + \frac{N_2}{M_0} \right], a \in (j,h]$$

from which it is obvious that

$$\bigcup_{a \in (j,h]} [L_{1,a}, R_{1,a}] \subset \left[M_1 + \frac{M_2}{N_0}, N_1 + \frac{N_2}{M_0} \right]. \quad (26)$$

(26) implies that $\overline{\bigcup_{a \in (j,h]} [L_{1,a}, R_{1,a}]} \subset (0,\infty)$ is compact. By induction method, we will prove $\overline{\bigcup_{a \in (j,h]} [L_{k,a}, R_{k,a}]}$ is compact. i.e.,

$$\overline{\bigcup_{a \in (j,h]} [L_{k,a}, R_{k,a}]} \subset (0,\infty), \quad k \in [0,\infty). \quad (27)$$

Therefore, from theorem 3.3 ,equations (23), (27) and since $L_{k,a}, R_{k,a}$ are left continuous. We have that [$L_{k,a}, R_{k,a}$] determines a function of positive fuzzy numbers $u(k)$ such that (21) holds. We prove now that $u(k)$ is the solution of (1) with initial interval value $u(j)$. For all $a \in (j,h]$, $[u(k+h)]_a = [L_{k+h,a}, R_{k+h,a}]$

$$= \left[A_{\ell,a} + \frac{B_{\ell,a}}{R_{k,a}}, A_{r,a} + \frac{B_{r,a}}{L_{k,a}} \right] = \left[A + \frac{B}{u(k)} \right]_a$$

we have that $u(k)$ is the solution of (1) with initial interval value $u(j)$.

Suppose that there exists another solution $\overline{u(k)}$ of (1) with initial interval value $u(j)$.

Then arguing as above, we can easily prove

$$\overline{[u(k)]}_a = [L_{k,a}, R_{k,a}], k \in [0,\infty), a \in (j,h]. (28)$$

From (21) and (28) we have that $[u(k)]_a = \overline{[u(k)]}_a, \quad a \in (j,h], \quad k \in [0,\infty)$. From which it holds $u(k) = \overline{u(k)}, k \in [0,\infty)$.

Thus the proof is completed.

Lemma 3.5. Let U,V be fuzzy numbers and $[U]_a = [U_{\ell,a}, U_{r,a}], [V]_a = [V_{\ell,a}, V_{r,a}],\ a \in (j,h]$ be the generalized a-cuts of U, V respectively. Let W be a fuzzy number such that $[W]_a = [W_{\ell,a}, W_{r,a}],\ a \in (j,h]$. Then $MIN\{U,V\} = W$ (resp. $MAX\{U,V\} = W$) if and only if $min\{U_{\ell,a}, V_{\ell,a}\} = W_{\ell,a}$, $min\{U_{r,a}, V_{r,a}\} = W_{r,a}$, (*resp.* $\max\{U_{\ell,a}, V_{\ell,a}\} = W_{\ell,a}$, $\max\{U_{r,a}, V_{r,a}\} = W_{r,a}$).

Proposition 3.6. Every positive solution of (1) where A *and* B are positive fuzzy numbers is bounded and persists.

Proof. Let $u(k)$ be a positive solution of (1) and the solution u(k) satisfied equation (19). From (20) it is obvious that

$$A_{\ell,a} \le L_{k,a}, A_{r,a} \le R_{k,a}, k \in [h,\infty), a \in (j,h]. (29)$$

Then, from (29) we get $[\min\{L_{k,a}, A_{\ell,a}\}, \min\{R_{k,a}, A_{r,a}\}] = [A_{\ell,a}, A_{r,a}]$.(30) So, from (30) and Lemma 3.5, it follows that

$$MIN\{u(k), A\} = A,\ k \in [0,\infty). \quad (31)$$

Moreover relations (20) and (29) imply that

$L_{k,a} \le D_{\ell,a},\ R_{k,a} \le D_{r,a},\ k \in [2h,\infty),\ a \in (j,h]$

$$D_{\ell,a} = A_{\ell,a} + \frac{B_{\ell,a}}{A_{r,a}},\ D_{r,a} = A_{r,a} + \frac{B_{r,a}}{A_{\ell,a}}. \quad (32)$$

