OSCILLATION OF GENERALIZED SECOND KIND LINEAR DIFFERENCE EQUATION

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Abstract: In this paper, the authors obtain new sufficient conditions of all solutions the generalized second kind difference equation (1)where and are generalized difference equation of second kind and first kind respectively, p(k) is nonnegative real valued functions. Our results improve the know results in the literature.

Key words: Oscillation, Second kind, Generalized difference equation
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Introduction: Difference equation usually describes the evaluation of some certain phenomena over time and is also important in describing dynamics for fundamentally discrete system, see [1]. For example, in the numerical integration, the standard approach is to use the difference equation, similarly, the population dynamics have discrete generations; the size of the generation is a function of the kth generation.This can be expressed as $u(k+1) = f(u(k))$.

Further, the concepts of difference equations with many examples in application such as asymptotic behavior of solution of difference equations were studied extensively by Elayadi [9] where the analytic and geometric approaches were also combined in order to studying difference equations. Further, in [8], both classical modern treatments of the difference equations were presented in excellent form.

Consequently the basic theory of difference equations is based on operator Δ is

$$\Delta u(k) = u(k+1) - u(k), \quad k \in N \qquad (2)$$

where N={0,1,2,3,4..........}. Even though many authors [1, 8, 9] have suggested the definition of Δ as

$$\Delta u(k) = u(k+\ell) - u(k), k \in N, \ell \in R - \{0\} \quad (3)$$

and there are several research took place on this line. By defining its inverse Δ_ℓ^{-1}, many interesting results and applications in number theory as well as in fluid dynamics can be obtained. By extending the study for sequences of complex numbers and ℓ to be real, some new qualitative properties like rotatory, expending, shrinking, spiral and weblike structures were studied for the solutions of difference equations involving $\Delta_{\ell.}$ for similar results, refer to [2-6].

In particular, when $\ell_1 = \ell_2 = \ell$ the generalized second kind linear difference equations becomes a generalized second order linear difference equations and also by taking $\ell_1 = \ell_2 = \ell = 1$, the above equations becomes the second order linear difference equations, were discussed in [10].

In this paper, we discuss the sufficient conditions for oscillation of generalized second kind linear difference equation(1).

Throughout this paper, we use the following notations.

$(i)\ \ u_n(\alpha) = n^{1-\alpha} \infty \sum_{k=n+1} k^\alpha p(k), \ \ for\ \alpha < 1,$

$(ii)\ \ p_*(\alpha) = \liminf u_n(\alpha), p^*(\alpha) = \limsup u_n(\alpha),$

$(iii)\ \ q = \liminf_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} k^2 p(k).$

$(iv)\ k$ means by either $kl_1 + j\ (or)\ kl_2 + j$

Preliminaries:

Definition 2.1.[7] Let $u(k)$ be real valued function. Then the generalized difference operator of second kind is defined as

$$\Delta_{\ell_1,\ell_2} u(k) = u(k+\ell_1+\ell_2) - [u(k+\ell_1) + u(k+\ell_2)] + u(k). \qquad (4)$$

Definition 2.2.[7] The inverse of generalized difference operator of the second kind denoted by $\Delta_{\ell_1,\ell_2}^{-1}$ is defined as follows. If $\Delta_{\ell_1,\ell_2} v(k) = u(k),$ then

$$v(k) = \Delta_{\ell_1,\ell_2}^{-1} u(k) + \frac{c_{1j}}{\ell} k + c_{0j}. \qquad (5)$$

Theorem 2.3. If

$$\liminf_{n\to\infty} n \sum_{k=n+1}^{\infty} p(k) > \frac{1}{4}. \qquad (6)$$

Then every solution of equation (1) is oscillatory.

Theorem 2.4. If for all large n

$$n \sum_{k=n+1}^{\infty} p(k) < \frac{1}{4}. \qquad (7)$$

Then the equation (1) has a nonoscillatory solution.

