A STUDY OF FS-FUNCTIONS AND PROPERTIES OF IMAGES OF FS-SUBSETS UNDER VARIOUS FS-FUNCTIONS

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Abstract: Vaddiparthi Yogesara, G.Srinivas and Biswajit Rath introduced the concept of Fs-set ,Fs-subset, complement an of Fs-subset and proved important results like De Morgan laws for Fs-sets which are called Fs-De Morgan laws. In this paper we introduce the concept of Fs-Function between Fs-sets and separate the collection of all Fs-functions into three categories , increasing Fs-Functions, decreasing Fs- Functions and preserving Fs-Functions. For any Fs-subset of a given Fs-set, we define image of Fs-subset under (1) Increasing Fs-Function, (2) Decreasing Fs-Function and (3) Preserving Fs-Function. Also we establish some properties of images of Fs- subsets under various Fs-Functions mentioned above.

Keywords: Fs-set, Fs-subset, Fs-empty set, Fs-union, Fs-intersection, Fs-complement, Fs-De Morgan laws and Fs-Function

Introduction: Murthy[1] introduced F-set in order to prove Axiom of choice for fuzzy sets which is not true for L-fuzzy sets introduced by Goguen[2]. In the paper[3], Tridiv discussed fuzzy compl-ement of an extended fuzzy subset and proved De Morgan laws etc. The extended Fuzzy sets Tridiv considered contains the membership value $\mu_1(x) - \mu_2(x)$. $-\mu_2(x)$, a term in this expression will not be in the interval [0,1]. In the paper[4], Vaddiparthi Yogeswara , G.Srinivas and Biswajit Rath introduced the concept of Fs-set and developed the theory of Fs-sets in order to prove collection of all Fs-subsets of given Fs-set is a complete Boolean algebra under Fs-unions, Fs-intersections and Fs-complements. The Fs-sets they introduced contain Boolean valued membership functions .All most they are successful in their efforts in proving that result with some conditions. In this paper we introduce the concept of Fs-Function between given Fs-sets and define various kinds of Fs-Functions which are increasing Fs-Function, decreasing Fs- Functions and preserving Fs-Functions. Also we introduce the concepts of images of Fs-subsets of given Fs-set under Fs-Functions of various kinds and prove some properties. For convenience of readers before beginning of the paper we mention various definitions and results in paper[4]. We denote the largest element of a complete Boolean algebra L_A[1.1] byM_A,the complement of b in L_A by b^c. For any crisp subset B, the usual set complement of B, is denoted by B^c and $B^c \cup A$ is denoted by $C_A B$ .Complete Boolean algebras in this paper are generally represented by suitable diagrams. We denote Fs-union and crisp set union by same symbol ∪ and similary Fs-intersection and crisp set intersection by the same symbol ∩.For all lattice theoretic properties and Boolean algebraic properties we refer Szasz [7], Garret Birkhoff[8],Steven Givant • Paul Halmos[8] and Thomas Jech[9]

I.Theory of Fs-sets

1.1 Fs-set: Let U be a universal set, $A_1 \subseteq U$ and let A⊆U be non-empty. A four tuple $\mathcal{A} = (A_1, A, \bar{A}(\mu_{1A_1}, \mu_{2A}), L_A)$ is said be an Fs-set if, and only if

(1) $A \subseteq A_1$

(2) L_A is a complete Boolean Algebra

(3) $\mu_{1A_1}: A_1 \longrightarrow L_A$, $\mu_{2A}: A \longrightarrow L_A$,are functions such that $\mu_{1A_1}|A \geq \mu_{2A}$

(4) $\bar{A}$:A⟶ L_A is defined by

$\bar{A}x = \mu_{1A_1}x \wedge (\mu_{2A}x)^c$,for each $x \in A$

1.2 Fs-subset

Let $\mathcal{A}=(A_1, A, \bar{A}(\mu_{1A_1},\mu_{2A}), L_A)$ and $\mathcal{B}=(B_1, B, \bar{B}(\mu_{1B_1},\mu_{2B}), L_B)$ be a pair of Fs-sets. $\mathcal{B}$ is said to be an Fs-subset of $\mathcal{A}$, denoted by $\mathcal{B}\subseteq\mathcal{A}$, if, and only if

(1) $B_1 \subseteq A_1$, $A \subseteq B$

(2) L_B is a complete subalgebra of L_A or $L_B \leq L_A$

(3) $\mu_{1B_1} \leq \mu_{1A_1}|B_1$, and $\mu_{2B}|A \geq \mu_{2A}$

1.3 Proposition: Let $\mathcal{B}$ and $\mathcal{A}$ be a pair of Fs-sets such that $\mathcal{B}\subseteq \mathcal{A}$. Then $\bar{B}x \leq \bar{A}x$ is true for each x ∈A

1.4 Definition: For some L_X, such that $L_X \leq L_A$ a four tuple $\mathcal{X} = (X_1, X, \bar{X}(\mu_{1X_1}, \mu_{2X}), L_X)$ is not an Fs-set if, and only if

(a)$X \nsubseteq X_1$ or

(b) $\mu_{1X_1}x \ngeq \mu_2 x$, for some $x \in X \cap X_1$

Here onwards, any object of this type is called an Fs-empty set of first kind and we accept that it is an Fs-subset of $\mathcal{B}$ for any $\mathcal{B} \subseteq \mathcal{A}$.

Definition: An Fs-subset $\mathcal{Y}=(Y_1, Y, \bar{Y}(\mu_{1Y_1}, \mu_{2Y}), L_Y)$ of $\mathcal{A}$, is said to be an Fs-empty set of second kind if, and only if

(a') $Y_1 = Y = A$

(b') $L_Y \leq L_A$

(c') $\bar{Y} = 0$

1.4.1 Remark: we denote Fs-empty set of first kind or

Fs-empty set of second kind by $\Phi_{\mathcal{A}}$ and we prove later (1.15), $\Phi_{\mathcal{A}}$ is the least Fs-subset among all Fs-subsets of $\mathcal{A}$.

Let $\mathcal{B}_1 = (B_{11}, B_1, \bar{B}_1(\mu_{1B_{11}}, \mu_{2B_1}), L_{B_1})$ and $\mathcal{B}_2 = (B_{12}, B_2, \bar{B}_2(\mu_{1B_{12}}, \mu_{2B_2}), L_{B_2})$ be a pair of Fs-sets. We say that $\mathcal{B}_1$ and $\mathcal{B}_2$ are equal, denoted by $\mathcal{B}_1 = \mathcal{B}_2$ if, only if

(1) $B_{11} = B_{12}$, $B_1 = B_2$

(2) $L_{B_1} = L_{B_2}$

(3) (a) $(\mu_{1B_{11}} = \mu_{1B_{12}}$ and $\mu_{2B_1} = \mu_{2B_2})$,or (b) $\bar{B}_1 = \bar{B}_2$

1.5.1Remark: We can easily observed that 3(a) and 3(b) not equivalent statements.

1.6 Proposition: $\mathcal{B}_1 = (B_{11}, B_1, \bar{B}_1(\mu_{1B_{11}}, \mu_{B_1}), L_{B_1})$ *and* $\mathcal{B}_2 = (B_{12}, B_2, \bar{B}_2(\mu_{1B_{12}}, \mu_{B_2}), L_{B_2})$ *are equal if, only if* $\mathcal{B}_1 \subseteq \mathcal{B}_2$ and $\mathcal{B}_2 \subseteq \mathcal{B}_1$

1.7 Definition of Fs-union for a given pair of Fs-subsets of $\mathcal{A}$:

Let $\mathcal{B}=(B_1, B, \bar{B}(\mu_{1B_1}, \mu_{2B}), L_B)$ and $\mathcal{C}= (C_1, C, \bar{C}(\mu_{1C_1}, \mu_{2C}), L_C)$, be a pair of Fs-subsets of $\mathcal{A}$. Then, the Fs-union of $\mathcal{B}$ and $\mathcal{C}$,denoted by $\mathcal{B}\cup\mathcal{C}$ is defined as

$\mathcal{B}\cup\mathcal{C}=\mathcal{D}=(D_1, D, \bar{D}(\mu_{1D_1}, \mu_{2D}), L_D)$,where

(1) $D_1 = B_1 \cup C_1$, $D = B \cap C$

(2) $L_D = L_B \vee L_C$=complete subalgebra generated by $L_B \cup L_C$

(3) $\mu_{1D_1}: D_1 \longrightarrow L_D$ is defined by $\mu_{1D_1}x = (\mu_{1B_1} \vee \mu_{1C_1})x$

$\mu_{2D} : D \longrightarrow L_D$ is defined by $\mu_{2D}x = \mu_{2B}x \wedge \mu_{2C}x$

$\bar{D}: D \longrightarrow L_D$ is defined by $\bar{D}x = \mu_{1D_1}x \wedge (\mu_{2D}x)^c$

1.8 Proposition: *$\mathcal{B}\cup\mathcal{C}$ is an Fs-subset of $\mathcal{A}$.*

1.9 Definition of Fs-intersection for a given pair of Fs-subsets of $\mathcal{A}$:

Let $\mathcal{B}=(B_1, B, \bar{B}(\mu_{1B_1}, \mu_{2B}), L_B)$ and $\mathcal{C}= (C_1, C, \bar{C}(\mu_{1C_1}, \mu_{2C}), L_C)$ be a pair of Fs-subsets of $\mathcal{A}$ satisfying the following conditions:

(i) $B_1 \cap C_1 \supseteq B \cup C$

(ii) $\mu_{1B_1}x \wedge \mu_{1C_1}x \geq (\mu_{2B} \vee \mu_{2C})x$,for each $x \in A$

Then, the Fs-intersection of $\mathcal{B}$ and $\mathcal{C}$, denoted by $\mathcal{B}\cap\mathcal{C}$ is defined as

$\mathcal{B}\cap\mathcal{C} = \mathcal{E}=(E_1, E, \bar{E}(\mu_{1E_1}, \mu_{2E}), L_E)$,where

(a) $E_1 = B_1 \cap C_1$, $E = B \cup C$

(b) $L_E = L_B \wedge L_C = L_B \cap L_C$

(c) $\mu_{1E_1}: E_1 \longrightarrow L_E$ is defined by $\mu_{1E_1}x = \mu_{1B_1}x \wedge \mu_{1C_1}x$

$\mu_{2E} : E \longrightarrow L_E$ is defined by $\mu_{2E}x = (\mu_{2B} \vee \mu_{2C})x$

$\bar{E}: E \longrightarrow L_E$ is defined by $\bar{E}x = \mu_{1E_1}x \wedge (\mu_{2E}x)^c$.

1.9.1 Remark: If (i) or (ii) fails we define $\mathcal{B}\cap\mathcal{C}$ as $\mathcal{B}\cap\mathcal{C}=\Phi_{\mathcal{A}}$, which is the Fs-empty set of first kind.

1.10 Proposition: *For any pair of Fs-subsets* $\mathcal{B}=(B_1, B, \bar{B}(\mu_{1B_1}, \mu_{2B}), L_B)$ *and* $\mathcal{C}=(C_1, C, \bar{C}(\mu_{1C_1}, \mu_{2C}), L_C)$ *of $\mathcal{A}$, the following results are true*

(1) $\mathcal{B}\subseteq\mathcal{B}\cup\mathcal{C}$ and $\mathcal{C}\subseteq\mathcal{B}\cup\mathcal{C}$

(2) $\mathcal{B}\cap\mathcal{C}\subseteq\mathcal{B}$ and $\mathcal{B}\cap\mathcal{C}\subseteq\mathcal{C}$ provided $\mathcal{B}\cap\mathcal{C}$ exists

(3) $\mathcal{B}\subseteq\mathcal{C}$ implies $\mathcal{B}\cup\mathcal{C}=\mathcal{C}$

(4) $\mathcal{B}\cap\mathcal{C}=\mathcal{B}$ when $\mathcal{B}\neq \Phi_{\mathcal{A}}$ and $\mathcal{B}\subseteq\mathcal{C}$ and $\Phi_{\mathcal{A}} \cap\mathcal{C}=\Phi_{\mathcal{A}}$

(5) $\mathcal{B}\cup\mathcal{C}= \mathcal{C} \cup \mathcal{B}$ (commutative law of Fs-union)

(6) $\mathcal{B}\cap\mathcal{C}= \mathcal{C} \cap \mathcal{B}$ provided $\mathcal{B}\cap\mathcal{C}$ exists. (commutative law of Fs-intersection)

(7) $\mathcal{B}\cup\mathcal{B}=\mathcal{B}$

(8) $\mathcal{B}\cap\mathcal{B}=\mathcal{B}$ ((7)and (8) are Idempotent laws of Fs-union and Fs-intersection respectively)

1.11 Proposition: For any Fs-subsets $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ of $\mathcal{A} = (A_1, A, \bar{A}(\mu_{1A_1}, \mu_{2A}), L_A)$, the following associative laws are true:

(I) $\mathcal{B} \cup (\mathcal{C} \cup \mathcal{D}) = (\mathcal{B} \cup \mathcal{C}) \cup \mathcal{D}$

(II) $\mathcal{B} \cap (\mathcal{C} \cap \mathcal{D}) = (\mathcal{B} \cap \mathcal{C}) \cap \mathcal{D}$, whenever Fs-intersections exist.