Using (22) for $0 < a_1 \le a_2$, we get

$$0 < D_{\ell,a_1} \le D_{\ell,a_2} \le D_{r,a_2} \le D_{r,a_1}. \quad (33)$$

From Theorem 3.3 and (32) we have that $D_{\ell,a}, D_{r,a}$ are left continuous. Moreover from (32) and (25) we get

$$[D_{\ell,a}, D_{r,a}] \subset \left[M_1 + \frac{M_2}{N_1}, N_1 + \frac{N_2}{M_1} \right]$$

from which it is obvious that $\overline{\bigcup_{a \in (j,h]} [D_{\ell,a}, R_{r,a}]}$ is compact. Hence from Theorem 3.3, (33) and since $D_{\ell,a}, D_{r,a}$ are left continuous there exists a fuzzy number D such that $[D]_a = [D_{\ell,a}, D_{r,a}]$. Using (32) and Lemma 3.5, we have

$$MAX\{u(k), D\} = D,\ k \in [2h,\infty). \quad (34)$$

Therefore, from (31) and (34) the proof of the proposition is completed.

In the following proposition we study the existence of a unique positive equilibrium u of (1) and the convergence of the positive solutions of (1).

Proposition 3.7. For the equation (1) satisfies the following statements.

(i) Equation (1) has a unique positive equilibrium u.

(ii) Every positive solution $u(k)$ of (1) tends to the positive equilibrium u as $k \to \infty$.

Proof. (i) Consider the equation

$$L_a = A_{\ell,a} + \frac{B_{\ell,a}}{R_a}, R_a = A_{r,a} + \frac{B_{r,a}}{L_a}, a \in (j,h]. (35)$$

Then the positive solution (L_a, R_a) of (35) is given by

$$L_a = \frac{A_{\ell,a}A_{r,a} + B_{\ell,a} - B_{r,a}}{2A_{r,a}} + \frac{\sqrt{(A_{\ell,a}A_{r,a} + B_{\ell,a} - B_{r,a})^2 + 4A_{\ell,a}A_{r,a}B_{r,a}}}{2A_{r,a}},$$

$$R_a = \frac{A_{\ell,a}A_{r,a} + B_{r,a} - B_{\ell,a}}{2A_{\ell,a}} + \frac{\sqrt{(A_{l,a}A_{r,a} + B_{r,a} - B_{l,a})^2 + 4A_{l,a}A_{r,a}B_{l,a}}}{2A_{\ell,a}}. \quad (36)$$

Let $u(k)$ be a positive solution of (1) such that $[u(k)]_a = [L_{k,a}, R_{k,a}], a \in (j,h], k \in [0,\infty)$.

Then applying Lemma 3.1 to (20) , we have

$$\lim_{k\to\infty} L_{k,a} = L_a, \quad \lim_{k\to\infty} R_{k,a} = R_a. \quad (37)$$

Hence, equation (23) and (37) for $0 < a_1 \le a_2 \le 1$ imply that

$$0 < L_{a_1} \le L_{a_2} \le R_{a_2} \le R_{a_1}. \quad (38)$$

Since $A_{\ell,a}, A_{r,a}, B_{\ell,a}, B_{r,a}$ are left continuous from (36) L_a, R_a are also left continuous. From (25) and (36) we get

$$R_a \le d = \frac{N_1^2 + N_2 - M_2}{2M_1} + \frac{\sqrt{(N_1^2 + N_2 - M_2)^2 + 4N_1^2 N_2}}{2M_1}. \quad (39)$$

Then, from (25), (35) and (39) we get

$$L_a \ge c = M_1 + \frac{M_2}{d}. \quad (40)$$

Therefore, equation (39) and (40) imply that $[L_a, R_a] \subset [c,d]$ from which it is obvious that $\overline{\bigcup_{a\in(j,h]} [L_a, R_a]}$ is compact and

$$\overline{\bigcup_{a\in(j,h]} [L_a, R_a]} \subset (0,\infty). \quad (41)$$

So from Theorem 3.3, equations (19), (35), (38), (41) and since are L_a, R_a left continuous, we have that $[L_a, R_a],\ a \in (j,h]$, determines a fuzzy number u such that $u = A + \frac{B}{u},\ [u]_a = [L_a, R_a],\ a \in (j,h]$ and so u is a positive equlibrium (1).