Main Results:

Lemma 3.1. Assume that $\alpha \in [0,1)$. Then

$$\sum_{k=n+1}^{\infty} \Delta_{\ell_1,\ell_2} k^\alpha \le \frac{\alpha^2}{1-\alpha} n^{\alpha-1}, \qquad (8)$$

$$\frac{(n+1)^{1-\alpha}}{1-\alpha} \le \sum_{k=n+1}^{\infty} k^{\alpha-2} \le \frac{n^{\alpha-1}}{1-\alpha}. \qquad (9)$$

Proof. By the mean value theorem, there exist $\xi_k \in (k, k+\ell_1+\ell_2), \eta_k \in (k-\ell_1-\ell_2)$ such that

$$\frac{\Delta_{\ell_1,\ell_2} k^{\alpha}}{k^{\alpha}} = \frac{\alpha^2 \xi_k^{2\alpha-2}}{k^{\alpha}} \le \frac{\alpha^2 k^{2\alpha-2}}{k^{\alpha}}, \qquad (10)$$

and $\dfrac{(n+1)^{1-\alpha}}{1-\alpha} = \eta_k^{-\alpha} \ge \dfrac{1}{k^{\alpha}}. \qquad (11)$

By (10) and (11), we have

$$\sum_{k=n+1}^{\infty} (\Delta_{\ell_1} k^{\alpha})^2 \le \frac{\alpha^2}{\ell_1-\alpha} \sum_{k=n+1}^{\infty} \frac{\Delta_{\ell_1}(k-\ell_1)^{1-\alpha}}{k^{2-2\alpha}}. (12)$$

We define for $k-\ell_1 \le t \le k$,

$$r(t) = (k-\ell_1)^{1-\alpha} + (t-k+\ell_1)\Delta_{\ell_1}(k-\ell_1)^{1-\alpha},$$

Then $r'(t) = \Delta_{\ell_1}(k-\ell_1)^{1-\alpha}$

and $(k-\ell_1)^{1-\alpha} \le r(t) \le k^{1-\alpha}, \quad k-\ell_1 \le t \le k.$

Hence,

$$\frac{\Delta_{\ell_1}(k-\ell_1)^{1-\alpha}}{k^{2-2\alpha}} = \int_{k-1}^{k} \frac{\Delta_{\ell_1}(k-\ell_1)^{1-\alpha}}{k^{2-2\alpha}} dt$$

$$\le \int_{k-1}^{k} \frac{r'(t)}{r^2(t)} dt = \frac{1}{(k-1)^{1-\alpha}} - \frac{1}{k^{1-\alpha,}}$$

which, together with (12), yields

$$\sum_{k=n+1}^{\infty} \frac{\Delta_{\ell_1,\ell_2} k^{\alpha}}{k^{\alpha}} \le \frac{\alpha^2}{1-\alpha} n^{\alpha-1,}$$

since, $\displaystyle\int_{n+1}^{\infty} \frac{1}{t^{2-\alpha}} dt < \int_{n}^{\infty} \frac{1}{t^{2-\alpha}} dt.$

Hence, (9) holds. The proof is complete.

Corollary 3.2 Assume that $\alpha \in [0,1)$. Then

$$\sum_{k=n+1}^{\infty} \Delta_{\ell}^2 k^{\alpha} \le \frac{\alpha^2}{1-\alpha} n^{\alpha-1},$$

$$\frac{(n+1)^{1-\alpha}}{1-\alpha} \le \sum_{k=n+1}^{\infty} k^{\alpha-2} \le \frac{n^{\alpha-1}}{1-\alpha}. \qquad (13)$$

Proof. The proof follows by substituting $\ell_1 = \ell_2 = \ell$ in the equations (8) and (9) respectively.