1.12 Arbitrary Fs-unions and arbitrary Fs-intersections:

Given a family $(\mathcal{B}_i)_{i\in I}$ of Fs-subsets of $\mathcal{A}=(A_1, A, \bar{A}(\mu_{1A_1}, \mu_{2A}), L_A)$, where $\mathcal{B}_i = (B_{1i}, B_i, \bar{B}_i(\mu_{1B_{1i}}, \mu_{2B_i}), L_{B_i})$,for any $i \in I$

1.13 Definition of Fs-union is as follows

Case (1): For I=Φ,define Fs-union of $(\mathcal{B}_i)_{i\in I}$,denoted by $\bigcup_{i\in I} \mathcal{B}_i$ as $\bigcup_{i\in I} \mathcal{B}_i = \Phi_{\mathcal{A}}$, which is the Fs-empty set

Case (2): Define for I≠Φ, Fs-union of $(\mathcal{B}_i)_{i\in I}$ denoted by $\bigcup_{i\in I} \mathcal{B}_i$ as follow

$$\bigcup_{i\in I} \mathcal{B}_i = \mathcal{B} = (B_1, B, \bar{B}(\mu_{1B_1}, \mu_{2B}), L_B),$$

where

(a) $B_1 = \bigcup_{i\in I} B_{1i}, B = \bigcap_{i\in I} B_i$

(b) $L_B = \bigvee_{i\in I} L_{B_i}$ = complete subalgebra generated by $\bigcup L_i (L_i = L_{B_i})$

(c) $\mu_{1B_1}: B_1 \to L_B$ is defined by $\mu_{1B_1}x = (\bigvee_{i\in I} \mu_{1B_{1i}})x = \bigvee_{i\in I_x} \mu_{1B_{1i}}x$, where $I_x = \{i \in I | x \in B_i\}$

$\mu_{2B}: B \longrightarrow L_B$ is defined by $\mu_{2B}x = (\bigwedge_{i\in I} \mu_{2B_i})x = \bigwedge_{i\in I} \mu_{2B_i}x$

$\bar{B}: B \longrightarrow L_B$ is defined by $\bar{B}x = \mu_{1B_1}x \wedge (\mu_{2B}\, x)^c$

1.13.1Remark: We can easily show that (d) $B_1 \supseteq B$ and $\mu_{1B_1}|B \geq \mu_{2B}$.

1.14 Definition of Fs-intersection:

Case (1): For I=Φ, we define Fs-intersection of $(\mathcal{B}_i)_{i\in I}$, denoted by $\bigcap_{i\in I} \mathcal{B}_i$ as $\bigcap_{i\in I} \mathcal{B}_i = \mathcal{A}$

Case (2): Suppose $\bigcap_{i\in I} B_{1i} \supseteq \bigcup_{i\in I} B_i$ and $\bigwedge_{i\in I} \mu_{1B_{1i}}|(\bigcup_{i\in I} B_i) \geq \bigvee_{\in I} \mu_{2B_i}$

Then, we define Fs-intersection of $(\mathcal{B}_i)_{i\in I}$, denoted by $\bigcap_{i\in I} \mathcal{B}_i$ as follows

$$\bigcap_{i\in I} \mathcal{B}_i = \mathcal{C} = (C_1, C, \bar{C}(\mu_{1C_1}, \mu_{2C}), L_C)$$

(a') $C_1 = \bigcap_{i\in I} B_{1i}, C = \bigcup_{i\in I} B_i$

(b') $L_C = \wedge_{i\in I} L_{B_i}$

(c') $\mu_{1C_1}: C_1 \longrightarrow L_C$ is defined by $\mu_{1C_1}x = (\wedge_{i\in I}\mu_{1B_{1i}})x = \wedge_{i\in I}\mu_{1B_{1i}}x$

$\mu_{2C}: C \longrightarrow L_C$ is defined by $\mu_{2C}x = (\vee_{i\in I}\mu_{2B_i})x = \vee_{i\in I_x}\mu_{2B_i}x,$

where, $I_x = \{i \in I \mid x \in B_i\}$

$\bar{C}: C \longrightarrow L_C$ is defined by $\bar{C}x = \mu_{1C_1}x \wedge (\mu_{2C}\,x)^c$

Case (3): $\cap_{i\in I} B_{1i} \not\supseteq \cup_{i\in I} B_i$ or $\wedge_{i\in I}\mu_{1B_{1i}}|(\cup_{i\in I} B_i) \not\geq \vee_{i\in I}\mu_{2B_i}$

We define

$$\bigcap_{i\in I}\mathcal{B}_i = \Phi_{\mathcal{A}}$$

1.14.1Lemma: For any Fs-subset $\mathcal{B}=(B_1, B, \bar{B}(\mu_{1B_1}, \mu_{2B}), L_A)$ and $\mathcal{B}\subseteq \mathcal{B}_i = (B_{1i}, B_i, \bar{B}_i(\mu_{1B_{1i}}, \mu_{2B_i}), L_{B_i})$

1.15 Proposition: $(\mathcal{L}(\mathcal{A}), \cap)$ is $\wedge$-complete lattics.

1.15.1 Corollary: For any Fs-subset $\mathcal{B}$ of $\mathcal{A}$, the following results are true

(i) $\Phi_{\mathcal{A}} \cup \mathcal{B} = \mathcal{B}$

(ii) $\Phi_{\mathcal{A}} \cap \mathcal{B} = \Phi_{\mathcal{A}}$.

1.16 Proposition:$(\mathcal{L}(\mathcal{A}), \cup)$ is $\vee$-complete lattics.

1.16.1 Corollary: $(\mathcal{L}(\mathcal{A}), \cup, \cap)$ is a complete lattice with $\vee$ and $\wedge$

1.17 Proposition: Let $\mathcal{B}=(B_1, B, \bar{B}(\mu_{1B_1}, \mu_{2B}), L_B)$, $\mathcal{C}=(C_1, C, \bar{C}(\mu_{1C_1}, \mu_{2C}), L_C)$ and $\mathcal{D}=(D_1, D, \bar{D}(\mu_{1D_1}, \mu_{2D}), L_D)$. Then $\mathcal{B}\cup(\mathcal{C}\cap\mathcal{D})=(\mathcal{B}\cup\mathcal{C})\cap(\mathcal{B}\cup\mathcal{D})$ provided $\mathcal{C}\cap\mathcal{D}$ exists.

1.18 Proposition: Let $\mathcal{B}=(B_1, B, \bar{B}(\mu_{1B_1}, \mu_{2B}), L_B)$, $\mathcal{C}=(C_1, C, \bar{C}(\mu_{1C_1}, \mu_{2C}), L_C)$ and $\mathcal{D}=(D_1, D, \bar{D}(\mu_{1D_1}, \mu_{2D}), L_D)$. Then $\mathcal{B}\cap(\mathcal{C}\cup\mathcal{D})=(\mathcal{B}\cap\mathcal{C})\cup(\mathcal{B}\cap\mathcal{D})$ provided in R.H.S $(\mathcal{B}\cap\mathcal{C})$ and $(\mathcal{B}\cap\mathcal{D})$ exist.

1.19 Definition of Fs-complement of an Fs-subset: Consider a particular Fs-set $\mathcal{A} = (A_1, A, \bar{A}(\mu_{1A_1}, \mu_{2A}), L_A)$, $A \neq \Phi$, where

(i) $A \subseteq A_1$

(ii) $L_A = [0, M_A]$, $M_A = \vee\bar{A} = \vee_{a\in A}\bar{A}$

(iii) $\mu_{1A_1} = M_A, \mu_{2A} = 0$,

$\bar{A}x = \mu_{1A_1}x \wedge (\mu_{2A}x)^c = M_A$, for each $x \in A$

Given $\mathcal{B}=(B_1, B, \bar{B}(\mu_{1B_1}, \mu_{2B}), L_B)$. We define Fs-complement of $\mathcal{B}$, denoted by $\mathcal{B}^{C_\mathcal{A}}$ for B=A and $L_B = L_A$ as follows:

$\mathcal{B}^{C_\mathcal{A}} = \mathcal{D}=(D_1, D, \bar{D}(\mu_{1D_1}, \mu_{2D}), L_D)$, where

(a')$D_1 = C_A B_1 = B_1^c \cup A, D = B = A$

(b') $L_D = L_A$

(c')$\mu_{1D_1}: D_1 \longrightarrow L_A$, is defined by $\mu_{1D_1}x = M_A$

$\mu_{2D}: A \longrightarrow L_A$, is defined by

$\mu_{2D}x = \bar{B}x = \mu_{1B_1}x \wedge (\mu_{2B}x)^c$

$\bar{D}: A \longrightarrow L_A$, is defined by $\bar{D}x = \mu_{1D_1}x \wedge (\mu_{2D}x)^c = M_A \wedge (\bar{B}x)^c = (\bar{B}x)^c$.

1.20 Proposition: $\mathcal{A}^{C_\mathcal{A}} = \Phi_{\mathcal{A}}$

1.21 Definition: Define $(\Phi_{\mathcal{A}})^{C_\mathcal{A}} = \mathcal{A}$

1.22 Proposition: For $\mathcal{B}=(B_1, B, \bar{B}(\mu_{1B_1}, \mu_{2B}), L_B)$, $\mathcal{C}=(C_1, C, \bar{C}(\mu_{1C_1}, \mu_{2C}), L_C)$, which are non Fs-empty sets and $B = C = A, L_B = L_C = L_A$

(1) $\mathcal{B}\cap\mathcal{B}^{C_\mathcal{A}}=\Phi_{\mathcal{A}}$

(2) $\mathcal{B}\cup\mathcal{B}^{C_\mathcal{A}}=\mathcal{A}$

(3) $(\mathcal{B}^{C_\mathcal{A}})^{C_\mathcal{A}} = \mathcal{B}$

(4) $\mathcal{B}\subseteq\mathcal{C}$ if and only if $\mathcal{C}^{C_\mathcal{A}} \subseteq \mathcal{B}^{C_\mathcal{A}}$

1.23 Proposition: Fs-De-Morgan's laws for a given pair of Fs-subsets:

For any pair of Fs-sets $\mathcal{B}=(B_1, B, \bar{B}(\mu_{1B_1}, \mu_{2B}), L_B)$ and $\mathcal{C}=(C_1, C, \bar{C}(\mu_{1C_1}, \mu_{2C}), L_C)$, with $B = C = A$ and $L_B = L_C = L_A$, we will have

(i) $(\mathcal{B}\cup\mathcal{C})^{C_\mathcal{A}} = \mathcal{B}^{C_\mathcal{A}} \cap \mathcal{C}^{C_\mathcal{A}}$ if $(\bar{B}x)^c \wedge (\bar{C}x)^c \leq [(\mu_{1B_1}x)^c \vee \mu_{2C}x] \wedge [(\mu_{1C_1}x)^c \vee \mu_{2B}x]$, for each $x \in A$

(ii)$(\mathcal{B}\cap\mathcal{C})^{C_\mathcal{A}} = \mathcal{B}^{C_\mathcal{A}} \cup \mathcal{C}^{C_\mathcal{A}}$, whenever $\mathcal{B}\cap\mathcal{C}$ exists.

1.24 Fs-De Morgan laws for any given arbitrary family of Fs-sets:

Proposition: Given a family of Fs-subsets $(\mathcal{B}_i)_{i\in I}$ of $\mathcal{A} = (A_1, A, \bar{A}(\mu_{1A_1}, \mu_{2A}), L_A)$, where L_A=[0,M_A].

$\mu_{1A_1} = M_A, \mu_{2A} = 0, \bar{A}x = M_A$

(I) $(\cup_{i\in I}\mathcal{B}_i)^{C_\mathcal{A}} = \cap_{i\in I}\mathcal{B}^{C_\mathcal{A}}$, for I≠Φ, where $\mathcal{B}_i = (B_{1i}, B_i, \bar{B}_i(\mu_{1B_{1i}}, \mu_{2B_i}), L_{B_i})$ and

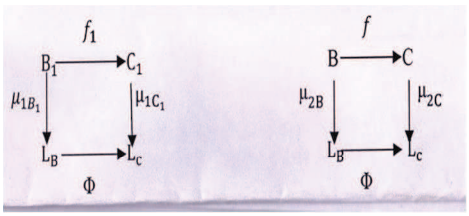
(1) $B_i = A, L_{B_i} = L_A$ provided $\wedge_{i\in I}(\bar{B}_i x)^c \leq \wedge_{i,j\in I,\, i\neq j}[(\mu_{1B_{1i}}x)^c \vee \mu_{2B_j}x]$

(II) $(\cap_{i\in I}\mathcal{B}_i)^{\mathcal{A}} = \cup_{i\in I}\mathcal{B}_i^{C_\mathcal{A}}$, whenever $\cap_{i\in I}\mathcal{B}_i$ exist

Theory Of Fs-Functions:

2.1 Fs-Function

A Triplet (f_1, f, Φ) is said to be is an Fs-Function between two given Fs-subsets $\mathcal{B} = (B_1, B, \bar{B}(\mu_{1B_1}, \mu_{2B}), L_B)$ and $\mathcal{C} = (C_1, C, \bar{C}(\mu_{1C_1}, \mu_{2C}), L_C)$ of $\mathcal{A}$, denoted by $(f_1, f, \Phi): \mathcal{B} = (B_1, B, \bar{B}(\mu_{1B_1}, \mu_{2B}), L_B) \longrightarrow \mathcal{C} = (C_1, C, \bar{C}(\mu_{1C_1}, \mu_{2C}), L_C)$ if, and only if (using the diagrams) .