Suppose that there exists another positive equilibrium $\bar{u}$ for (1). Then there exist functions $\overline{L_a} : (j,h] \to (0,\infty),\ \overline{R_a} : (j,h] \to (0,\infty)$, such that

$$\bar{u} = A + \frac{B}{\bar{u}},\ [\bar{u}]_a = [\overline{L_a}, \overline{R_a}],\ a \in (j,h]$$

from which we get $\overline{L_a} = A_{\ell,a} + \frac{B_{\ell,a}}{\overline{R_a}}$,

$\overline{R_a} = A_{r,a} + \frac{B_{r,a}}{\overline{L_a}}$ and so $L_a = \overline{L_a},\ R_a = \overline{R_a}$, $a \in (j,h]$. Hence $u = \bar{u}$. This completes (i).

(ii) From (37) we have

$$\ell\text{im}\, D(u(k),k) = \ell\text{im}\sup\{\max\{|L_{k,a} - L_a|, |R_{k,a} - R_a|\}\} = 0, k \to \infty$$

where sup is taken for all $a \in (j,h]$. This completes the proof of the proposition.

Lemma 3.8. Let us assume the system of difference equations (9). Then the following statements are true.

(i) System (9) has a unique positive equilibrium (u,v) .

(ii) Every positive solution $(u(k), v(k))$ of (9) oscillates about (u,v) , if and only if $u(0) \ge u,\ v(0) \ge v$ or $u(0) \le u,\ v(0) \le v$.

Proof. (i) From Lemma 3.1 we have (u,v) , where $y = \max\{\lambda_1, \lambda_2\},\ z = \max\{\mu_1, \mu_2\}$ is the unique positive equilibrium of (9).

(ii) Let $(u(k), v(k))$ be a solution of (9) such that

$$u(0) \ge u,\ v(0) \ge v. \quad (42)$$

Then, from (9) and (42) we get

$$u(h) = \alpha + \frac{\beta}{v(j)} \le \alpha + \frac{\beta}{v} = u,$$

$$v(h) = \gamma + \frac{\delta}{u(j)} \le \gamma + \frac{\delta}{u} = v$$

$$u(2h) = \alpha + \frac{\beta}{v(h)} \ge \alpha + \frac{\beta}{v} = u,$$

$$v(2h) = \gamma + \frac{\delta}{u(h)} \ge \gamma + \frac{\delta}{u} = v$$

and by induction method, we get

$$u(2k) \ge u, v(2k) \ge v,$$
$$u(2k+h) \le u, v(2k+h) \le v,\ k \in [0,\infty). \quad (43)$$

Similarly if $u(0) \le u,\ v(0) \le v$ then we can easily prove that for $k \in [0,\infty)$

$$u(2k) \le u,\ v(2k) \le v,$$
$$u(2k+h) \ge u,\ v(2k+h) \ge v. \quad (44)$$

Therefore from (43) and (44), we obtain $(u(k), v(k))$ oscillates about (u, v). Let $(u(k), v(k))$ be a solution of (9) such that $u(0) > u, v(0) < v \ or \ u(0) < u, v(0) > v.$ (45)
Then from (9), (45) and using the above relation, we have $u(k) > u,\ v(k) < v$ or $u(k) < u,\ v(k) > v,\ k \in [h, \infty)$ from which $(u(k), v(k))$ does not oscillates about (u, v). This completes the proof of the lemma.

Proposition 3.9. Consider Equation (1) where A, B are positive fuzzy numbers. Then a positive solution $u(k)$ of (1) oscillates about the positive equilibrium u of (1) if and only if
$L_{0,a} \ge L_a, R_{0,a} \ge R_a \ or \ L_{0,a} \le L_a, R_{0,a} \le R_a.$(46)

Proof. Suppose that equation (46) are satisfied. Then since (20), (35) hold, from Lemma 3.8 for every $k_0 \in K$ there exist $k_1, k_2 \in K, k_1, k_2 \ge k_0$ such that for $a \in (j, h]$

$$(L_{k_2,a} - L_a)(L_{k_1,a} - L_a) \le 0,$$
$$(R_{k_2,a} - R_a)(R_{k_1,a} - R_a) \le 0$$
$$(L_{k_2,a} - L_a)(R_{k_2,a} - R_a) \ge 0, \quad (47)$$
$$(L_{k_1,a} - L_a)(R_{k_1,a} - R_a) \ge 0.$$