Lemma 3.3. Assume that $u(k)$ is a nonosciiiatory solution of equation (1) such that $u(k-\ell_1) > 0$ for $k \ge k_o$. Let $w(k) = \dfrac{\Delta_{\ell_1} u(k-\ell_1)}{u(k-\ell_2)}$. Then

$$\Delta_{\ell_2} w(k) + w(k)w(k+\ell_1) + p(k) \le 0,$$

$$w(k) \ge w(k+\ell_2), (k-k_0)w(k) < 1, k \ge k_0, (14)$$

and $\quad p_*(0) \le r - r^2, \;\; q \le R - R^2, \qquad (15)$

where $r = \liminf\limits_{k\to\infty} kw(k+\ell_2)$ *and*

$$R = \limsup_{k\to\infty} kw(k+\ell_2). \qquad (16)$$

Proof. Since $u(k)$ is a nonoscillatory solution of equation (1) such that $u(k-\ell_1) > 0$ for $k \ge k_0$. By (1), we can easily show that for $k \ge k_0$.

$$\Delta_{\ell_1} u(k-\ell_1) \ge 0, \;\; \Delta_{\ell_1,\ell_2} u(k-\ell_1) \le 0, \qquad (17)$$

In view of the definition of $w(k)$, we obtain

$$\Delta_{\ell_2} w(k) = \frac{\Delta_{\ell_1,\ell_2} u(k-\ell_1)}{u(k)} - \frac{\Delta_{\ell_1} u(k-\ell_1)}{u(k)u(k+\ell_2)}, \quad \text{for}$$

$k \ge k_0$. (18)

which, together with (1) and (17), yields

$$\Delta_{\ell_2} w(k) + w(k)w(k+\ell_1) + p(k) \le 0, \; k \ge k_0. \; (19)$$

Since $w(k) > 0$ for $k \ge k_0$, it follows that

$$w(k) \ge w(k+\ell_1), \;\; k \ge k_0, \qquad (20)$$

and $\dfrac{\Delta_{\ell_2} w(k)}{w(k)w(k+\ell_1)} < -1. \qquad (21)$

Summing (21) from k_0 to $k-1$, we have

$$(k-k_0)w(k) < 1, \qquad k \ge k_0, \qquad (22)$$

which implies that

$$\lim_{k\to\infty} w(k) = 0 \quad and \quad \lim_{k\to\infty} \frac{1}{v} \sum_{n=k_0}^{k} w(n) = 0. \; (23)$$

By (16) and (22), we have

$$0 \le r \le 1 \qquad and \qquad 0 \le R \le 1. \qquad (24)$$

Hence

$$r - r^2 \ge 0 \quad and \quad R - R^2 \ge 0. \qquad (25)$$

We now prove that (15) holds. By (25), we may assume that $p_*(0) \ne 0$ *and* $q \ne 0$. From (19) and (20), we easily find that for any $k_1 > 0$

$$kw(k+\ell_1) \ge k \sum_{n=k+1}^{\infty} p(n)$$

$$+k \sum_{n=k+1}^{\infty} \frac{(n-\ell_1)w(n)nw(n+\ell_1)}{(n-\ell_1)n}, \; k > k_1 \qquad (26)$$

and for any $k_1 \ge k_0$, we have

$$\Delta_{l_2} w(k) + w^2(k+\ell_1) + p(k) \le 0. \qquad (27)$$

Multiplying (27) by k^2 summing it from k_1 to k, we obtain (13)

$$\sum_{n=k_1}^{k} n^2 p(n) \le -\sum_{n=k_1}^{k} n^2 \Delta_{\ell_2} w(n) - \sum_{n=k_1}^{k} n^2 w^2(n+\ell_1)$$
$$= -(k+\ell_1)^2 w(k+\ell_1) + k_1^2 w(k_1)$$
$$+\sum_{n=k_1}^{k} w(n+\ell_1)\Delta_{\ell_2} n^2 - \sum_{n=k_1}^{k} n^2 w^2(n+\ell_1)$$
$$= -(k+\ell_1)^2 w(k+\ell_1) + k_1^2 w(k_1) +$$
$$\sum_{n=k_1}^{k} w(n+\ell_1) + \sum_{n=k_1}^{k} nw(n+\ell_1)(\ell_1+\ell_2 - nw(n+\ell_1)).$$

It follows that

$$kw(k+\ell_1) < \frac{k_1^2 p(k_1) + \sum_{n=k_1}^{k} w(n+\ell_1)}{k} - \sum_{n=k_1}^{k} \frac{n^2 p(n)}{k}$$
$$+\frac{1}{k}\sum_{n=k_1}^{k} nw(n+\ell_1)(\ell_1+\ell_2 - nw(n+\ell_1)), k > k_1. \quad (28)$$

