HEXAGONAL MESH PYRAMID: SOME TOPOLOGICAL PROPERTIES

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Abstract: Pyramidal networks are desirable network topology, used as both software data-structure and hardware architecture. In this paper, we introduce a new pyramid network, Hexagonal mesh pyramid [HXP_n], where n is the number of levels. The proposed network is derived from hexagonal mesh with triangular tessellations as the base in each level. Some basic properties of HXP_n are studied. The Hamiltonian and pancyclic properties are proved. A brief comparison of the new network and the existing triangular mesh pyramid structure are analyzed. Results show that the proposed network has high node, scalability and connectivity and total network bandwidth.

Keywords: Hamiltonian, Hexagonal mesh, Hexagonal mesh pyramid, Pancyclic.

Introduction: Interconnection networks are currently being used for many different applications ranging from inter – ip connections in VLSI circuits to wide area computer networks [8, 10]. An interconnection network can be modeled by a graph where a processor is represented by a vertex, and a communication channel between two processing vertices is represented by edge between the corresponding vertices. The conventional pyramid network (a mesh pyramid) is one of the important network topologies as it has been used in both hardware architecture and software structures for parallel computing, graph theory, digital geometry, machine vision and image processing [10]. Various topologies for interconnection networks have been proposed in the literature: these include cubic networks(e.g meshes, tori, k-ary n-cubes, hypercubes, folded cubes and hypermeshes), hierarchical networks(e.g pyramids and trees) and recursive networks(e.g RTCC networks, OTIS networks, WK recursive networks and star graphs) that have been widely studied in the literature for the topological properties [5,9].

Desirable properties of interconnection network include [1] symmetry, small node degree d, diameter D, network cost (product of the node degree and the network diameter i,e$d \times D$), high connectivity, scalability (both form a hardware and performance point of view) and fault – tolerance. Most topologies introduced by researchers try to compromise between cost and performance, resulting in a wide range of different interconnection topologies, each with advantages and disadvantages [10].

In this paper, we propose a new pyramidal network, called as hexagonal mesh pyramid and denoted as *HXPn*, where n is the number of levels. This new network based on hexagonal mesh with triangular tessellation. The new network preserves many desirable properties of traditional pyramid networks, but unlike them, here a vertex may have more than one parent that can be exploited to effectively handle more communication patterns found in different parallel applications.

2. Hexagonal Mesh and Hexagonal Mesh Pyramid : Useful distributed processor architectures offer the advantage of improved connectivity and reliability. An important component of such a distributed system is the system topology, which defines the inter-processor communication architecture. Such system topology forms the interconnection networks.

HEXAGONAL MESH

Triangular, square and hexagon are the three existing regular plane tessellations which are composed of the same kind of regular polygons. To design direct interconnection networks we use any one of this, also these types of interconnection networks are highly competitive in overall performance.

The triangular tessellation is used to define hexagonal network and this type of hexagons are widely studied in [9, 10]. For further details it is better to refer papers [9, 10].

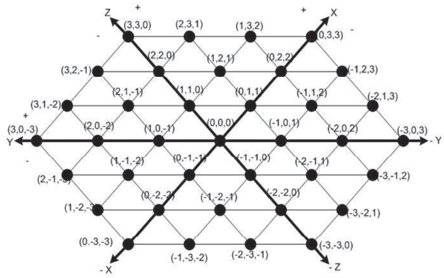

Figure 1

Topological Properties Of Hxn: HX_n has $3n^2-3n+1$ vertices and $9n^2 - 15n+6$ edges, where n is the number of vertices on one side of the hexagon [9, 10]. The diameter is $2(n-1)$. In this structure, there are 6 vertices with degree 3 which we call as corner vertices. The vertex, which is at $n-1$ distance from the corner vertices, is called the center vertex and it is unique. The degree of a vertex in this network can be 3,4 and 6.

This particular structure has wide application in the field of computer graphics, cellular phone base stations [6], image processing and in chemistry as the representation of benzenoid hydrocarbons. [8]

HEXAGONAL MESH PYRAMID(HXP_n)

A famous network topology that has been used as the base of both hardware architectures and software structures is the pyramid. By exploiting the inherent hierarchy at each level, pyramid structures can efficiently handle the communication requirements of various problems in graph theory, digital geometry, machine vision, and image processing [8].

The main problems with traditional pyramids are hardware scalability and poor network connectivity and bandwidth. To address these problems, in this paper we propose a new pyramidal network called as hexagonal mesh pyramid. The new network preserves many desirable properties of traditional pyramid network.