Relations (47) are equivalent to

$$[\min\{L_{k_1,a}, L_a\}, \min\{R_{k_1,a}, R_a\}] = [L_{k_1,a}, R_{k_1,a}]$$
$$[\min\{L_{k_2,a}, L_a\}, \min\{R_{k_2,a}, R_a\}] = [L_a, R_a] (48)$$
$$or\ [\min\{L_{k_1,a}, L_a\}, \min\{R_{k_1,a}, R_a\}] = [L_a, R_a]$$
$$[\min\{L_{k_2,a}, L_a\}, \min\{R_{k_2,a}, R_a\}] = [L_{k_2,a}, R_{k_2,a}].$$

Then using (8), (48) and Lemma 3.5 the solution $u(k)$ oscillates about u .

Conversely suppose that a solution $u(k)$ of (1) oscillates about u .Then from (8) and Lemma 3.5 we have that (48) are satisfied from which it is obvious that (47) hold. Therefore from Lemma 3.8 relations (46) are true. This completes the proof of the proposition.

4. Applications of Fuzzy Arithmetic Operations

Theorem 4.1. Let $u(k)$ be real valued function. Then,

$$\Delta_h^{-1} u(k)\Big|_j^k = \sum_{r=1}^{\left[\frac{k}{h}\right]} u(k - rh). \quad (49)$$

.*Proof.* The proof follows from relation

$$\Delta_h \sum_{r=1}^{\left[\frac{k}{h}\right]} u(k - rh) = u(k)$$

Theorem 4.2. If $k_h^{(n)} = k(k-h)\dots(k-(n-1)h)$ is the generalized polynomial factorial, then

$$\sum_{r=1}^{\left[\frac{k}{h}\right]} (k - rh)_h^{(n)} = \frac{k_h^{(n+1)}}{(n+1)h}\Bigg|_j^k. \quad (50)$$

Proof. The proof follows by substituting $u(k) = k_h^{(n)}$ in (49).

Theorem 4.3. If $k \in [0, \infty)$ and S_r^n's are stirling numbers of second kind, then

$$\sum_{r=1}^{\left[\frac{k}{h}\right]} (k - rh)^n = \sum_{r=1}^{n} S_r^n h^{n-r} \frac{k_h^{(r+1)}}{(r+1)h}\Bigg|_j^k. \quad (51)$$

Proof. The proof follows by taking $u(k) = k^n$ in (49).

Example 4.4. Let $S_1 = 2, 5, 8, \dots, 77$ and $S_2 = 3, 7, 11, \dots, 103$ are sequence of numbers, and $S_{1,2} = [4, 27] + [25, 343] + \dots + [5929, 1092727]$

Then by taking $n = 2$ in (51), we find

$$2^2 + 5^2 + 8^2 + \dots + 77^2 = S_1^2 \left[\frac{80_3^{(2)}}{2} - \frac{2_3^{(2)}}{2}\right] +$$
$$S_2^2 \left[\frac{80_3^{(3)}}{9} - \frac{2_3^{(3)}}{9}\right] = 53729 \quad (52)$$

and taking $n = 3$ in (51), we get

$$3^3 + 7^3 + 11^3 + \dots + 103^3 = S_1^3 2\left[107_4^{(2)} - 3_4^{(2)}\right] +$$
$$S_2^3 \left[\frac{107_4^{(3)}}{3} - \frac{3_4^{(3)}}{3}\right] + S_3^3 \left[\frac{107_4^{(4)}}{16} - \frac{3_4^{(4)}}{16}\right]$$
$$= 7591402. \quad (53)$$

From (51) and (52), we find

$$S_{1,2} = [2^2, 3^3] + [5^2, 7^3] + [8^2, 11^3] + \dots + [77^2, 103^3]$$
$$= [2^2 + 5^2 + \dots + 77^2, 3^3 + 7^3 + \dots + 103^3]$$
$$[2^2 + 5^2 + 8^2 + \dots + 77^2, 3^3 + 7^3 + 11^3 + \dots + 103^3]$$
$$= [53729, 7591402].$$