Since $\lim_{k\to\infty} \frac{k_1^2 p(k_1) + \sum_{n=k_1}^{k} w(n+\ell_1)}{k} = 0$ and

$nw(n+\ell_1)(\ell_1+\ell_2 - nw(n+\ell_1)) < 1.$

By (26) and (28), we have

$r \ge p_*(0)$ *and* $R \le 1-q.$

It is easy to see that for any $\min\{j_1, j_2\} < \in < \min\{r, \ell_1 - R, \ell_2 - R\}$ there exist $k_2 > k_1$ such that

$r - \in < kw(k+\ell_1) < R + \in,$

$k\sum_{n=k+1}^{\infty} p(n) > p_*(0) - \in$ *for* $k > k_2.$

and $\frac{1}{k}\sum_{n=k_1}^{k} n^2 p(n) > q - \in,$ *for* $k > k_2.$

This, together with (26) and (28), yields

$kw(k+\ell_1) > p_*(0) - \in + (r-\in)^2$, for $k > k_2$

$$kw(k+\ell_1) < \frac{1}{k}\left(k_2^2 p(k_2) + \sum_{n=k_2}^{k} w(n+\ell_1)\right)$$
$$-q + \in + (R+\in)(\ell_1+\ell_2 - R - \in),$$

for $k > k_2$, which implies that

$r \ge p_*(0) + r^2$ and $R \le -q + R(2-R)$.

Thus (13) holds and the proof is complete.

Theorem 3.4. If

$$q = \liminf_{k\to\infty} \frac{1}{k}\sum_{n=1}^{k} n^2 p(n) > \frac{1}{4}, \quad (29)$$

then every solution of (1) is oscillatory.

Proof. If not, let $u(k)$ be a oscillatory solution of equation (1) such that $u(k-\ell_1) > 0$ for $k > k_0$. Let $R = \limsup_{k\to\infty} kw(k+\ell_1)$. By lemma 3.3, we have $q \le R - R^2 \le \frac{1}{4}$, which contradicts (29) and the proof is complete.

Theorem 3.5. Let $q \le 1/4$ and assume that there exists $\alpha \in [j, \ell_1)$ such that

$$p^*(\alpha) > \frac{\alpha^2}{4(1-\alpha)} + \frac{1}{2}\left(1+\sqrt{1-4q}\right). \quad (30)$$

Then every solution of (1) oscillates.

Proof. If not, let $u(k)$ be a non oscillatory solution of equation (1) such that $u(k-\ell_1) > 0$ for $k > k_0$. By lemma 3.3, we have

$$\Delta_{\ell_2} w(k) + w^2(k+\ell_1) + p(k) \le 0, \; k > k_0, \quad (31)$$

and $q \le R - R^2,$ (32)

where $w(k) = \frac{\Delta_{\ell_1} u(k-\ell_1)}{u(k-\ell_2)}$ and

$R = \limsup_{k\to\infty} kw(k+\ell_1).$

Equation (32) implies that

$$R \le M = \frac{1}{2}\left(1+\sqrt{\ell_1 - 4q}\right). \quad (33)$$

Multiply (31) by k^α summing it from $k+1$ to ∞, we obtain

$$\sum_{n=k+1}^{\infty} n^\alpha p(n) \le -\sum_{n=k+1}^{\infty} n^\alpha \Delta_{\ell_2} w(n)$$
$$-\sum_{n=k+1}^{\infty} n^\alpha w^2(n+\ell_1) = (k+\ell_2)^\alpha w(k+\ell_1+\ell_2)$$
$$+\sum_{n=k+1}^{\infty} w(n+\ell_2)\Delta_{\ell_2} n^\alpha - \sum_{n=k+1}^{\infty} n^\alpha w^2(n+\ell_1) \quad (34)$$
$$= (k+\ell_2)^\alpha w(k+\ell_1+\ell_2) + \frac{1}{4}\sum_{n=k+1}^{\infty} \frac{(\Delta_{\ell_2} n^\alpha)^2}{n^\alpha}$$
$$-\sum_{n=k+1}^{\infty}\left(n^{\alpha/2} w(n+\ell_1) - \frac{1}{2} n^{-\alpha/2} \Delta_{\ell_1} n^\alpha\right)^2$$
$$< (k+\ell_2)^\alpha w(k+\ell_1+\ell_2) + \frac{1}{4}\sum_{n=k+1}^{\infty} \frac{(\Delta_{\ell_2} n^\alpha)^2}{n^\alpha}. \quad (35)$$