(Fig-1)

(1a) $f_1|_B = f$

(1b) $\Phi: L_B \longrightarrow L_c$ is complete homomorphius

2.2 Result: (i) $\mu_{1C_1}|_C \circ f_1|_B \geq \mu_{2C} \circ f$

(ii) $\Phi \circ \mu_{1B_1}|_B \geq \Phi \circ \mu_{2B}$

Proof (i): $f_1x = f$, for each $x \in B$

$(\mu_{1C_1}|_C \circ f_1|_B)x = \mu_{1C_1}(f_1x) = \mu_{1C_1}(fx) \geq \mu_{2C}(fx) = (\mu_{2C} \circ f)x$

Hence $\mu_{1C_1}|_C \circ f_1|_B \geq \mu_{2C} \circ f$.

Proof (ii): $\mu_{1B_1}x \geq \mu_{2B}x$

$\Longrightarrow \Phi(\mu_{1B_1}x) \geq \Phi(\mu_{2B}x)$

($\because$ Φ is a complete homomorphism)

$\Longrightarrow (\Phi \circ \mu_{1B_1})x \geq (\Phi \circ \mu_{2B})x$

Hence $\Phi \circ \mu_{1B_1}|_B \geq \Phi \circ \mu_{2B}$

2.2.1 Remark: Φ is a complete homomorphism between complete Boolean algebras implies $\Phi(0) = 0$ and $\Phi(1) = 1$ and $[\Phi(a)]^c = \Phi(a^c)$

Therefore $\Phi(a) \wedge \Phi(a^c) = \Phi(a \wedge a^c) = \Phi(0) = 0$

$\Phi(a) \vee \Phi(a^c) = \Phi(a \vee a^c) = \Phi(1) = 1$

2.3 Def: Increasing Fs-function

(f_1, f, Φ) is said to be an increasing Fs- function, and denoted by $(f_1, f, \Phi)_i$ if ,and only if(using fig-1)

(2a) $\mu_{1C_1}|_C \circ f_1|_B \geq \Phi \circ \mu_{1B_1}$

(2b) $\mu_{2C} \circ f \leq \Phi \circ \mu_{2B}$

2.4 Results: $\Phi \circ (\mu_{2B}x)^c = [(\Phi \circ \mu_{2B})x]^c$

Proof: LHS: $\Phi \circ (\mu_{2B}x)^c = \Phi[(\mu_{2B}x)^c] = [\Phi(\mu_{2B}x)]^c = [(\Phi \circ \mu_{2B})x]^c$

2.5 Result: $\Phi \circ \overline{B} \leq \overline{C} \circ f$,provided (f_1, f, Φ) is an increasing Fs-function

Proof: $\Phi(\overline{B}x) = \Phi(\mu_{1B_1}x \wedge (\mu_{2B}x)^c)$

$= \Phi(\mu_{1B_1}x) \wedge \Phi[(\mu_{2B}x)^c]$

$= \Phi(\mu_{1B_1}x) \wedge [\Phi(\mu_{2B}x)]^c$

$= (\Phi \circ \mu_{1B_1})x \wedge [(\Phi \circ \mu_{2B})x]^c \leq (\mu_{1C_1} \circ f_1)x \wedge [(\mu_{2C} \circ f)x]^c = \mu_{1C_1}(f_1x) \wedge [\mu_{2C}(fx)]^C$

$= \mu_{1C_1}(fx) \wedge [\mu_{2C}(fx)]^C = \overline{C}\,(fx)$

Hence $\Phi \circ \overline{B} \leq \overline{C} \circ f$

2.6 Def: Decreasing Fs-function

(f_1, f, Φ) is said to be decreasing Fs-function denoted as $(f_1, f, \Phi)_d$ and if and only if

(3a) $\mu_{1C_1}|_C \circ f_1|_B \leq \Phi \circ \mu_{1B_1}$

(3b) $\mu_{2C} \circ f \geq \Phi \circ \mu_{2B}$

2.7 Result: $\Phi \circ \overline{B} \geq \overline{C} \circ f$,provided (f_1, f, Φ) is a decreasing Fs-function

Proof: $\Phi(\overline{B}x) = \Phi(\mu_{1B_1}x \wedge (\mu_{2B}x)^c)$

$= \Phi(\mu_{1B_1}x) \wedge \Phi[(\mu_{2B}x)^c]$

$= \Phi(\mu_{1B_1}x) \wedge [\Phi(\mu_{2B}x)]^c$

$= (\Phi \circ \mu_{1B_1})x \wedge [(\Phi \circ \mu_{2B})x]^c \geq (\mu_{1C_1} \circ f_1)x \wedge [(\mu_{2C} \circ f)x]^c = \mu_{1C_1}(f_1x) \wedge [\mu_{2C}(fx)]^C$

$= \mu_{1C_1}(fx) \wedge [\mu_{2C}(fx)]^C = \overline{C}\,(fx)$

Hence $\Phi \circ \overline{B} \geq \overline{C} \circ f$

2.8 Def:Preserving Fs- function

(f_1, f, Φ) is said to be preserving Fs-function and denoted as $(f_1, f, \Phi)_p$ if ,and only if

(4a) $\mu_{1C_1}|_C \circ f_1|_B = \Phi \circ \mu_{1B_1}$

(4b) $\mu_{2C} \circ f = \Phi \circ \mu_{2B}$

2.9 Result: $\Phi \circ \overline{B} = \overline{C} \circ f$,provided (f_1, f, Φ) is Fs-preserving function

Proof: $\Phi(\overline{B}x) = \Phi(\mu_{1B_1}x \wedge (\mu_{2B}x)^c)$

$= \Phi(\mu_{1B_1}x) \wedge \Phi[(\mu_{2B}x)^c]$

$= \Phi(\mu_{1B_1}x) \wedge [\Phi(\mu_{2B}x)]^c$

$= (\Phi \circ \mu_{1B_1})x \wedge [(\Phi \circ \mu_{2B})x]^c$

$= (\mu_{1C_1} \circ f_1)x \wedge [(\mu_{2C} \circ f)x]^c$

$= \mu_{1C_1}(f_1x) \wedge [\mu_{2C}(fx)]^C$

$= \mu_{1C_1}(fx) \wedge [\mu_{2C}(fx)]^C = \overline{C}\,(fx)$

Hence $\Phi \circ \overline{B} = \overline{C} \circ f$

2.10 Proposition: The class of all Fs-sets as objects together with morphism sets Fs-functions under the partial operation denoted by $\circ$ is called composition between Fs-functions whenever it exists is a category denoted by $\mathbb{F}$s-SET

Where $(f_1, f, \Phi) \circ (g_1, g, \Psi) = (g_1 \circ f_1, g \circ f, \Psi \circ \Phi)$

Proof: Given objects $(B_1, B, \overline{B}(\mu_{1B_1}, \mu_{2B}), L_B)$ and $(C_1, C, \overline{C}(\mu_{1C_1}, \mu_{2C}), L_C)$ with an Fs-function

$(f_1, f, \Phi): \mathcal{B} = (B_1, B, \overline{B}(\mu_{1B_1}, \mu_{2B}), L_B) \longrightarrow \mathcal{C} = (C_1, C, \overline{C}(\mu_{1C_1}, \mu_{2C}), L_C)$

We can easily show that

(5a) $(f_1, f, \Phi) \circ (1_{B_1}, 1_B, 1_{L_B}) = (f_1, f, \Phi)$

(5b) $(1_{C_1}, 1_C, 1_{L_C}) \circ (f_1, f, \Phi) = (f_1, f, \Phi)$

Where $(1_{B_1}, 1_B, 1_{L_B}): (B_1, B, \overline{B}(\mu_{1B_1}, \mu_{2B}), L_B) \longrightarrow (B_1, B, \overline{B}(\mu_{1B_1}, \mu_{2B}), L_B)$ is identity Fs-function, where $1_{B_1}: B_1 \longrightarrow B_1$, $1_B: B \longrightarrow B$ and $1_{L_B}: L_B \longrightarrow L_B$ are identity functions

(2) For any given Fs-sets $(B_1, B, \overline{B}(\mu_{1B_1}, \mu_{2B}), L_B)$, $(C_1, C, \overline{C}(\mu_{1C_1}, \mu_{2C}), L_C)$, $(D_1, D, \overline{D}(\mu_{1D_1}, \mu_{2D}), L_D)$ and $(E_1, E, \overline{E}(\mu_{1E_1}, \mu_{2E}), L_E)$ and Fs-functions

$(f_1, f, \Phi_1): (B_1, B, \overline{B}(\mu_{1B_1}, \mu_{2B}), L_B) \longrightarrow (C_1, C, \overline{C}(\mu_{1C_1}, \mu_{2C}), L_C)$

$(g_1, g, \Phi_2): (C_1, C, \overline{C}(\mu_{1C_1}, \mu_{2C}), L_C) \longrightarrow (D_1, D, \overline{D}(\mu_{1D_1}, \mu_{2D}), L_D)$

$(h_1, h, \Phi_3): (D_1, D, \overline{D}(\mu_{1D_1}, \mu_{2D}), L_D) \longrightarrow (E_1, E, \overline{E}(\mu_{1E_1}, \mu_{2E}), L_E)$

We can easily show that

$[(h_1, h, \Phi_3) \circ (g_1, g, \Phi_2)] \circ (f_1, f, \Phi_1) = (h_1, h, \Phi_3) \circ [(g_1, g, \Phi_2) \circ (f_1, f, \Phi_1)]$

The class of all Fs-sets together with morphism sets Hom_i $[(B_1, B, \overline{B}(\mu_{1B_1}, \mu_{2B}), L_B), (C_1, C, \overline{C}(\mu_{1C_1}, \mu_{2C}), L_C)] = \{(f_1, f, \Phi) | (f_1, f, \Phi)_i: (B_1, B, \overline{B}(\mu_{1B_1}, \mu_{2B}), L_B) \longrightarrow (C_1, C, \overline{C}(\mu_{1C_1}, \mu_{2C}), L_C)\}$ for any pair of Fs-sets $(B_1, B, \overline{B}(\mu_{1B_1}, \mu_{2B}), L_B)$ and $(C_1, C, \overline{C}(\mu_{1C_1}, \mu_{2C}), L_C)$ defines a category when the partial operation denoted by $\circ$ is defined as follows

For any pair of Fs-functions $(f_1, f, \Phi)$and(g_1, g, Ψ)

$(f_1, f, \Phi) \circ (g_1, g, \Psi) = (g_1 \circ f_1, g \circ f, \Psi \circ \Phi)$

Whenever composition functions on the right hand side above are defined .The category defined above is called the category of Fs-sets with increasing Fs-functions and denoted by $\mathbb{F}$s-SET_i

The class of all Fs-sets together with morphism sets $Hom_d[(B_1, B, \overline{B}(\mu_{1B_1}, \mu_{2B}), L_B), (C_1, C, \overline{C}(\mu_{1C_1}, \mu_{2C}), L_C)] = \{(f_1, f, \Phi) | (f_1, f, \Phi)_d: (B_1, B, \overline{B}(\mu_{1B_1}, \mu_{2B}), L_B) \rightarrow$

$\left(C_1, C, \bar{C}(\mu_{1C_1},\mu_{2C}), L_C\right)\}$ for any pair of Fs-sets $\left(B_1, B, \bar{B}(\mu_{1B_1},\mu_{2B}), L_B\right)$ and $\left(C_1, C, \bar{C}(\mu_{1C_1},\mu_{2C}), L_C\right)$ defines a category when the partial operation denoted by $\circ$ is defined as follows

For any pair of Fs-functions (f_1, f, Φ) and (g_1, g, Ψ)

$(f_1, f, \Phi) \circ (g_1, g, \Psi) = (g_1 \circ f_1, g \circ f, \Psi \circ \Phi)$

Whenever composition functions on the right hand side above are defined .The category defined above is called the category of Fs-sets with decreasing Fs-functions and denoted by $\mathbb{F}$s-SET$_d$

The class of all Fs-sets together with morphism sets $Hom_p\left[\left(B_1, B, \bar{B}(\mu_{1B_1},\mu_{2B}), L_B\right), \left(C_1, C, \bar{C}(\mu_{1C_1},\mu_{2C}), L_C\right)\right]$ $= \{(f_1, f, \Phi) | (f_1, f, \Phi)_p : \left(B_1, B, \bar{B}(\mu_{1B_1},\mu_{2B}), L_B\right) \longrightarrow \left(C_1, C, \bar{C}(\mu_{1C_1},\mu_{2C}), L_C\right)\}$ for any pair of Fs-sets $\left(B_1, B, \bar{B}(\mu_{1B_1},\mu_{2B}), L_B\right)$ and $\left(C_1, C, \bar{C}(\mu_{1C_1},\mu_{2C}), L_C\right)$ defines a category when the partial operation denoted by $\circ$ is defined as follows

For any pair of Fs-functions (f_1, f, Φ) and (g_1, g, Ψ)

$(f_1, f, \Phi) \circ (g_1, g, \Psi) = (g_1 \circ f_1, g \circ f, \Psi \circ \Phi)$

Whenever composition functions on the right hand side above are defined .The category defined above is called the category of Fs-sets with preserving Fs-functions and denoted by $\mathbb{F}$s-SET$_p$

2.11 Proposition: Composition of two increasing Fs-function are increasing.

Proof: suppose $(f_1, f, \Phi)_i : \left(B_1, B, \bar{B}(\mu_{1B_1},\mu_{2B}), L_B\right) \longrightarrow \left(C_1, C, \bar{C}(\mu_{1C_1},\mu_{2C}), L_C\right)$ and $(g_1, g, \Psi)_i : \left(C_1, C, \bar{C}(\mu_{1C_1},\mu_{2C}), L_C\right) \longrightarrow \left(D_1, D, \bar{D}(\mu_{1D_1},\mu_{2D}), L_D\right)$ are two increasing functions