Definition: A hexagonal mesh pyramid of n levels denoted as HXP_n consists of a set of vertices, arranged as n layers of hexagonal mesh. A vertex is addressed as $(k(x, y, z))$ and is said to be a vertex at level k. The part (x, y, z) of the address determines the address of a vertex within the layer k, of the hexagonal network. The vertices at level k form a network of HX_k. The vertex $(k(x, y, z))$ is connected to the vertices $(k + 1(x, y, z))$ and 6 other vertices which are adjacent to $(k + 1(x, y, z))$ in HX_{k+1}.

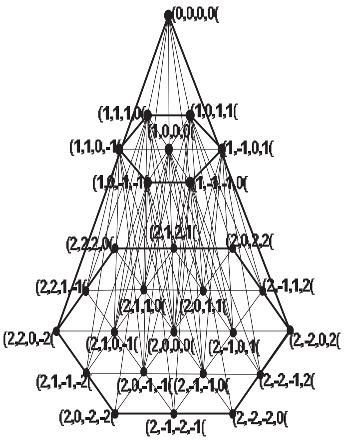

Figure 2

LEMMA: The number of vertices of aHXP_n is n^3

Proof: The number of vertices of a hexagonal mesh HX_n is $3n^2 - 3n + 1$

$$\therefore |V(HXP_n)| = 3\sum n^2 - 3\sum n + n$$

$$= 3\left[\frac{n(n+1)(2n+1)}{6}\right] - 3\left[\frac{n(n+1)}{2}\right] + n$$

$= n^3$,where n is the number of level.

Lemma: The number of edges of a HXP_n is $3n^2(n-1) + 6(n-1)^3$

Proof: The number of edges of a hexagonal mesh is $9n^2 - 15n + 6$

$\therefore |E(HXP_n)|$ = The number of horizontal edges + number of vertical edges

$$= 9\sum n^2 - 15\sum n + 6n + 6(n-1$$

$$= 9\left[\frac{n(n+1)(2n+1)}{6}\right] - 15\left[\frac{n(n+1)}{2}\right] + 6n + 6(n-1)^3$$

$= 3n^2(n-1) + 6(n-1)^3$, where n is the number of level.

Hamiltonian Properties :

DEFINITION [4]

A given network G = (V,E), is Hamiltonian connected if it contains a Hamiltonian path starting at any node $x \in V$ and ending at any node $y \in V - x$.

THEOREM

Any Hexagonal mesh network HXn is Hamiltonian.

In the following figure shows how a Hamiltonian cycle can be constructed in a hexagonal mesh network.

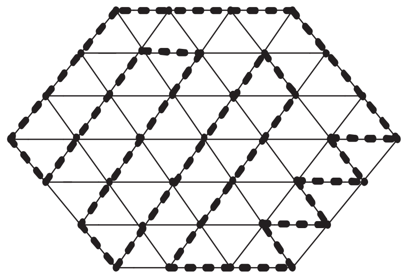

Figure 3

Theorem: A hexagonal mesh pyramid of level n, HXPn is Hamiltonian connected.

PROOF Proof is going by induction.

Let x and y be any two vertices in HXP_2. The following figures show that how this x and y are Hamiltonian connected in HXP_2.

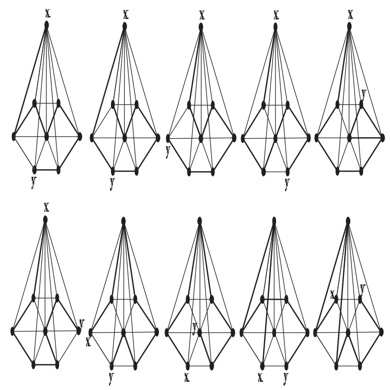

FIGURE.4

Assume that the theorem holds forHXP_n. Next, we prove that the theorem is true for HXP_{n+1}. To complete the proof we consider three cases.

CASE I

Assume that x and y are in level $2,3...n$. To add a new network of level $n+1$ to the previous network we have to show that it also preserves the Hamiltonian path. By making a Hamiltonian path in the hexagonal mesh at the new level and connecting this to the Hamiltonian path constructed in HXP_n.

Figure 5 shows how to connect the Hamiltonian path constructed in the new level to the Hamiltonian path constructed in the upper *n*-level from source vertex x to destination vertex *y* in order to construct a Hamiltonian path from *x*to *y* to the HXP_{n+1}.