It follows that

$$k^{1-\alpha}\sum_{n=k+1}^{\infty} n^{\alpha}p(n) < \left(\frac{k+\ell_2}{k}\right)^{\alpha} kw(k+\ell_1+\ell_2)$$

By

$$+\frac{1}{4}k^{1-\alpha}\sum_{n=k+1}^{\infty}\frac{\left(\Delta_{\ell_2}n^{\alpha}\right)^2}{n^{\alpha}}. \qquad (36)$$

(33), (36), and lemma 3.1, we have

$p^*(\alpha) \le \dfrac{\alpha^2}{4(1-\alpha)} + \dfrac{1}{2}(1+\sqrt{1-4q})$, which contradicts (30). The proof is complete.

Corollary 3.6. Assume that

$$q \le \frac{1}{4} \text{ and } p^*(0) > \frac{1}{2}\left(1+\sqrt{1-4q}\right). \qquad (37)$$

Then every solution of (1) oscillates.

Theorem 3.7. Let $p_*(0) \le 1/4$ and $q \le 1/4$ assume that there exists $\alpha \in [0,1)$ such that

$$p_*(0) > \frac{\alpha(2-\alpha)}{4} \qquad (38)$$

and

$$p^*(\alpha) > \frac{p_*(0)}{1-\alpha} + \frac{1}{2}\left(\sqrt{1-4p_*(0)}+\sqrt{1-4q}\right). (39)$$

Then every solution of (1) oscillates.

Proof. If not, let $u(k)$ be a non oscillatory solution of equation (1) such that $u(k-\ell_1)$ for $k > k_0$. By Lemma 3.3, we have

$$\Delta_{\ell_2}w(k) + w^2(k+\ell_1) + p(k) \le 0,\ k \ge k_0, (40)$$

and $\quad p_*(0) \le r - r^2$ and $q \le R - R^2 \qquad (41)$

where $r = \limsup_{k\to\infty} kw(k+\ell_1)$ $R = \limsup_{k\to\infty} kw(k+\ell_1)$.

Equation (40) implies that

$$r \ge m = \frac{1}{2}\left(1-\sqrt{1-4p_*(0)}\right) \qquad (42)$$

and $\quad R \le M = \dfrac{1}{2}\left(1+\sqrt{1-4q}\right). \qquad (43)$

By (37) and (41), we have $m > \dfrac{\alpha}{2}$. Hence, for any $0 < \in < m - \alpha/2$ there exists $k_1 > k_0$ such that

$$m - \in < kw(k+\ell_1+\ell_2) < \left(\frac{k+\ell_2}{k}\right)^{\alpha} kw(k+\ell_1+\ell_2) < M + \in,$$

for $k > k_1$. $\qquad (44)$

Multiplying (39) by k^{α}, summing it from $k+1$ to ∞, we have

$$\sum_{n=k+1}^{\infty} n^{\alpha}p(n) \qquad (45)$$

$$\le -\sum_{n=k+1}^{\infty} n^{\alpha}\Delta_{\ell_2}w(n) - \sum_{n=k+1}^{\infty} n^{\alpha}w^2(n+\ell_1)$$

$$= (k+\ell_2)^{\alpha_1}w(k+\ell_1) + \sum_{n=k+1}^{\infty} w(n+\ell_1)\Delta_{\ell_2}n^{\alpha}$$