Implies (1) $\mu_{1C_1}|_C \circ f_1|_B \geq \Phi \circ \mu_{1B_1}$

(2) $\mu_{2C} \circ f \leq \Phi \circ \mu_{2B}$

And (3) $\mu_{1D_1}|_D \circ g_1|_c \geq \Psi \circ \mu_{1c_1}$

(4) $\mu_{2D} \circ g \leq \Psi \circ \mu_{2C}$

Need to prove that

(5) $\mu_{1D_1}|_D \circ g_1|_c f_1|_B \geq (\Psi \circ \Phi) \circ \mu_{1B_1}$

(6) $\mu_{2D} \circ gf \leq (\Psi \circ \Phi) \circ \mu_{2C}$

Proof (5): $\left(\mu_{1D_1}|_D \circ g_1|_c f_1|_B\right)x$

$= \left(\mu_{1D_1}|_D \circ (g_1|_c \circ f_1|_B)\right)x$

$= \left((\mu_{1D_1}|_D \circ g_1|_c) \circ f_1|_B\right)x$

$= \left(\mu_{1D_1}|_D \circ g_1|_c\right)(f_1 x)$

$\geq \left(\Psi \circ \mu_{1c_1}\right)(f_1 x)$

$= \Psi\left(\mu_{1c_1}(f_1 x)\right) = \Psi\left[\left(\mu_{1C_1} \circ f_1|_B\right)\right]x$

($\because$ Ψ is a homomorphism)

$\geq \Psi\left[\left(\Phi \circ \mu_{1B_1}\right)\right]x = \left[\Psi \circ \left(\Phi \circ \mu_{1B_1}\right)\right]x$

$= \left[(\Psi \circ \Phi) \circ \mu_{1B_1}\right]x$

Hence $\mu_{1D_1}|_D \circ g_1|_c f_1|_B \geq (\Psi \circ \Phi) \circ \mu_{1B_1}$

Proof (6): $[\mu_{2D} \circ (g \circ f)]x$

$= [(\mu_{2D} \circ g) \circ f]x = (\mu_{2D} \circ g)(fx)$

$\leq (\Psi \circ \mu_{2C})(fx)$

$= \Psi(\mu_{2C}(fx)) = \Psi[(\mu_{2C} \circ f)x]$

$\leq \Psi[(\Phi \circ \mu_{2B})x] = [\Psi \circ (\Phi \circ \mu_{2B})]x = [(\Psi \circ \Phi) \circ \mu_{2B}]x$

Hence $\mu_{2D} \circ gf \leq (\Psi \circ \Phi) \circ \mu_{2C}$

Hence $(f_1, f, \Phi)_i \circ (g_1, g, \Psi)_i = [g_1 \circ f_1, g \circ f, \Psi \circ \Phi]_i$

2.12 Proposition: Composition of two decreasing Fs-function are decreasing.

Proof: suppose $(f_1, f, \Phi)_d : \left(B_1, B, \bar{B}(\mu_{1B_1},\mu_{2B}), L_B\right) \longrightarrow \left(C_1, C, \bar{C}(\mu_{1C_1},\mu_{2C}), L_C\right)$ and $(g_1, g, \Psi)_d : \left(C_1, C, \bar{C}(\mu_{1C_1},\mu_{2C}), L_C\right) \longrightarrow \left(D_1, D, \bar{D}(\mu_{1D_1},\mu_{2D}), L_D\right)$ are two decreasing functions

Implies (a) $\mu_{1C_1}|_C \circ f_1|_B \leq \Phi \circ \mu_{1B_1}$

(b) $\mu_{2C} \circ f \geq \Phi \circ \mu_{2B}$

And (c) $\mu_{1D_1}|_D \circ g_1|_c \leq \Psi \circ \mu_{1c_1}$

(d) $\mu_{2D} \circ g \geq \Psi \circ \mu_{2C}$

Need to prove that

(e) $\mu_{1D_1}|_D \circ (g_1|_c \circ f_1|_B) \leq (\Psi \circ \Phi) \circ \mu_{1B_1}$

(f) $\mu_{2D} \circ (g \circ f) \geq (\Psi \circ \Phi) \circ \mu_{2C}$

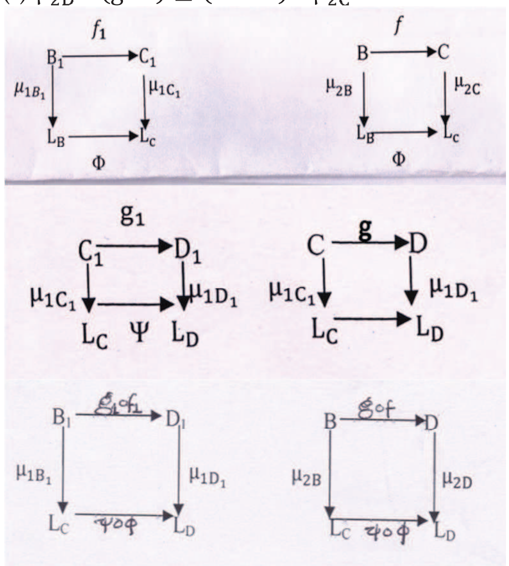


Fig-2

Proof (e): $\left(\mu_{1D_1}|_D \circ g_1|_c f_1|_B\right)x$

$= \left(\mu_{1D_1}|_D \circ (g_1|_c \circ f_1|_B)\right)x$

$= \left((\mu_{1D_1}|_D \circ g_1|_c) \circ f_1|_B\right)x$

$= \left(\mu_{1D_1}|_D \circ g_1|_c\right)(f_1 x) \leq \left(\Psi \circ \mu_{1c_1}\right)(f_1 x)$

$= \Psi\left(\mu_{1c_1}(f_1 x)\right) = \Psi\left[\left(\mu_{1C_1} \circ f_1|_B\right)\right]x$

($\because$ Ψ is a homomorphism)

$\leq \Psi\left[\left(\Phi \circ \mu_{1B_1}\right)\right]x = \left[\Psi \circ \left(\Phi \circ \mu_{1B_1}\right)\right]x$

$= \left[(\Psi \circ \Phi) \circ \mu_{1B_1}\right]x$

Hence $\mu_{1D_1}|_D \circ g_1|_c f_1|_B \leq (\Psi \circ \Phi) \circ \mu_{1B_1}$

Proof (f): $[\mu_{2D} \circ (g \circ f)]x$

$= [(\mu_{2D} \circ g) \circ f]x$

$= (\mu_{2D} \circ g)(fx) \geq (\Psi \circ \mu_{2C})(fx)$

$= \Psi(\mu_{2C}(fx)) = \Psi[(\mu_{2C} \circ f)x]$

$\geq \Psi[(\Phi \circ \mu_{2B})x] = [\Psi \circ (\Phi \circ \mu_{2B})]x$

$= [(\Psi \circ \Phi) \circ \mu_{2B}]x$

Hence $\mu_{2D} \circ (g \circ f) \geq (\Psi \circ \Phi) \circ \mu_{2C}$

Hence $(f_1, f, \Phi)_d \circ (g_1, g, \Psi)_d = [g_1 \circ f_1, g \circ f, \Psi \circ \Phi]_d$

2.13 Proposition: Composition of two preserving Fs-function are preserving.

Proof: suppose $(f_1, f, \Phi)_p: (B_1, B, \overline{B}(\mu_{1B_1}, \mu_{2B}), L_B) \longrightarrow (C_1, C, \overline{C}(\mu_{1C_1}, \mu_{2C}), L_C)$ and $(g_1, g, \Psi)_p: (C_1, C, \overline{C}(\mu_{1C_1}, \mu_{2C}), L_C) \longrightarrow (D_1, D, \overline{D}(\mu_{1D_1}, \mu_{2D}), L_D)$ are two decreasing functions

Implies (a) $\mu_{1C_1}|_C \circ f_1|_B = \Phi \circ \mu_{1B_1}$

(b) $\mu_{2C} \circ f = \Phi \circ \mu_{2B}$

And (c) $\mu_{1D_1}|_D \circ g_1|_c = \Psi \circ \mu_{1c_1}$

(d) $\mu_{2D} \circ g = \Psi \circ \mu_{2C}$

Need to prove that

(e) $\mu_{1D_1}|_D \circ (g_1|_c \circ f_1|_B) = (\Psi \circ \Phi) \circ \mu_{1B_1}$

(f) $\mu_{2D} \circ (g \circ f) = (\Psi \circ \Phi) \circ \mu_{2C}$

Proof (e): $(\mu_{1D_1}|_D \circ (g_1|_c \circ f_1|_B))x$

$= ((\mu_{1D_1}|_D \circ g_1|_C) \circ f_1|_B)x$

$= (\mu_{1D_1}|_D \circ g_1|_C)(f_1 x) = (\Psi \circ \mu_{1c_1})(f_1 x)$

$= \Psi(\mu_{1c_1}(f_1 x)) = \Psi[(\mu_{1C_1} \circ f_1|_B)]x$ $(\because \Psi$ is a homomorphism)

$= \Psi[(\Phi \circ \mu_{1B_1})]x = [\Psi \circ (\Phi \circ \mu_{1B_1})]x$

$= [(\Psi \circ \Phi) \circ \mu_{1B_1}]x$

Hence

$\mu_{1D_1}|_D \circ (g_1|_c \circ f_1|_B) = (\Psi \circ \Phi) \circ \mu_{1B_1}$

Proof (f): $[\mu_{2D} \circ (g \circ f)]x$

$= [(\mu_{2D} \circ g) \circ f]x$

$= (\mu_{2D} \circ g)(fx) = (\Psi \circ \mu_{2C})(fx)$

$= \Psi(\mu_{2C}(fx)) = \Psi[(\mu_{2C} \circ f)x]$

$= \Psi[(\Phi \circ \mu_{2B})x] = [\Psi \circ (\Phi \circ \mu_{2B})]x = [(\Psi \circ \Phi) \circ \mu_{2B}]x$

Hence $\mu_{2D} \circ (g \circ f) = (\Psi \circ \Phi) \circ \mu_{2C}$

Hence $(f_1, f, \Phi)_p \circ (g_1, g, \Psi)_p = [g_1 \circ f_1, g \circ f, \Psi \circ \Phi]_p$

2.13.1 Remark: (f_1, f, Φ) is preserving if, and only if (f_1, f, Φ) simultaneously both increasing and decreasing

2.14 Def: Fs-image of an Fs-subset under increasing Fs-function:

Let

$(f_1, f, \Phi)_i: (B_1, B, \overline{B}(\mu_{1B_1}, \mu_{2B}), L_B) \longrightarrow (C_1, C, \overline{C}(\mu_{1C_1}, \mu_{2C}), L_C)$

Let

$\mathcal{D} = (D_1, D, \overline{D}(\mu_{1D_1}, \mu_{2D}), L_D) \subseteq \mathcal{B} = (B_1, B, \overline{B}(\mu_{1B_1}, \mu_{2B}), L_B)$ then

(a) $D_1 \subseteq B_1$, $B \subseteq D$

(b) $L_D \le L_B$

(c) $(\mu_{1D_1} \le \mu_{1B_1}|D_1$, and $\mu_{2D}|B \ge \mu_{2B})$ or $\overline{D}x \le \overline{B}x$ for each $x \in B$

Define$(f_1, f, \Phi)_i(\mathcal{D}) = \mathcal{E} = (E_1, E, \overline{E}(\mu_{1E_1}, \mu_{2E})L_E)$, where

(d) $E_1 = f_1(D_1)$,$E = f(D)$

(e) $L_E = \Phi L_D$

(f) $\mu_{1E_1}: E_1 \longrightarrow L_E$ is defined by $\mu_{1E_1} y = \bigvee_{\substack{y=f_1 x \\ x \in D_1}} \Phi\mu_{1D_1} x$

$\mu_{2E}: E \longrightarrow L_E$ is defined by $\mu_{2E} y = \bigwedge_{\substack{y=fx \\ x \in D}} \Phi\mu_{2D} x$

$\overline{E}: E \longrightarrow L_E$ is defined by

$\overline{E}y = (\mu_{1E_1} y) \wedge (\mu_{2E} y)^c$

$= \left(\bigvee_{\substack{y=f_1 x \\ x \in D_1}} \Phi\mu_{1D_1} x\right) \wedge \left(\bigwedge_{\substack{y=fx \\ x \in D}} \Phi\mu_{2D} x\right)^c$

2.15 Result:$(f_1, f, \Phi)_i(\mathcal{D})$ is an Fs-subset of $\mathcal{C} = (C_1, C, \overline{C}(\mu_{1C_1}, \mu_{2C}), L_C)$

Proof: Let $(f_1, f, \Phi)_i(\mathcal{D}) = \mathcal{E} = (E_1, E, \overline{E}(\mu_{1E_1}, \mu_{2E})L_E)$

(1) $E_1 = f_1(D_1) \subseteq C$,$E = f(D) \supseteq C$

(2) $L_E = \Phi L_D \le L_C$

We have to prove that $\mu_{1E_1} \le \mu_{1C_1}$ and $\mu_{2E} \ge \mu_{2C}$

$y = f_1 x \Longrightarrow (\Phi \circ \mu_{1D_1})x \le (\Phi \circ \mu_{1B_1})x \le (\mu_{1C_1} \circ f_1)x = \mu_{1C_1} y$

$\therefore \bigvee_{\substack{y=f_1 x \\ x \in D_1}} \Phi\mu_{1D_1} x \le \mu_{1C_1} y$ ....(I)

$y = fx \Longrightarrow (\Phi \circ \mu_{2D})x \ge (\Phi \circ \mu_{2B})x \ge (\mu_{2C} \circ f)x = \mu_{2C} y$