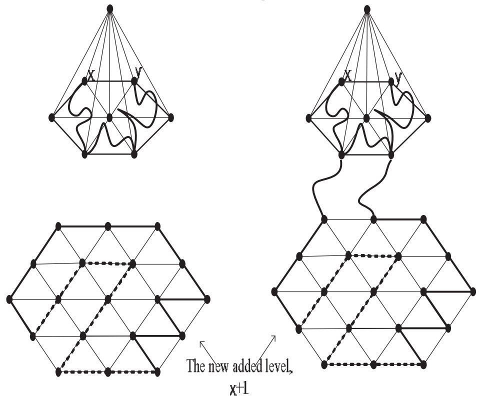

Figure 5

CASE II: Assume that *x* is in the upper *n* – level of the HXP_n (in levels *1,2,3…n*) and *y* is in the newly added level, *n+1*. In this case we can make a Hamiltonian path between vertex *x* and the parent vertex of *z*, where *z* is the neighboring vertex of *y* in the Hamiltonian cycle constructed in the hexagonal mesh added at level *n+1*. We can remove the edge between *z* and *y* and connect the parent vertex of *z* to *z*. Now the Hamiltonian path from *x*to *y* in the hexagonal mesh pyramid of *n+ 1* level is completed.

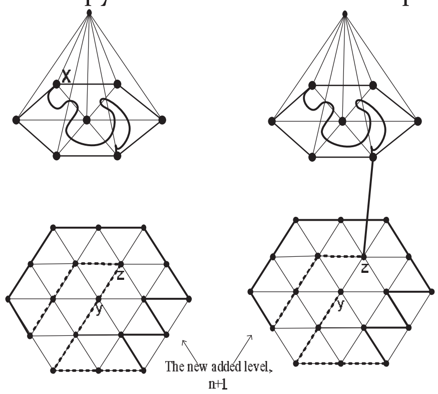

Figure 6

CASE III: Both x and y are in the newly added hexagonal mesh at level n+1. To construct the Hamiltonian path between x and y, is similar to the previous case. A Hamiltonian path in the upper n-level hexagonal mesh pyramid is constructed between the parent of z and the parent of w, where z and w are the neighboring vertices of y and x respectively, in the Hamiltonian cycle constructed in the hexagonal mesh at level n+1. Remove the edge between z and y and the edge between x and w, and connecting the parent of z to z and the parent of w to w. Thus the desired Hamiltonian path between x and y in the new hexagonal mesh pyramid of n+1 level is constructed.

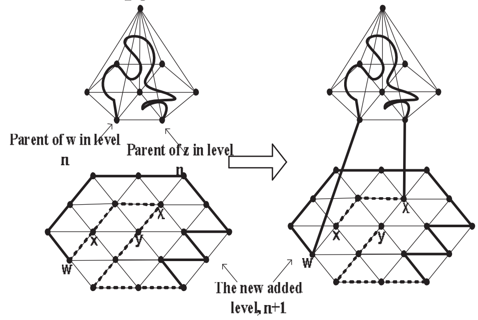

Figure 7

Corollary 3.1 Any hexagonal mesh pyramid of n levels is Hamiltonian.

Proof: It comes directly from the above theorem. Choose any two neighboring vertices x and y in hexagonal mesh pyramid. Construct the Hamiltonian path between the chosen vertices. Finally connecting x to y to form a Hamiltonian cycle.

Pancyclicty: Definition 4.1 [2]

A given network *G = (V, E)* and *n* = | *V(G)* | . If *G* contains cycles of length *3, 4, 5…n* then *G* is called to be pancyclic.

Lemma 4.1 The hexagonal mesh HX_n is pancyclic.

Proof: We shall prove this lemma by induction.

Obviously the hexagonal mesh HX_2 is pancyclic. i,eHX_2 contain cycles of length *3,4…7*(See Fig. 8)

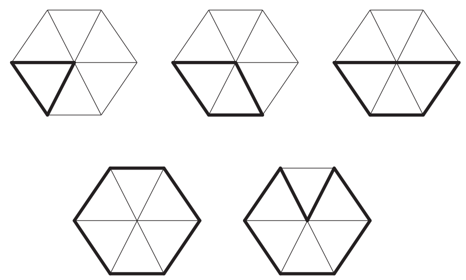

Figure 8

We have to assume that this lemma holds for HX_n. Next we shall prove, this is true for the hexagonal mesh HX_{n+1}, i,e level at *n+1*.

