FIXED POINT THEOREMS IN CONE METRIC SPACES BY USING COMMON LIMIT RANGE PROPERTY

GINISWAMY , MAHESHWARI

Abstract: The aim of this paper is to establish some common coincidence and common fixed point theorems under different types of contractive conditions by using CLRg property. These results extend and improve several well known results in the literature of metric fixed point theory.

Keywords: CLRg property, cone metric space, fixed points, weakly compatible.

Introduction: Jungck [3] defined a pair of self mappings to be weakly compatible, if they commute at their coincidence points. Recently, as a generalization of metric space, Huang and Zhang[1] introduced the concept of cone metric space by replacing the set of real numbers by an ordered Banach space and obtained some fixed point theorems for mappings satisfying different contractive conditions in normal cone metric space. Later on, Rezapour and Hamlbarani[2] proved the results of [1] by omitting the assumption of normality of cone, which is a milestone in the literature of fixed point theory in cone metric space. Since then, the study of common fixed points in cone metric spaces under different types of contractive conditions has been at the centre of vigorous research activity. In 2011, Sintunavarat et.al.[4] have introduced the notion of Common Limit in the Range of g (CLRg) property in fuzzy metric space. The importance of this property is, it ensures that one does not require the closeness of the range subspaces and hence, now a days, authors are giving much attention to this property for generalizing the results present in the literature(see [11]-[13] and the references therein).

In the present paper, we extend and generalize some common fixed point theorems under different types of contractive conditions in metric spaces into the context of cone metric spaces, by employing the notions of CLRg property and weak compatibility.

The following are the basic definitions needed in the main results.

Definition 1: Let E be a real Banach space. A subset P of E is called a *Cone* if

1. P is closed, non-empty and $P \neq \{\theta\}$
2. $a, b \in R, a, b \geq 0, x, y \in P$ imply that $ax + by \in P$.
3. $P \cap (-P) = \{\theta\}$.

Definition 2: For a given cone $P \subset E$, we define the following,

1. The *Partial ordering* $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$.
2. $x \ll y$ for $y - x \in intP$ where $intP$ denotes the interior of P and use $x < y$ for $x \leq y$ and $x \neq y$.
3. P is *Solid* if $intP \neq \theta$ and *normal* whenever there is a number $K > 0$ such that $\forall x, y \in E, 0 \leq x \leq y$ implies $\| x \| \leq K \| y \|$.

Definition 3: Let X be a nonempty set. Suppose that the mapping $d: X \times X \to E$ satisfies

1. $0 \leq d(x, y), \forall x, y \in X$ and $d(x, y) = 0$ iff $x = y$.
2. $d(x, y) = d(y, x)$ for all $x, y \in X$.
3. $d(x, y) \leq d(x, z) + d(z, y)\ \forall x, y, z \in X$.

Then d is called a *Cone metric* on X and *(X,d)* is called a *Cone metric space*.

Definition 4: Let f and g be two self maps defined on a set X. Then f and g are said to be *weakly compatible* if they commute at their coincidence point.

Definition 5: Let f and g be two self maps defined on a metric space *(X, d)*. Then f and g are said to satisfy the *Common Limit in the Range of g* (CLRg) property, if there exist a sequence $\{x_n\}$ in X such that $lim_{n\to\infty} fx_n = lim_{n\to\infty} gx_n = gx$ for some x in X.

Proposition 1:[3] Let f and g be two weakly compatible self maps of a set X. If f and g have a unique point of coincidence $w = fx = gx$, then w is the unique common fixed point of f and g.

Main Results:

The following two theorems extend the results of [5] into the context of cone metric space.

Theorem 1: *Let f and g be two weakly compatible self mappings of a cone metric space (X, d) satisfying*

(i) CLRg property

$$(ii)\, d(fx, fy) < \max\left\{ \begin{array}{c} d(gx, gy), \frac{[d(fx,gx)+d(fy,gy)]}{2}, \\ \frac{[d(fy,gx)+(fx,gy)]}{2} \end{array} \right\}$$

$\forall x \neq y \in X$.

Then f and g have a unique common fixed point.

Proof: Since f and g satisfies CLRg property, there exist a sequence $\{x_n\}$ in X such that $lim_{n\to\infty} fx_n = lim_{n\to\infty} gx_n = gx$ for some x in X.