$\Longrightarrow \bigwedge_{\substack{y=fx \\ x \in D}} \Phi\mu_{2D} x \ge \mu_{2C} y \Longrightarrow \left(\bigwedge_{\substack{y=fx \\ x \in D}} \Phi\mu_{2D} x\right)^c \le (\mu_{2C} y)^c$ ....(II)

(I)and(II) implies

$\left(\bigvee_{\substack{y=f_1 x \\ x \in D_1}} \Phi\mu_{1D_1} x\right) \wedge \left(\bigwedge_{\substack{y=fx \\ x \in D}} \Phi\mu_{2D} x\right)^c \le \mu_{1C_1} y \wedge (\mu_{2C} y)^c \Longrightarrow \overline{E}y \le \overline{C}y$

2.16 Result: $[(\mu_{1C_1} \circ f_1)x]^c = [\mu_{1C_1}(f_1 x)]^c \ge [\Phi(\mu_{1B_1} x)]^c = \Phi[(\mu_{1B_1} x)^c]$

2.17 Result: $[(\mu_{2C} \circ f)x]^c = [\mu_{2C}(fx)]^c \ge [\Phi(\mu_{2B} x)]^c = \Phi[(\mu_{2B} x)^c]$

2.18 Fs-image of an Fs-subset under decreasing Fs-function:

Let

$(f_1, f, \Phi)_d: (B_1, B, \overline{B}(\mu_{1B_1}, \mu_{2B}), L_B) \longrightarrow (C_1, C, \overline{C}(\mu_{1C_1}, \mu_{2C}), L_C)$

Let

$\mathcal{D} = (D_1, D, \overline{D}(\mu_{1D_1}, \mu_{2D}), L_D) \subseteq \mathcal{B} = (B_1, B, \overline{B}(\mu_{1B_1}, \mu_{2B}), L_B)$ then

(1) $D_1 \subseteq B_1$, $B \subseteq D$

(2) $L_D \le L_B$

(3) $(\mu_{1D_1} \le \mu_{1B_1}|D_1$, and $\mu_{2D}|B \ge \mu_{2B})$ or $\overline{D}x \le \overline{B}x$,for each $x \in B$

Define$(f_1, f, \Phi)_d(\mathcal{D}) = \mathcal{F} = (F_1, F, \overline{F}(\mu_{1F_1}, \mu_{2F}), L_F)$, where

(4) $F_1 = f_1(D_1)$,$F = f(D)$

(5) $L_F = \Phi L_D$

(6) $\mu_{1F_1}: F_1 \longrightarrow L_F$ is defined by $\mu_{1F_1} y = \overline{C}y \wedge \left(\bigvee_{\substack{y=f_1 x \\ x \in D_1}} \Phi\mu_{1D_1} x\right)$

$\mu_{2F}: F \longrightarrow L_F$ is defined by

$\mu_{2E} y = \bigwedge_{\substack{y=fx \\ x \in D}} \Phi\mu_{2D} x$

$\overline{F}: F \longrightarrow L_F$ is defined by

$\overline{F}y = (\mu_{1F_1} y) \wedge (\mu_{2F} y)^c$

$= \overline{C}y \wedge \left(\bigvee_{\substack{y=f_1 x \\ x \in D_1}} \Phi\mu_{1D_1} x\right) \wedge \left(\bigwedge_{\substack{y=fx \\ x \in D}} \Phi\mu_{2D} x\right)^c$

2.19 Result:$(f_1, f, \Phi)_d(\mathcal{D})$ is an Fs-subset of $\mathcal{C}=(C_1, C, \bar{C}(\mu_{1C_1}, \mu_{2C}), L_C)$

Proof: Let $(f_1, f, \Phi)_d(\mathcal{D}) = \mathcal{F} = (F_1, F, \bar{F}(\mu_{1F_1}, \mu_{2F}), L_F)$, where

(a) $F_1 = f_1(D_1)$, $F = f(D)$

(b) $L_F = \Phi L_D$

We have to prove that $\mu_{1F_1} \leq \mu_{1C_1}$ and $\mu_{2F} \geq \mu_{2C}$

$y = f_1 x \Longrightarrow (\Phi \circ \mu_{1D_1})x \leq (\Phi \circ \mu_{1B_1})x \leq (\mu_{1C_1} \circ f_1)x = \mu_{1C_1} y$

$\therefore \bar{C}y \wedge \left(\vee_{\substack{y=f_1 x \\ x\in D_1}} \Phi\mu_{1D_1} x\right) \leq \mu_{1C_1} y$ ....(I)

$y = fx \Longrightarrow (\Phi \circ \mu_{2D})x \geq (\Phi \circ \mu_{2B})x \geq (\mu_{2C} \circ f)x = \mu_{2C} y$

$\Longrightarrow \wedge_{\substack{y=fx \\ x\in D}} \Phi\mu_{2D} x \geq \mu_{2C} y \Longrightarrow \left(\wedge_{\substack{y=fx \\ x\in D}} \Phi\mu_{2D} x\right)^c \leq (\mu_{2C} y)^c$ ....(II)

(I)and(II) implies $\bar{C}y \wedge \left(\vee_{\substack{y=f_1 x \\ x\in D_1}} \Phi\mu_{1D_1} x\right) \wedge \left(\wedge_{\substack{y=fx \\ x\in D}} \Phi\mu_{2D} x\right)^c \leq \mu_{1C_1} y \wedge (\mu_{2C} y)^c$

$\Longrightarrow \bar{F}y \leq \bar{C}y$

2.20 Fs-image of an Fs-subset under preserving Fs-function:

Let $(f_1, f, \Phi)_p: (B_1, B, \bar{B}(\mu_{1B_1}, \mu_{2B}), L_B) \longrightarrow (C_1, C, \bar{C}(\mu_{1C_1}, \mu_{2C}), L_C)$

Let $\mathcal{D} = (D_1, D, \bar{D}(\mu_{1D_1}, \mu_{2D}), L_D) \subseteq \mathcal{B} = (B_1, B, \bar{B}(\mu_{1B_1}, \mu_{2B}), L_B)$ then

(g) $D_1 \subseteq B_1$, $B \subseteq D$

(h) $L_D \leq L_B$

(i) $(\mu_{1D_1} \leq \mu_{1B_1}|D_1$, and $\mu_{2D}|B \geq \mu_{2B})$ or $\bar{D}x \leq \bar{B}x$ for each $x\in B$

Define$(f_1, f, \Phi)_i(\mathcal{D}) = \mathcal{G} = (G_1, G, \bar{G}(\mu_{1G_1}, \mu_{2G}), L_G)$, where

(j) $G_1 = f_1(D_1)$, $G = f(D)$

(k) $L_G = \Phi L_D$

(l) $\mu_{1G_1}: G_1 \longrightarrow L_G$ is defined by $\mu_{1G_1} y = \vee_{\substack{y=f_1 x \\ x\in D_1}} \Phi\mu_{1D_1} x$

$\mu_{2G}: G \longrightarrow L_G$ is defined by $\mu_{2G} y = \wedge_{\substack{y=fx \\ x\in D}} \Phi\mu_{2D} x$

$\bar{G}: G \longrightarrow L_G$ is defined by $\bar{G}y = (\mu_{1G_1} y) \wedge (\mu_{2G} y)^c = \left(\vee_{\substack{y=f_1 x \\ x\in D_1}} \Phi\mu_{1D_1} x\right) \wedge \left(\wedge_{\substack{y=fx \\ x\in D}} \Phi\mu_{2D} x\right)^c$

2.21 Result:$(f_1, f, \Phi)_P(\mathcal{D})$ is an Fs-subset of $\mathcal{C}=(C_1, C, \bar{C}(\mu_{1C_1}, \mu_{2C}), L_C)$

Proof: Let $(f_1, f, \Phi)_p(\mathcal{D}) = \mathcal{G} = (G_1, G, \bar{G}(\mu_{1G_1}, \mu_{2G}), L_G)$

(3) $G_1 = f_1(D_1) \subseteq C_1$, $G = f(D) \supseteq C$

(4) $L_G = \Phi L_D \leq L_C$

We have to prove that $\mu_{1G_1} \leq \mu_{1C_1}$ and $\mu_{2G} \geq \mu_{2C}$

$y = f_1 x \Longrightarrow (\Phi \circ \mu_{1D_1})x \leq (\Phi \circ \mu_{1B_1})x \leq (\mu_{1C_1} \circ f_1)x = \mu_{1C_1} y$

$\therefore \vee_{\substack{y=f_1 x \\ x\in D_1}} \Phi\mu_{1D_1} x \leq \mu_{1C_1} y$ ....(I)

$y = fx \Longrightarrow (\Phi \circ \mu_{2D})x \geq (\Phi \circ \mu_{2B})x \geq (\mu_{2C} \circ f)x = \mu_{2C} y$

$\Longrightarrow \wedge_{\substack{y=fx \\ x\in D}} \Phi\mu_{2D} x \geq \mu_{2C} y \Longrightarrow \left(\wedge_{\substack{y=fx \\ x\in D}} \Phi\mu_{2D} x\right)^c \leq (\mu_{2C} y)^c$ ....(II)

(I)and(II) implies

$\left(\vee_{\substack{y=f_1 x \\ x\in D_1}} \Phi\mu_{1D_1} x\right) \wedge \left(\wedge_{\substack{y=fx \\ x\in D}} \Phi\mu_{2D} x\right)^c \leq \mu_{1C_1} y \wedge (\mu_{2C} y)^c$

$\Longrightarrow \bar{G}y \leq \bar{C}y$

IV. Properties of images of Fs-subsets

3.1 Proposition: For any pair of Fs-functions and for $\mathcal{H} \subseteq \mathcal{B}$

$(f_1, f, \Phi): (B_1, B, \bar{B}(\mu_{1B_1}, \mu_{2B}), L_B) \longrightarrow (C_1, C, \bar{C}(\mu_{1C_1}, \mu_{2C}), L_C)$ and $(g_1, g, \Psi): (C_1, C, \bar{C}(\mu_{1C_1}, \mu_{2C}), L_C) \longrightarrow (D_1, D, \bar{D}(\mu_{1D_1}, \mu_{2D}), L_D)$ the following are true

(I) $[(f_1, f, \Phi)_* \circ (g_1, g, \Psi)_*](\mathcal{H}) = (g_1, g, \Psi)_*[(f_1, f, \Phi)_*(\mathcal{H})]$, whenever $*=$ i or p

(II) $[(f_1, f, \Phi)_d \circ (g_1, g, \Psi)_d](\mathcal{H}) \supseteq (g_1, g, \Psi)_d[(f_1, f, \Phi)_d(\mathcal{H})]$

Proof:(I):Let$*=$ i .Then

LHS:$[(f_1, f, \Phi)_i \circ (g_1, g, \Psi)_i](\mathcal{H}) = [g_1 \circ f_1, g \circ f, \Psi \circ \Phi]_i = \mathcal{G} = (G_1, G, \bar{G}(\mu_{1G_1}, \mu_{2G}), L_G)$ say where

[1] $G_1 = (g_1 \circ f_1)(H_1)$, $G = (g \circ f)(H)$

[2] $L_G = (\Psi \circ \Phi)L_H$

[3] $\mu_{1G_1}: G_1 \longrightarrow L_G$ is defined by $\mu_{1G_1} a = \vee_{\substack{a=(g_1 \circ f_1)h \\ h\in H_1}} (\Psi \circ \Phi)\mu_{1H_1} h$

$\mu_{2G}: G \longrightarrow L_G$ is defined by $\mu_{2G} a = \wedge_{\substack{a=(g\circ f)h \\ h\in H}} (\Psi \circ \Phi)\mu_{2H} h$

RHS: $(g_1, g, \Psi)_i[(f_1, f, \Phi)_i(\mathcal{H})]$

Let$(f_1, f, \Phi)_i(\mathcal{H}) = \mathcal{K} = (K_1, K, \bar{K}(\mu_{1K_1}, \mu_{2K}), L_K)$, where

[4] $K_1 = f_1(H_1)$, $K = f(H)$

[5] $L_K = \Phi L_H$

[6] $\mu_{1K_1}: K_1 \longrightarrow L_K$ is defined by $\mu_{1K_1} k = \vee_{\substack{k=f_1 h \\ h\in H_1}} \Phi\mu_{1H_1} h$

$\mu_{2K}: K \longrightarrow L_K$ is defined by $\mu_{2K} k = \wedge_{\substack{k=fh \\ h\in H}} \Phi\mu_{2H} h$

Now $(g_1, g, \Psi)_i[(f_1, f, \Phi)_i(\mathcal{H})] = (g_1, g, \Psi)_i(\mathcal{K}) = \mathcal{M} = (M_1, M, \bar{M}(\mu_{1M_1}, \mu_{2M}), L_M)$, where

[7] $M_1 = g_1(K_1) = g_1(f_1(H_1)) = (g_1 \circ f_1)(H_1)$, $M = g(K) = g(f(H)) = (g \circ f)(H)$

[8] $L_M = \Psi L_K = \Psi\Phi L_H = (\Psi \circ \Phi)L_H$

[1] and [7] imply $M_1 = G_1$ and $M = G$ ......(I)

[2] and [8] imply $L_M = L_G$ ....(II)

[9] $\mu_{1M_1}: M_1 \longrightarrow L_M$ is defined by

$\mu_{1M_1}a = \bigvee_{\substack{a=g_1k \\ k\in K_1}} \Psi\mu_{1K_1}k$

$= \bigvee_{\substack{a=g_1k \\ k\in K_1}} \left[\Psi\left(\bigvee_{\substack{k=f_1h \\ h\in H_1}} \Phi\mu_{1H_1}h\right)\right]$