It is easy to suppose that the full size cycle is embedded. It is nothing but the Hamiltonian cycle. But we must show how a hexagonal mesh of level *n+1* can embed all cycles of length $3,4,5\ldots 3(n+1)^2 - 3(n+1) + 1$. Obviously cycles of length $3,4,5\ldots 3n^2 - 3n + 1$ can be constructed in the inner hexagon. To construct cycles of length $3n^2 - 3n + 2 \ldots 3(n+1)^2 - 3(n+1) + 1$we can add new vertices from outer hexagon to the Hamiltonian cycle constructed in the inner hexagon.

Figure 9(a) Hamiltonian cycle in the inner hexagon. Figure9(b) shows how one vertex in the outer hexagonal mesh is added to the Hamiltonian cycle of the inner hexagonal mesh, After this we get a cycle of length $3n^2 - 3n + 2$. Repeat this process until we got the cycles of length $3n^2 - 3n + 3 \dots 3(n+1)^2 - 3(n+1)$. The only remaining cycle is the one, of length $3(n+1)^2 - 3(n+1) + 1$ is the Hamiltonian cycle.

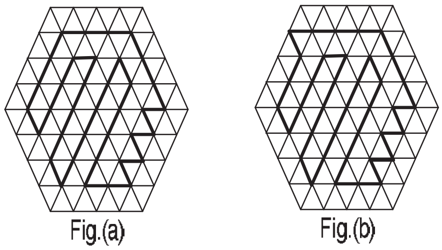

Figure 9

Theorem 4.1 The hexagonal mesh pyramid of n levels is pancyclic.
Proof: The proof is by induction.
Obviously HXP_2 is pancyclic. i,eHXP_2 have 8 vertices. So there exist cycles of length $3,4,5 \dots 8$
Assume that HXP_n is pancyclic.
Next we have to prove that HXP_{n+1}ispancyclic. We use the above lemma, to construct a cycles of length $3,4 \dots (n+1)^3$. Obviously cycles of length $3, 4 \dots n^3$ can constructed in the upper n – level . To construct cycles of length $n^3+1, n^3+2 \dots (n+1)^3$, we can use the new vertices(added as the basement hexagonal mesh at level $n+1$) to the Hamiltonian cycle constructed in the upper n – level as shown in the figure.
Figure 10(a) shows the constructed Hamiltonian cycle of the upper n – level HXPn .Figure 10(b) shows how one node (in the basement hexagonal mesh at level n+1) is added to the Hamiltonian cycle constructed in the upper HXPn. Thus we get a cycle of length $n^3 + 1$. Repeat the process until we add all cycle upto the length $n^3 + 3n^2 + 3n$. The remaining cycle is nothing but the Hamiltonian cycle of length $n^3 + 3n^2 + 3n + 1$. Using corollary 3.1, we can construct the Hamiltonian cycle in HXP_{n+1}. Figure 10(c) shows the Hamiltonian cycle in HXP_{n+1}.

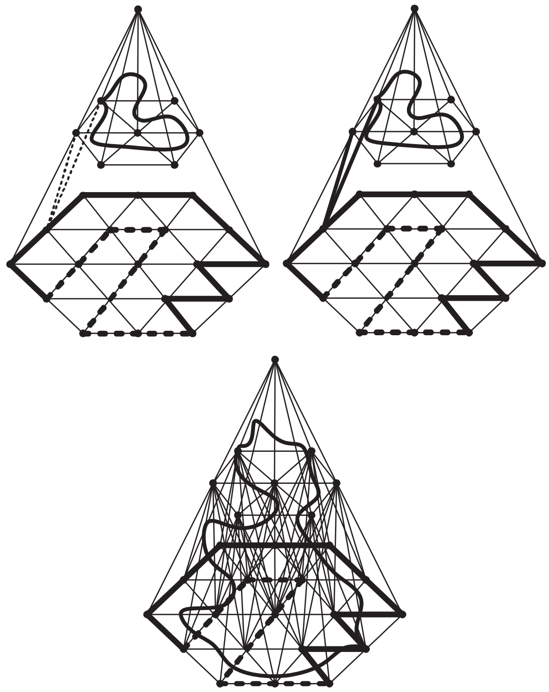

Figure 10

Comparisons: In this section, we compare the hexagonal mesh pyramid with the other well – known pyramid network, named as triangular pyramid(Trippy). Node degree, Network diameter and network cost are the three basic topological properties of a network. Figure 11(a, b) briefly describes about node degree and network diameter.