$$-\sum_{n=k+1}^{\infty} n^{\alpha}w^2(n+\ell_1). \qquad (46)$$

By the mean value theorem, we can show that

$$\Delta_{\ell_1}n^{\alpha} = (n+\ell_1)^{\alpha} < \alpha n^{\alpha-1}, \qquad (47)$$

which, together with (45), yields

$$k^{1-\alpha}\sum_{n=k+\ell_1}^{\infty} n^{\alpha}p_n \le \left(\frac{k+\ell_2}{k}\right)^{\alpha} kw(n+\ell_1) +$$

$$k^{1-\alpha}\sum_{n=k+\ell_1}^{\infty} n^{\alpha-2}[n\,\mathrm{w}(n+\ell_1)(\alpha - n\,\mathrm{w}(n+\ell_1+\ell_2))]. (48)$$

By (41), (45), and Lemma 3.1, we obtain

$$k^{1-\alpha}\sum_{n=k+1}^{\infty} n^{\alpha}p_n \le \varepsilon + (\mathrm{m}-\varepsilon)(\alpha - \mathrm{m}+\varepsilon)k^{1-\alpha}$$

$$\sum_{n=k+1}^{\infty} n^{\alpha-2} < M + \varepsilon + \frac{(\mathrm{m}-\varepsilon)(\alpha-\mathrm{m}+\varepsilon)}{1-\alpha}, k > k_1$$

which, together with (41) and (42), implies that

$$p^*(\alpha) \le \mathrm{M} + \frac{m(\alpha-\mathrm{m})}{1-\alpha} = \frac{p_*(0)}{1-\alpha}$$

(42)

$$+\frac{1}{2}\left(\sqrt{1-4p_*(0)}+\sqrt{1-4q}\right).$$

Which contradicts (38) and hence the proof.

Corollary 3.8. Let $0 < p_*(0) \le \dfrac{1}{4}$, $q \le \dfrac{1}{4}$, and

$$p^*(0) > p_*(0) + \frac{1}{2}\left(\sqrt{1-4p_*(0)}+\sqrt{1-4q}\right). (47)$$

Then every solution (1) oscillates.

References:

1. R.P.Agarwal, Differences Equations and Inequalities, Marcel Dekkar, New Yark, 2000.
2. M.Maria Susai Manuel, G.Britto Antony Xavier and E.Thandapani, "Theory of Generalized Difference Operator and its Applications", For East Journal of Mathematical Sciences, 20(2)

(2006), 163-17.
3. M.Maria Susai Manuel, G.Britto Antony Xavier and E.Thadapani, "Qualitative Properties Of Solutions of Certain Class of Difference Equations", For East Journal of Mathematical Sciences, 23(3) (2006), 295-304.
4. M.Maria Susai Manuel, G.Britto Antony Xavier and E.Thadapani, "Generalized Bernoulli Polynomials Through Weighted Pochhammer Symbols", Far East Journal of Applied Mathematics, 26(3) (2007), 321-333.
5. M.Maria Susai Manuel, A.George Maria Selvam and G.Britto Antony Xavier, "Rotatory and Boundedness of solutions o Certain Class of Difference Equations", International J. of Pure and Mathematics, 33(3) (2006), 333-343.
6. M.Maria Susai Manuel, G.Britto Antony Xavier, "Recessive Dominant and Spiral Behaviours of Solutions of Certain Class of Difference Equations", International Journal of Differential Equations and Applications, 10(4) (2007), 423-433.
7. Ronald E.Mickens, Difference Equations, Van Nostrand Reinhold Company, New Yark, 1990.
8. Saber N. Elayadi, An Introduction to Difference Equations, Third Edition, Spinger, USA, 2000.
9. Xiaoping Li, Jianchu Jiang, "Oscillation of Second-Order Linear Difference Equations", Mathematical and Computer Modeling, 35(2002), 983-990.
10. B.G.Zhang and S.S.Cheng, "Oscillation Criteria and Comparison Theorems for Delay Difference Equations", Fasciculi Mathematici, 25 (1995), 13-32.
11. Y.Zhou and B.G.Zhang, "Oscillation of Delay Difference Equations in a Critical State", Comp. Math. Applic., 39 (7/8) (2000), 71-80.

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