$= \bigvee_{\substack{a=g_1(f_1h) \\ h\in H_1}} (\Psi\circ\Phi)\mu_{1H_1}h$

$\mu_{2M}: M \longrightarrow L_M$ is defined by

$\mu_{2M}a = \bigwedge_{\substack{a=g(k) \\ k\in K}} \Psi\mu_{2K}k$

$= \bigwedge_{\substack{a=g(k) \\ k\in K}} \left[\Psi\left(\bigwedge_{\substack{k=fh \\ h\in H}} \Phi\mu_{2H}h\right)\right]$

$= \bigwedge_{\substack{a=g(fh) \\ h\in H}} (\Psi\circ\Phi)\mu_{2H}h$

[3] and [9] imply $\mu_{1M_1} = \mu_{1G_1}$ and $\mu_{2M} = \mu_{2G}$ ....(III)

(I),(II) and (III) imply $\mathcal{M} = \mathcal{G}$

Hence

$[(f_1,f,\Phi)_i \circ (g_1,g,\Psi)_i](\mathcal{H}) = (g_1,g,\Psi)_i[(f_1,f,\Phi)_i(\mathcal{H})]$

For $*=$ p the proof is obvious

Proof:(II): LHS:$[(f_1,f,\Phi)_d \circ (g_1,g,\Psi)_d](\mathcal{H}) = [g_1\circ f_1, g\circ f, \Psi\circ\Phi]_d = \mathcal{G} = \left(G_1, G, \overline{G}\left(\mu_{1G_1},\mu_{2G}\right), L_G\right)$ say where

[11] $G_1 = (g_1\circ f_1)(H_1)$, $G = (g\circ f)(H)$

[12] $L_G = (\Psi\circ\Phi)L_H$

[13] $\mu_{1G_1}: G_1 \longrightarrow L_G$ is defined by $\mu_{1G_1}a = \overline{D}a \wedge \left(\bigvee_{\substack{a=(g_1\circ f_1)h \\ h\in H_1}} (\Psi\circ\Phi)\mu_{1H_1}h\right)$

$\mu_{2G}: G \longrightarrow L_G$ is defined

$\mu_{2G}a = \bigwedge_{\substack{a=(g\circ f)h \\ h\in H}} (\Psi\circ\Phi)\mu_{2H}h$

RHS: $(g_1,g,\Psi)_d[(f_1,f,\Phi)_d(\mathcal{H})]$

Let $(f_1,f,\Phi)_d(\mathcal{H}) = \mathcal{K} = \left(K_1, K, \overline{K}\left(\mu_{1K_1},\mu_{2K}\right), L_K\right)$, where

[14] $K_1 = f_1(H_1)$,$K = f(H)$

[15] $L_K = \Phi L_H$

[16] $\mu_{1K_1}: K_1 \longrightarrow L_K$ is defined by $\mu_{1K_1}k = \overline{C}k \wedge \left(\bigvee_{\substack{k=f_1h \\ h\in H_1}} \Phi\mu_{1H_1}h\right)$

$\mu_{2K}: K \longrightarrow L_K$ is define as $\mu_{2K}k = \bigwedge_{\substack{k=fh \\ h\in H}} \Phi\mu_{2H}h$

Now$(g_1,g,\Psi)_d[(f_1,f,\Phi)_d(\mathcal{H})] = (g_1,g,\Psi)_d(\mathcal{K}) = \mathcal{M} = \left(M_1, M, \overline{M}\left(\mu_{1M_1},\mu_{2M}\right), L_M\right)$, where

[17] $M_1 = g_1(K_1) = g_1\left(f_1(H_1)\right) = (g_1\circ f_1)(H_1)$, $M = g(K) = g\left(f(H)\right) = (g\circ f)(H)$

[18] $L_M = \Psi L_K = \Psi\Phi L_H = (\Psi\circ\Phi)L_H$

[11] and [17] imply $M_1 = G_1$ and $M = G$ ....(VI)

[12] and [18] imply $L_M = L_G$ ....(V)

It is sufficient to show that $\mu_{1M_1} \le \mu_{1G_1}$ and $\mu_{2M} \ge \mu_{2G}$

[19] $\mu_{1M_1}: M_1 \longrightarrow L_M$ is defined by

$\mu_{1M_1}a = \overline{D}a \wedge \left(\bigvee_{\substack{a=g_1k \\ k\in K_1}} \Psi\mu_{1K_1}k\right)$

$= \overline{D}a \wedge \left[\bigvee_{\substack{a=g_1k \\ k\in K_1}} \Psi\left(\overline{C}k\left(\bigvee_{\substack{k=f_1h \\ h\in H_1}} \Phi\mu_{1H_1}h\right)\right)\right]$

For particular $k \in K_1$

$\mu_{1K_1}k = \overline{C}k \wedge \left(\bigvee_{\substack{k=f_1h \\ h\in H_1}} \Phi\mu_{1H_1}h\right) \le \left(\bigvee_{\substack{k=f_1h \\ h\in H_1}} \Phi\mu_{1H_1}h\right)$

$\Longrightarrow \Psi\left(\overline{C}k \wedge \left(\bigvee_{\substack{k=f_1h \\ h\in H_1}} \Phi\mu_{1H_1}h\right)\right) \le \Psi\left(\bigvee_{\substack{k=f_1h \\ h\in H_1}} \Phi\mu_{1H_1}h\right)$

$\Longrightarrow \Psi\overline{C}k \wedge \left(\bigvee_{\substack{k=f_1h \\ h\in H_1}} \Psi\Phi\mu_{1H_1}h\right) \le \left(\bigvee_{\substack{k=f_1h \\ h\in H_1}} \Psi\Phi\mu_{1H_1}h\right)$

$\Longrightarrow \bigvee_{\substack{a=g_1k \\ k\in K_1}} \left[\Psi\overline{C}k \wedge \left(\bigvee_{\substack{k=f_1h \\ h\in H_1}} \Psi\Phi\mu_{1H_1}h\right)\right] \le \bigvee_{\substack{a=g_1k \\ k\in K_1}} \left(\bigvee_{\substack{k=f_1h \\ h\in H_1}} \Psi\Phi\mu_{1H_1}h\right)$

$\Longrightarrow \overline{D}a \wedge \left[\bigvee_{\substack{a=g_1k \\ k\in K_1}} \Psi\left(\overline{C}k\left(\bigvee_{\substack{k=f_1h \\ h\in H_1}} \Phi\mu_{1H_1}h\right)\right)\right]$

$\le \overline{D}a \wedge \left(\bigvee_{\substack{a=(g_1\circ f_1)h \\ h\in H_1}} (\Psi\circ\Phi)\mu_{1H_1}h\right)$

$\Longrightarrow \mu_{1M_1}a \le \mu_{1G_1}a$

$\mu_{2M}: M \longrightarrow L_M$ is defined by

$\mu_{2M}a = \bigwedge_{\substack{a=g(k) \\ k\in K}} \Psi\mu_{2K}k = \bigwedge_{\substack{a=g(k) \\ k\in K}} \left[\Psi\left(\bigwedge_{\substack{k=fh \\ h\in H}} \Phi\mu_{2H}h\right)\right] = \bigwedge_{\substack{a=g(fh) \\ h\in H}} (\Psi\circ\Phi)\mu_{2H}h$

[13] and [19] imply $\mu_{1M_1} \le \mu_{1G_1}$ and $\mu_{2M} = \mu_{2G}$ ....(VI)

(IV),(V) and (IV) imply $\mathcal{M} \subseteq \mathcal{G}$

Hence

$[(f_1,f,\Phi)_d \circ (g_1,g,\Psi)_d](\mathcal{H}) \supseteq (g_1,g,\Psi)_d[(f_1,f,\Phi)_d(\mathcal{H})]$

3.2 Proposition: For any Fs-function $(f_1,f,\Phi): \left(B_1, B, \overline{B}\left(\mu_{1B_1},\mu_{2B}\right), L_B\right) \longrightarrow \left(C_1, C, \overline{C}\left(\mu_{1C_1},\mu_{2C}\right), L_C\right)$ and for any pair of Fs-subsets $\mathcal{H}_1 = \left(H_{11}, H_1, \overline{H}_1\left(\mu_{1H_{11}},\mu_{2H_1}\right), L_{H_1}\right)$ and $\mathcal{H}_2 = \left(H_{12}, H_2, \overline{H}_2\left(\mu_{1H_{12}},\mu_{2H_2}\right), L_{H_2}\right)$ of $\mathcal{B} = \left(B_1, B, \overline{B}\left(\mu_{1B_1},\mu_{2B}\right), L_B\right)$ such that $\mathcal{H}_1 \subseteq \mathcal{H}_2$,

$(f_1,f,\Phi)_*(\mathcal{H}_1) \subseteq (f_1,f,\Phi)_*(\mathcal{H}_2)$

holds whenever $*=$ i or d or p .

Proof: Let $*=$ i. Then $(f_1,f,\Phi)_i(\mathcal{H}_1) \subseteq (f_1,f,\Phi)_i(\mathcal{H}_2)$

Suppose

$(f_1,f,\Phi)_i(\mathcal{H}_1) = \mathcal{G}_1 = \left(G_{11}, G_1, \overline{G}_1\left(\mu_{1G_{11}},\mu_{2G_1}\right) L_{G_1}\right)$,where

(a) $G_{11} = f_1(H_{11})$,$G_1 = f(H_1)$

(b) $L_{G_1} = \Phi L_{H_1}$

(c) $\mu_{1G_{11}}: G_{11} \longrightarrow L_{G_1}$ is defined by $\mu_{1G_{11}}g =$

$\bigvee_{\substack{g=f_1h\\ h\in H_{11}}} \Phi\mu_{1H_{11}}h$

$\mu_{2G_1}: G_1 \longrightarrow L_{G_1}$ is defined by $\mu_{2G_1} = \bigwedge_{\substack{g=fh\\ h\in H_1}} \Phi\mu_{2H_1}h$

Again suppose Suppose $(f_1, f, \Phi)_i(\mathcal{H}_2) = \mathcal{G}_2 = (G_{12}, G_2, \overline{G}_2(\mu_{1G_{12}}, \mu_{2G_2})L_{G_2})$ where

(d) $G_{12} = f_1(H_{12})$,$G_2 = f(H_2)$

(e) $L_{G_2} = \Phi L_{H_2}$

(f) $\mu_{1G_{12}}: G_{12} \longrightarrow L_{G_2}$ is defined as $\mu_{1G_{12}}g = \bigvee_{\substack{g=f_1h\\ h\in H_{12}}} \Phi\mu_{1H_{12}}h$

$\mu_{2G_2}: G_2 \longrightarrow L_{G_2}$ is defined by $\mu_{2G_2}g = \bigwedge_{\substack{g=fh\\ h\in H_2}} \Phi\mu_{2H_2}h$

From definition of Fs-subsets $\mathcal{H}_1 \subseteq \mathcal{H}_2$ imply

(g) $H_{11} \subseteq H_{12} \Rightarrow f_1(H_{11}) \subseteq f_1(H_{12})$,$H_1 \supseteq H_2 \Rightarrow f(H_1) \supseteq f(H_2)$

(h) $L_{H_1} \leq L_{H_2} \Rightarrow \Phi L_{H_1} \leq \Phi L_{H_2}$

(i) $\mu_{1H_{11}} \leq \mu_{1H_{12}} \Rightarrow \mu_{1H_{11}}h \leq \mu_{1H_{12}}h \Rightarrow \Phi\mu_{1H_{11}}h \leq \Phi\mu_{1H_{12}}h$, for each $h \in H_{11}$

$\Rightarrow \bigvee_{g=f_1h} \Phi\mu_{1H_{11}}h \leq \bigvee_{g=f_1h} \Phi\mu_{1H_{12}}h$, for each $h \in H_{11}$

$\mu_{2H_1} \geq \mu_{2H_2} \Rightarrow \mu_{2H_1}h \geq \mu_{2H_2}h \Rightarrow \Phi\mu_{2H_1}h \geq \Phi\mu_{2H_2}h$, for each $h \in H_2$

$\Rightarrow \bigwedge_{g=fh} \Phi\mu_{2H_1}h \geq \bigwedge_{g=fh} \Phi\mu_{2H_2}h$, for each $h \in H_2$

(a),(d) and(g),
imply $G_{11} \subseteq G_{12}$, $G_1 \supseteq G_2$ ....(I)

(b),(e) and (h) imply $L_{G_1} \leq L_{G_2}$.........(II)

(c),(f) and (i),
imply $\mu_{1G_{11}} \leq \mu_{1G_{12}}, \mu_{2G_1} \geq \mu_{2G_2}$ ....(III)

(I),(II) and (III) imply $\mathcal{G}_1 \subseteq \mathcal{G}_2$

Hence $(f_1, f, \Phi)_i(\mathcal{H}_1) \subseteq (f_1, f, \Phi)_i(\mathcal{H}_2)$

For $*=$ p the proof is obvious

Let $*=$ d. Then $(f_1, f, \Phi)_d(\mathcal{H}_1) \subseteq (f_1, f, \Phi)_d(\mathcal{H}_2)$

Suppose
$(f_1, f, \Phi)_d(\mathcal{H}_1) = \mathcal{G}_1 = (G_{11}, G_1, \overline{G}_1(\mu_{1G_{11}}, \mu_{2G_1})L_{G_1})$ where