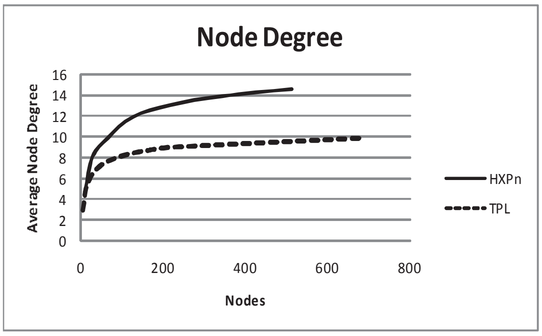

Figure 11(a)

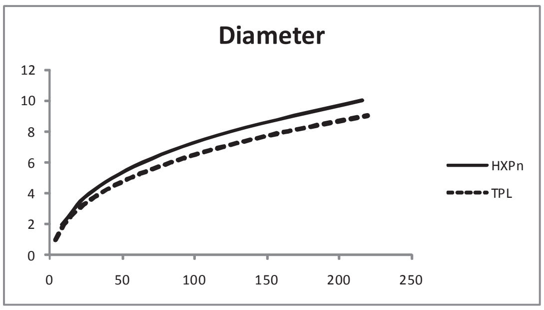

Figure 11(b)

The node degree (N_D) is defined as the number of physical channels originating from a node. This attribute is a measure of a node's input – output complexity [6, 7, 8]. For hexagonal mesh pyramid, the node degree is independent of the network size. Figure 12(a) compares the average node degree of HXP_n and Trippy. HXP^n has larger node degree compared to the Trippy and the difference becomes larger as the network size increases. Note that, for very large network size the average node degree is greater than 14, since most of the node has the degree 13 and are surrounded with much lower degrees 4, 6, 10, 11. While in the Trippy, the average node degree is 12. The diameter is commonly used to measure to compare the static network performance of a system. The diameter of a network is the maximum inter – node distance, i,e the maximum number of edges that must be traversed to send a message to all vertices along a shortest path. The diameter is small, then the network is more fast and efficient to sent the message. i,e the message dissemination becomes faster. The diameter of HXP_n is 2(n-1). Figure 11(b) shows that HXP_n has greater diameter than the Trippy.Network cost is defined as the product of diameter and vertex degree. This is a rough metric to measure both hardware cost and the pessimistic performance of a network. Figure 12 shows that the network cost of HXP_n and Trippy. Form the figure, we conclude that HXP_n has costly than the Trippy.

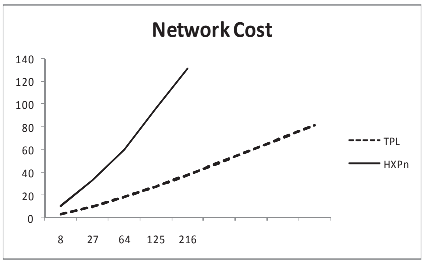

Figure 12

The network throughput measured in the worst – case is another performance that can be used for early performance comparison. Network throughput is the total network bandwidth (which is proportional to the number of edges in the network) divided by the diameter. Figure 13 shows that HXP_n has better throughput compared to the Trippy.

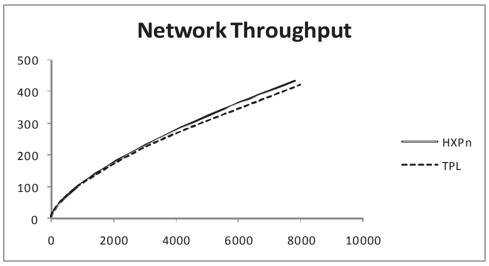

Figure 14

Hardware scalability is an important property for a network topology. Hardware scalability shows how easy a network can scaled to the next possible size, in other words, how many nodes must be added to a given network to make the network size. The network is more scalable when we add the smaller number of nodes. For example, a linear array has the best hardware scalability as the next size of the network is obtained by adding only one node to the existing network. In HXP_n we add $3n^2 - 3n + 1$ nodes while in the Trippy it becomes $\frac{n(n+1)}{2}$. From this we conclude that HXPn has poor scalability.

Connectivity degree is a measure that shows how well the nodes are connected within a network topology. This measure is a real number larger than zero but no more than 1, and is calculated by equivalent complete network (with $\frac{N(N-1)}{2}$ edges). For complete graph the connectivity degree is 1 because all the nodes are connected to each other. HXP_n has greater connectivity degree as compared to the Trippy because it has 7 edges from one node to another while the Trippy has only 3.

Conclusions: In literature, there are many network topologies have been proposed for multicomputer interconnection networks. In this paper we introduced a new pyramidal topology based on hexagonal mesh networks. The high connectivity, greater node degree, greater diameter and network throughput of the proposed network have made it a good cost – effective candidate for interconnecting processing nodes in a multiprocessor computer.

Future work can consider in finding an efficient broadcasting and routing algorithms.

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