Example 4.5. Let $S_1 = 10, 13, 16, \dots, 46$ and $S_2 = 9, 13, \dots, 61$ are sequence of numbers, $S^{(1),(2)} = [70, 45] + [130, 585] + \dots + [1978, 184281].$

Then by taking $n = 2$ in (50), we have

$$10_3^{(2)} + 13_3^{(2)} .... + 46_3^{(2)} = \frac{49_3^{(3)}}{9} - \frac{10_3^{(3)}}{9} = 10738 \quad (54)$$

and substituting $n = 3$ in (50), we obtain

$$9_4^{(3)} + 13_4^{(3)} + ... + 61_4^{(3)}$$

$$= \frac{65_4^{(4)}}{16} - \frac{9_4^{(4)}}{16} = 748650. \qquad (55)$$

Using (54) and (55), we find

$$S^{(1),(2)} = [10_3^{(2)} + ... + 46_3^{(2)}, 9_4^{(3)} + ... + 61_4^{(3)}]$$

$$= [10_3^{(2)}, 9_4^{(3)}] + [13_3^{(2)}, 13_4^{(3)}] + ... + [46_3^{(2)}, 61_4^{(3)}]$$

$$= [10738, 748650].$$

References:

1. Agarwal R.P., “Difference Equations and Inequalities,” Marcel Dekker, New York, 1992.
2. Deeba E, De Korvin A, Koh EL., “A Fuzzy Difference Equation With an Application,” J. Difference Eqn. Appl, 2 (1996), 365-374.
3. Deeba E, De Korvin A., “Analysis by fuzzy difference equations of a model of CO_2 level in the blood,” Appl. Math. Lett., 12 (1999), 33-40.
4. Diamond P., “Brief note on the Variation of Constants Formula for Fuzzy Differential Equations,” Fuzzy Sets and Systems, 129 (2002), 65-71.
5. Elaydi S.N., “An Introduction to Difference Equations,” Springer-Verlag, New York, 1996.
6. Gnana Bhaskar T, Lakshmikantham V, Vasundhara Devi1 Revisiting., “Fuzzy Differential Equations,” Nonlinear Analysis, 58 (2004), 351-358.
7. James J. Buckleya, Thomas Feuring, “Fuzzy Differential Equations,” Fuzzy Sets and Systems, 110 (2000), 43-54.
8. Kocic VL,Ladas G., “Global Behavior of Nonlinear Difference Equations of Higher Order with Applications,” Kluwer, Academic Publishers, Dordrecht,1993.
9. Lakshmikantham V and A.S. Vatsala, “Basic Theory of Fuzzy Difference Equations,” J. Difference Equations and Applications, 8 (11)(2002), 978-989.
10. M.M.S.Manuel, G.B.A.Xavier and E.Thandpani, “Theory of Generalized Difference Operator and Its Applications,” Far East Journal of Mathematical Sciences, 20 (2) (2006), 163-171.
11. Negoita C.V, Ralescu D., “Applications of Fuzzy Sets to Systems Analysis,” Birkhauser, Verlag, Besel, 1975.
12. Osmo Kaleva, “A note on Fuzzy Differential Equations,” Nonlinear Analysis, 64, (2006), 895-900.
13. Pearson D.W., “A Property of Linear Fuzzy Differential Equations,” Appl. Math. Lett., 10(3) (1997), 99-103.
14. Ronald E Mickens, “Difference Equtions,” Van Nostrand Reinhold Company, New York, 1990.
15. Shiv Raj Shingh, Caman Singh, Fuzzy Set Theory, KRISHNA Prakashan Media (P) Ltd, First Edition, 2012.
16. W.G.Kelley,A.C. Peterson, “Difference Equations: An Introduction with Applications,”Academic Press, 1991.
17. Wu C, Zhang B., “Embedding Problem of Noncompact Fuzzy Number Space E,” Fuzzy Sets and Systems 105 (1999), 165-169.

***

Department of Mathematics,SKP Engineering College,
Tiruvannamalai , Tamil Nadu, S.India.
drchanmaths@gmail.com,jkathirmaths@gmail.com