(j) $G_{11} = f_1(H_{11})$,$G_1 = f(H_1)$

(k) $L_{G_1} = \Phi L_{H_1}$

(l) $\mu_{1G_{11}}: G_{11} \longrightarrow L_{G_1}$ is defined by $\mu_{1G_{11}}g = \bar{C}h \wedge \left(\bigvee_{\substack{g=f_1h\\ h\in H_{11}}} \Phi\mu_{1H_{11}}h\right)$

$\mu_{2G_1}: G_1 \longrightarrow L_{G_1}$ is defined by $\mu_{2G_1} = \bigwedge_{\substack{g=fh\\ h\in H_1}} \Phi\mu_{2H_1}h$

Again suppose
$(f_1, f, \Phi)_d(\mathcal{H}_2) = \mathcal{G}_2 = (G_{12}, G_2, \overline{G}_2(\mu_{1G_{12}}, \mu_{2G_2})L_{G_2})$ where

(m) $G_{12} = f_1(H_{12})$,$G_2 = f(H_2)$

(n) $L_{G_2} = \Phi L_{H_2}$

(o) $\mu_{1G_{12}}: G_{12} \longrightarrow L_{G_2}$ is defined by $\mu_{1G_{12}}g = \bar{C}h \wedge \left(\bigvee_{\substack{g=f_1h\\ h\in H_{12}}} \Phi\mu_{1H_{12}}h\right)$

$\mu_{2G_2}: G_2 \longrightarrow L_{G_2}$ is defined by $\mu_{2G_2}g = \bigwedge_{\substack{g=fh\\ h\in H_2}} \Phi\mu_{2H_2}h$

$\mathcal{H}_1 \subseteq \mathcal{H}_2$ implies

(p) $H_{11} \subseteq H_{12} \Rightarrow f_1(H_{11}) \subseteq f_1(H_{12})$,$H_1 \supseteq H_2 \Rightarrow f(H_1) \supseteq f(H_2)$

(q) $L_{H_1} \leq L_{H_2} \Rightarrow \Phi L_{H_1} \leq \Phi L_{H_2}$

(r) $\mu_{1H_{11}} \leq \mu_{1H_{12}}$

$\Rightarrow \mu_{1H_{11}}h \leq \mu_{1H_{12}}h$

$\Rightarrow \Phi\mu_{1H_{11}}h \leq \Phi\mu_{1H_{12}}h$, for each $h \in H_{11}$

$\Rightarrow (\bigvee_{g=f_1h} \Phi\mu_{1H_{11}}h)$
$\leq (\bigvee_{g=f_1h} \Phi\mu_{1H_{12}}h)$, for each $h \in H_{11}$

$\Rightarrow \bar{C}h \wedge (\bigvee_{g=f_1h} \Phi\mu_{1H_{11}}h)$
$\leq \bar{C}h \wedge (\bigvee_{g=f_1h} \Phi\mu_{1H_{12}}h)$, for each $h \in H_{11}$

$\mu_{2H_1} \geq \mu_{2H_2} \Rightarrow \mu_{2H_1}h \geq \mu_{2H_2}h \Rightarrow \Phi\mu_{2H_1}h \geq \Phi\mu_{2H_2}h$, for each $h \in H_2$

$\Rightarrow \bigwedge_{g=fh} \Phi\mu_{2H_1}h \geq \bigwedge_{g=fh} \Phi\mu_{2H_2}h$, for each $h \in H_2$

(j),(m) and(p)
imply $G_{11} \subseteq G_{12}$, $G_1 \supseteq G_2$ ....(IV)

(k),(n) and (q) imply $L_{G_1} \leq L_{G_2}$...(V)

(l),(o) and (r) imply
$\mu_{1G_{11}} \leq \mu_{1G_{12}}, \mu_{2G_1} \geq \mu_{2G_2}$........(VI)

(IV),(V) and (VI) imply $\mathcal{G}_1 \subseteq \mathcal{G}_2$

Hence $(f_1, f, \Phi)_d(\mathcal{H}_1) \subseteq (f_1, f, \Phi)_p(\mathcal{H}_2)$

3.4 Proposition: For any Fs-function $(f_1, f, \Phi): (B_1, B, \overline{B}(\mu_{1B_1}, \mu_{2B}), L_B) \longrightarrow (C_1, C, \bar{C}(\mu_{1C_1}, \mu_{2C}), L_C)$ and for any pair of Fs-subsets $\mathcal{H}_1 = (H_{11}, H_1, \overline{H}_1(\mu_{1H_{11}}, \mu_{2H_1})L_{H_1})$ and $\mathcal{H}_2 = (H_{12}, H_2, \overline{H}_2(\mu_{1H_{12}}, \mu_{2H_2})L_{H_2})$ of $\mathcal{B} = (B_1, B, \overline{B}(\mu_{1B_1}, \mu_{2B}), L_B)$,

$(f_1, f, \Phi)_*(\mathcal{H}_1 \cup \mathcal{H}_2) = (f_1, f, \Phi)_*(\mathcal{H}_1) \cup (f_1, f, \Phi)_*(\mathcal{H}_2)$

holds, whenever $*=$ i or d or p , provided f is one-one

Proof: Let $*=$ i. Then

LHS: $(f_1, f, \Phi)_i(\mathcal{H}_1 \cup \mathcal{H}_2)$

Let $\mathcal{H}_1 \cup \mathcal{H}_2 = \mathcal{K} = (K_1, K, \overline{K}(\mu_{1K_1}, \mu_{2K}), L_K)$, where

[1] $K_1 = H_{11} \cup H_{12}$,$K = H_1 \cap H_2$

[2] $L_K = L_{H_1} \vee L_{H_2}$

[3] $\mu_{1K_1}: K_1 \longrightarrow L_K$ is defined by $\mu_{1K_1}x = (\mu_{1H_{11}} \vee \mu_{1H_{12}})x$

$\mu_{2K}: K \longrightarrow L_K$ is defined by $\mu_{2K}x = \mu_{2H_1}x \wedge \mu_{2H_2}x$

Suppose $(f_1, f, \Phi)_i(\mathcal{H}_1 \cup \mathcal{H}_2) = (f_1, f, \Phi)_i(\mathcal{K}) = \mathcal{M} = (M_1, M, \overline{M}(\mu_{1M_1}, \mu_{2M}), L_M)$, where

[4] $M_1 = f_1(K_1) = f_1(H_{11} \cup H_{12}) = f_1(H_{11}) \cup f_1(H_{12})$,
$M = f(K) = f(H_1 \cap H_2) = f(H_1) \cap f(H_2)$

[5] $L_M = \Phi L_K = \Phi(L_{H_1} \vee L_{H_2}) = \Phi L_{H_1} \vee \Phi L_{H_2}$

[6] $\mu_{1M_1}: M_1 \longrightarrow L_M$ is defined as

$\mu_{1M_1}y = \bigvee_{\substack{y=f_1x\\ x\in K_1}} \Phi\mu_{1K_1}x$

$= \bigvee_{\substack{y=f_1x\\ x\in K_1}} \Phi[(\mu_{1H_{11}} \vee \mu_{1H_{12}})x]$

$= \left(\bigvee_{\substack{y=f_1x\\ x\in H_{11}-H_{12}}} \Phi\mu_{1H_{11}}\right) \vee \left[\bigvee_{\substack{y=f_1x\\ x\in H_{11}\cap H_{12}}} \Phi(\mu_{1H_{11}}x \vee \mu_{1H_{12}}x)\right] \vee \left(\bigvee_{\substack{y=f_1x\\ x\in H_{12}-H_{11}}} \Phi\mu_{1H_{12}}\right)$

$\mu_{2M}: M \longrightarrow L_M$ is defined as

$\mu_{2M}y = \wedge_{\substack{y=fx\\x\in K}} \Phi\mu_{2K}x = \wedge_{\substack{y=fx\\x\in H_1\cap H_2}} \Phi(\mu_{2H_1}x \wedge \mu_{2H_2}x) =$ $\Phi\mu_{2K}x$ for fixed $x\in K$ as f is one-one.

RHS: $(f_1, f, \Phi)_i(\mathcal{H}_1) \cup (f_1, f, \Phi)_i(\mathcal{H}_2)$

Let $(f_1, f, \Phi)_i(\mathcal{H}_1) = \mathcal{G}_1 = (G_{11}, G_1, \overline{G}_1(\mu_{1G_{11}}, \mu_{2G_1})L_{G_1})$,where

[7] $G_{11} = f_1(H_{11})$, $G_1 = f(H_1)$

[8] $L_{G_1} = \Phi L_{H_1}$

[9] $\mu_{1G_{11}}: G_{11} \longrightarrow L_{G_1}$ is defined as $\mu_{1G_{11}}y = \vee_{\substack{y=f_1x\\x\in H_{11}}} \Phi\mu_{1H_{11}}x$

$\mu_{2G_1}: G_1 \longrightarrow L_{G_1}$ is defined as $\mu_{2G_1}y = \wedge_{\substack{y=fx\\x\in H_1}} \Phi\mu_{2H_1}x =$ $\Phi\mu_{2H_1}x$ for fixed $x\in H_1$(∵f is one-one)

Again

Let $(f_1, f, \Phi)_i(\mathcal{H}_2) = \mathcal{G}_2 = (G_{12}, G_2, \overline{G}_2(\mu_{1G_{12}}, \mu_{2G_2})L_{G_2})$, where

[10] $G_{12} = f_1(H_{12})$, $G_1 = f(H_2)$

[11] $L_{G_2} = \Phi L_{H_2}$

[12] $\mu_{1G_{12}}: G_{12} \longrightarrow L_{G_2}$ is defined as $\mu_{1G_{12}}y = \vee_{\substack{y=f_1x\\x\in H_{12}}} \Phi\mu_{1H_{12}}x$

$\mu_{2G_2}: G_2 \longrightarrow L_{G_2}$ is defined as $\mu_{2G_2}y = \wedge_{\substack{y=fx\\x\in H_2}} \Phi\mu_{2H_2}x = \Phi\mu_{2H_2}x$ for fixed $x\in H_2$(∵f is one-one)

Suppose $(f_1, f, \Phi)_i(\mathcal{H}_1) \cup (f_1, f, \Phi)_i(\mathcal{H}_2) = \mathcal{G}_1 \cup \mathcal{G}_2 = \mathcal{N} = (N_1, N, \overline{N}(\mu_{1N_1}, \mu_{2N}), L_N)$,

Where

[13] $N_1 = G_{11} \cup G_{12} = f_1(H_{11}) \cup f_1(H_{12}) = f_1(H_{11} \cup H_{12})$,

$N = G_1 \cap G_2 = f(H_1) \cap f(H_2) = f(H_1 \cap H_2)$

[14] $L_N = L_{G_1} \vee L_{G_2} = \Phi L_{H_1} \vee \Phi L_{H_2}$

[15] $\mu_{1N_1}: N_1 \longrightarrow L_N$ is defined as $\mu_{1N_1}y = (\mu_{1G_{11}} \vee \mu_{1G_{12}})y$

$\mu_{2N}: N \longrightarrow L_N$ is defined as $\mu_{2N}y = \mu_{2G_1}y \wedge \mu_{2G_2}y$

Clearly,(4) and(13)imply$M_1 = N_1, M = N$

(5) and (14) imply$L_M = L_N$

Sufficient to show that

$\mu_{1M_1} = \mu_{1N_1}, \mu_{2M} = \mu_{2N}$

Now,$\mu_{1N_1}y = (\mu_{1G_{11}} \vee \mu_{1G_{12}})y$, $y\in N_1 = f_1(H_{11}) \cup f_1(H_{12}) = f_1(H_{11} \cup H_{12})$

Let $X_y = \{x \in H_{11} \cup H_{12} | f_1x = y\} = A_y \cup B_y \cup C_y$, where

$A_y = \{x \in H_{11} - H_{12} | f_1x = y\}, B_y = \{x \in H_{11} \cap H_{12} | f_1x = y\}$

$A_y = \{x \in H_{12} - H_{11} | f_1x = y\}$

Case(1): $y\in f_1(H_{11}) \cap f_1(H_{12}) \Rightarrow \mu_{1N_1}y = \mu_{1G_{11}}y \vee \mu_{1G_{12}}y =$

$$\left(\vee_{\substack{y=f_1x\\x\in H_{11}}} \Phi\mu_{1H_{11}}x\right) \vee \left(\vee_{\substack{y=f_1x\\x\in H_{12}}} \Phi\mu_{1H_{12}}x\right)$$

$$\therefore \mu_{1N_1}y = \left(\vee_{\substack{y=f_1x\\x\in A_y}} \Phi\mu_{1H_{11}}\right) \vee \left[\vee_{\substack{y=f_1x\\x\in B_y}} \Phi(\mu_{1H_{11}}x \vee \mu_{1H_{12}}x)\right] \vee \left(\vee_{\substack{y=f_1x\\x\in C_y}} \Phi\mu_{1H_{12}}\right)$$

Clearly $\mu_{1M_1}y = \mu_{1N_1}y$

Case(2):$y\in f_1(H_{11}) - f_1(H_{12})$,then $y = f_1x, x \in A_y, x \notin B_y \cup C_y$

$$\Rightarrow \mu_{1N_1}y = \mu_{1M_1}y = \left(\vee_{\substack{y=f_1x\\x\in H_{11}-H_{12}}} \Phi\mu_{1H_{11}}\right)$$

Observe that, $\mu_{1M_1}y = \left(\vee_{\substack{y=f_1x\\x\in H_{11}-H_{12}}} \Phi\mu_{1H_{11}}\right)$

$\therefore \mu_{1M_1}y = \mu_{1N_1}y$

Case(3): $y\in f_1(H_{12}) - f_1(H_{11})$ as in case(2) we get$\mu_{1M_1}y = \mu_{1N_1}y$

$\mu_{2N}y = \mu_{2N_1}y \wedge \mu_{2N_2}y, y \in N = f(H_1 \cap H_2)$

$$= \left(\wedge_{\substack{y=fx\\x\in H_1\cap H_2}} \Phi\mu_{2H_1}x\right) \wedge \left(\wedge_{\substack{y=fx\\x\in H_1\cap H_2}} \Phi\mu_{2H_2}x\right)$$

$= \wedge_{\substack{y=fx\\x\in H_1\cap H_2}} \Phi(\mu_{2H_1}x \wedge \mu_{2H_2}x) = \Phi\mu_{2K}x$ for fixed $x\in K$ as f is one-one.

$\therefore \mu_{2M} = \mu_{2N}$

Let $*= d$. Then

LHS: $(f_1, f, \Phi)_d(\mathcal{H}_1 \cup \mathcal{H}_2)$

Let $\mathcal{H}_1 \cup \mathcal{H}_2 = \mathcal{K} = (K_1, K, \overline{K}(\mu_{1K_1}, \mu_{2K}), L_K)$, where

(a) $K_1 = H_{11} \cup H_{12}$,$K = H_1 \cap H_2$

(b) $L_K = L_{H_1} \vee L_{H_2}$

(c) $\mu_{1K_1}: K_1 \longrightarrow L_K$ is defined as $\mu_{1K_1}x = (\mu_{1H_{11}} \vee \mu_{1H_{12}})x$

$\mu_{2K}: K \longrightarrow L_K$ is defined as $\mu_{2K}x = \mu_{2H_1}x \wedge \mu_{2H_2}x$

Suppose $(f_1, f, \Phi)_d(\mathcal{H}_1 \cup \mathcal{H}_2) = (f_1, f, \Phi)_d(\mathcal{K}) = \mathcal{M} = (M_1, M, \overline{M}(\mu_{1M_1}, \mu_{2M}), L_M)$, where

(d) $M_1 = f_1(K_1) = f_1(H_{11} \cup H_{12}) = f_1(H_{11}) \cup f_1(H_{12})$,

$M = f(K) = f(H_1 \cap H_2) = f(H_1) \cap f(H_2)$

(e) $L_M = \Phi L_K = \Phi(L_{H_1} \vee L_{H_2}) = \Phi L_{H_1} \vee \Phi L_{H_2}$

(f) $\mu_{1M_1}: M_1 \longrightarrow L_M$ is defined as

$$\mu_{1M_1}y = \overline{C}y \wedge \left(\vee_{\substack{y=f_1x\\x\in K_1}} \Phi\mu_{1K_1}x\right) = \overline{C}y \wedge \left[\vee_{\substack{y=f_1x\\x\in K_1}} \Phi\{(\mu_{1H_{11}} \vee \mu_{1H_{12}})x\}\right]$$

$$=\overline{C}y \wedge \left(\vee_{\substack{y=f_1x\\x\in H_{11}-H_{12}}} \Phi\mu_{1H_{11}}\right) \vee \left[\vee_{\substack{y=f_1x\\x\in H_{11}\cap H_{12}}} \Phi(\mu_{1H_{11}}x \vee \mu_{1H_{12}}x)\right] \vee \left(\vee_{\substack{y=f_1x\\x\in H_{12}-H_{11}}} \Phi\mu_{1H_{12}}\right)$$

$\mu_{2M}: M \longrightarrow L_M$ is defined as

$\mu_{2M}x = \wedge_{\substack{y=fx\\x\in K}} \Phi\mu_{2K}x = \wedge_{\substack{y=fx\\x\in H_1\cap H_2}} \Phi(\mu_{2H_1}x \wedge \mu_{2H_2}x) =$ $\Phi\mu_{2K}x$ for fixed $x\in K$ as f is one-one.

RHS: $(f_1, f, \Phi)_d(\mathcal{H}_1) \cup (f_1, f, \Phi)_d(\mathcal{H}_2)$

Let $(f_1, f, \Phi)_d(\mathcal{H}_1) = \mathcal{G}_1 =$

$\left(G_{11}, G_1, \overline{G}_1\left(\mu_{1G_{11}}, \mu_{2G_1}\right)L_{G_1}\right)$,where

(g) $G_{11} = f_1(H_{11})$, $G_1 = f(H_1)$

(h) $L_{G_1} = \Phi L_{H_1}$

(i) $\mu_{1G_{11}}: G_{11} \longrightarrow L_{G_1}$ is defined as $\mu_{1G_{11}}y = \bar{C}y \wedge \left(\bigvee_{\substack{y=f_1x \\ x\in H_{11}}} \Phi\mu_{1H_{11}}x\right)$

$\mu_{2G_2}: G_2 \longrightarrow L_{G_2}$ is defined as $\mu_{2G_1}y = \bigwedge_{\substack{y=fx \\ x\in H_1}} \Phi\mu_{2H_1}x = \Phi\mu_{2H_1}x$ for fixed $x\in H_1$ (∵f is one-one)

Again

Let $(f_1, f, \Phi)_d(\mathcal{H}_2) = \mathcal{G}_2 = \left(G_{12}, G_2, \overline{G}_2\left(\mu_{1G_{12}}, \mu_{2G_2}\right)L_{G_2}\right)$, where

(j) $G_{12} = f_1(H_{12})$, $G_1 = f(H_2)$

(k) $L_{G_2} = \Phi L_{H_2}$

(l) $\mu_{1G_{12}}: G_{12} \longrightarrow L_{G_2}$ is defined as $\mu_{1G_{12}}y = \bar{C}y \wedge \left(\bigvee_{\substack{y=f_1x \\ x\in H_{12}}} \Phi\mu_{1H_{12}}x\right)$

$\mu_{2G_2}: G_2 \longrightarrow L_{G_2}$ is defined as $\mu_{2G_2}y = \bigwedge_{\substack{y=fx \\ x\in H_2}} \Phi\mu_{2H_2}x = \Phi\mu_{2H_2}x$ for fixed $x\in H_2$ (∵f is one-one)

Suppose $(f_1, f, \Phi)_d(\mathcal{H}_1) \cup (f_1, f, \Phi)_d(\mathcal{H}_2) = \mathcal{G}_1 \cup \mathcal{G}_2 = \mathcal{N} = \left(N_1, N, \overline{N}\left(\mu_{1N_1}, \mu_{2N}\right), L_N\right)$,

Where

(m) $N_1 = G_{11} \cup G_{12} = f_1(H_{11}) \cup f_1(H_{12}) = f_1(H_{11} \cup H_{12})$, $N = G_1 \cap G_2 = f(H_1) \cap f(H_2) = f(H_1 \cap H_2)$

(n) $L_N = L_{G_1} \vee L_{G_2} = \Phi L_{H_1} \vee \Phi L_{H_2}$

(o) $\mu_{1N_1}: N_1 \longrightarrow L_N$ is defined as $\mu_{1N_1}y = \left(\mu_{1G_{11}} \vee \mu_{1G_{12}}\right)y$

$\mu_{2N}: N \longrightarrow L_N$ is defined as $\mu_{2N}y = \mu_{2N_1}y \wedge \mu_{2N_2}y$

Clearly,(d) and (m)imply $M_1 = N_1, M = N$

(e) and (n) imply $L_M = L_N$

Sufficient to show that

$$\mu_{1M_1} = \mu_{1N_1}, \mu_{2M} = \mu_{2N}$$

Now, $\mu_{1N_1}y = \left(\mu_{1G_{11}} \vee \mu_{1G_{12}}\right)y$, $y\in N_1 = f_1(H_{11}) \cup f_1(H_{12}) = f_1(H_{11} \cup H_{12})$

Let $X_y = \{x \in H_{11} \cup H_{12} | f_1x = y\} = A_y \cup B_y \cup C_y$, where

$A_y = \{x \in H_{11} - H_{12} | f_1x = y\}$, $B_y = \{x \in H_{11} \cap H_{12} | f_1x = y\}$

$A_y = \{x \in H_{12} - H_{11} | f_1x = y\}$

Case(1): $y\in f_1(H_{11}) \cap f_1(H_{12}) \Rightarrow \mu_{1N_1}y = \mu_{1G_{11}}y \vee \mu_{1G_{12}}y$

$$= \left[\bar{C}y \wedge \left(\bigvee_{\substack{y=f_1x \\ x\in H_{11}}} \Phi\mu_{1H_{11}}x\right)\right] \vee \left[\bar{C}y \wedge \left(\bigvee_{\substack{y=f_1x \\ x\in H_{12}}} \Phi\mu_{1H_{12}}x\right)\right]$$

$$= \bar{C}y \wedge \left(\bigvee_{\substack{y=f_1x \\ x\in H_{11}}} \Phi\mu_{1H_{11}}x\right) \vee \left(\bigvee_{\substack{y=f_1x \\ x\in H_{12}}} \Phi\mu_{1H_{12}}x\right)$$

$$\therefore \mu_{1N_1}y = \bar{C}y \wedge \left(\bigvee_{\substack{y=f_1x \\ x\in A_y}} \Phi\mu_{1H_{11}}\right) \vee \left[\bigvee_{\substack{y=f_1x \\ x\in B_y}} \Phi\left(\mu_{1H_{11}}x \vee \mu_{1H_{12}}x\right)\right] \vee \left(\bigvee_{\substack{y=f_1x \\ x\in C_y}} \Phi\mu_{1H_{12}}\right)$$

Clearly $\mu_{1M_1}y = \mu_{1N_1}y$

Case(2): $y\in f_1(H_{11}) - f_1(H_{12})$,then $y = f_1x$, $x \in A_y$, $x \notin B_y \cup C_y$

$$\Rightarrow \mu_{1N_1}y = \mu_{1M_1}y = \bar{C}y \wedge \left(\bigvee_{\substack{y=f_1x \\ x\in H_{11}-H_{12}}} \Phi\mu_{1H_{11}}\right)$$

Observe that, $\mu_{1M_1}y = \bar{C}y \wedge \left(\bigvee_{\substack{y=f_1x \\ x\in H_{11}-H_{12}}} \Phi\mu_{1H_{11}}\right)$

$\therefore \mu_{1M_1}y = \mu_{1N_1}y$

Case(3): $y\in f_1(H_{12}) - f_1(H_{11})$ as in case(2) we get $\mu_{1M_1}y = \mu_{1N_1}y$

$\mu_{2N}y = \mu_{2N_1}y \wedge \mu_{2N_2}y$,

$y \in N = f(H_1 \cap H_2)$

$$= \left(\bigwedge_{\substack{y=fx \\ x\in H_1\cap H_2}} \Phi\mu_{2H_1}x\right) \wedge \left(\bigwedge_{\substack{y=fx \\ x\in H_1\cap H_2}} \Phi\mu_{2H_2}x\right)$$

$$= \left[\bigwedge_{\substack{y=fx \\ x\in H_1\cap H_2}} \Phi\left(\mu_{2H_1}x \wedge \mu_{2H_2}x\right)\right]$$

$= \Phi\left(\mu_{2H_1}x \wedge \mu_{2H_2}x\right) = \Phi\mu_{2K}x$ for fixed $x \in K$

$\therefore \mu_{2M} = \mu_{2N}$

3.5 Proposition: For any Fs-function $(f_1, f, \Phi): \left(B_1, B, \overline{B}\left(\mu_{1B_1}, \mu_{2B}\right), L_B\right) \longrightarrow \left(C_1, C, \overline{C}\left(\mu_{1C_1}, \mu_{2C}\right), L_C\right)$ and for any pair of Fs-subsets $\mathcal{H}_1 = \left(H_{11}, H_1, \overline{H}_1\left(\mu_{1H_{11}}, \mu_{2H_1}\right)L_{H_1}\right)$ and $\mathcal{H}_2 = \left(H_{12}, H_2, \overline{H}_2\left(\mu_{1H_{12}}, \mu_{2H_2}\right)L_{H_2}\right)$ of $\mathcal{B} = \left(B_1, B, \overline{B}\left(\mu_{1B_1}, \mu_{2B}\right), L_B\right)$,

$(f_1, f, \Phi)_*(\mathcal{H}_1 \cap \mathcal{H}_2) \subseteq (f_1, f, \Phi)_*(\mathcal{H}_1) \cap (f_1, f, \Phi)_*(\mathcal{H}_2)$,holds, whenever $*=$ i or d or p

The proof follows from the relevant definitions.

3.6 Proposition: For any Fs-function $(f_1, f, \Phi): \left(B_1, B, \overline{B}\left(\mu_{1B_1}, \mu_{2B}\right), L_B\right) \longrightarrow \left(C_1, C, \overline{C}\left(\mu_{1C_1}, \mu_{2C}\right), L_C\right)$ and for any pair of Fs-subsets $\mathcal{H}_1 = \left(H_{11}, H_1, \overline{H}_1\left(\mu_{1H_{11}}, \mu_{2H_1}\right)L_{H_1}\right)$ and $\mathcal{H}_2 = \left(H_{12}, H_2, \overline{H}_2\left(\mu_{1H_{12}}, \mu_{2H_2}\right)L_{H_2}\right)$ of $\mathcal{B} = \left(B_1, B, \overline{B}\left(\mu_{1B_1}, \mu_{2B}\right), L_B\right)$,

$(f_1, f, \Phi)_*(\mathcal{H}_1 \cap \mathcal{H}_2) = (f_1, f, \Phi)_*(\mathcal{H}_1) \cap (f_1, f, \Phi)_*(\mathcal{H}_2)$,holds, whenever $*=$ i or d or p. . Provided f_1 is bijective and Φ is injective

The proof follows from the relevant definitions.

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