Consider

$$d(fx_n, fx) < \max\left\{ \begin{array}{c} d(gx_n, gx), \frac{[d(fx_n,gx_n)+ d(fx,gx)]}{2}, \\ \frac{[d(fx,gx_n)+ d(fx_n,gx)]}{2} \end{array} \right\}$$

On letting $n \to \infty$, we obtain

$fx = gx$. Thus x is the coincidence point of f and g. let $z = fx = gx$. Since f and g are weakly compatible we have

$$fz = fgx = gfx = gz.$$

Now we prove that $fz = z$. Suppose that $fz \neq z$, then

$d(fz,z) = d(fz,fx) <$ $\max\left\{ d(gz,gx), \frac{[d(fz,gz)+d(fx,gx)]}{2}, \frac{[d(fx,gz)+d(fz,gx)]}{2} \right\} = d(fz,z)$, which is a contradiction.
Hence $fz = z = gz$. i.e. z is the common fixed point of f and g.
Uniqueness of the fixed point can be proved easily.

Corollary 1.1: *Let f and g be two weakly compatible self mappings of a cone metric space (X, d) satisfying*
(i) CLRg property
(ii) $d(fx,fy) < \max\left\{ d(gx,gy), \frac{[d(fy,gx)+d(fx,gy)]}{2} \right\}$, $\forall x \neq y \in X$.
Then f and g have a unique common fixed point.

The next theorem involves a function ϕ, which is defined by Mathkowski[10] as follows:
Let Φ be the set of all auxiliary functions $\phi: [0,\infty) \to [0,\infty)$ is a nondecreasing function satisfying $lim_{n\to\infty}\phi^n(t) = 0$ for all $t \in (0,\infty)$. If $\phi \in \Phi$, then ϕ is called a $\Phi -$map. Further $\phi(t) < t$ for all $t \in (0,\infty)$ and $\phi(0) = 0$.

Theorem 2: *Let f ,g, S and T be four self mappings of a cone metric space (X, d) such that*
(i) $d(fx,gy) \leq \phi(\max\{d(Sx,Ty), d(Sx,gy), d(Ty,gy)\})$ *for all* $x,y \in X$.
(ii) (f,S) and (g,T) satisfies CLR(S) and CLR(T) property respectively.
(iii)(f,S) and (g,T) are weakly compatible.
Then f ,g, S and T have a unique common fixed point.

Proof: Since f and S satisfies CLR(S) property, there exist a sequence $\{x_n\}$ in X such that $lim_{n\to\infty} fx_n = lim_{n\to\infty} Sx_n = Sx$ for some x in X.
Similarly, g and T satisfies CLR(T) property implies there exist a sequence $\{y_n\}$ in X such that $lim_{n\to\infty} gy_n = lim_{n\to\infty} Ty_n = Ty$ for some y in X.
Consider
$d(fx_n,gy_n) \leq \phi(\max\{d(Sx_n,Ty_n), d(Sx_n,gy_n), d(Ty_n,gy_n)\})$. As $n \to \infty$, we get $Sx = Ty$.
Now consider
$d(fx,gy_n) \leq \phi(\max\{d(Sx,Ty_n), d(Sx,gy_n), d(Ty_n,gy_n)\})$
Letting $n \to \infty$, we obtain $fx = Sx$. Thus x is the coincidence point of f and S. Since (f,S) are weakly compatible, we have $ffx = fSx = Sfx = SSx$. Similarly, y is the coincidence point of g and T. Again, (g,T) are weakly compatible implies $TTy = Tgy = gTy = ggy$.
Now we prove that $ffx = fx$. Consider,

$$d(ffx,fx) = d(ffx,gy) \leq \phi(\max\{d(Sfx,Ty), d(Sfx,gy), d(Ty,gy)\})$$

$= \phi d(ffx,fx)$, which implies $ffx = fx = Sfx$. Thus fx is the common fixed point of f and S. Similarly, gy is the common fixed point of g and T. Since $fx = gy$, fx is the common fixed point of f, g, S and T. Uniqueness of the fixed point can be proved easily.
By restricting the mappings f,g,S and T suitably, we can derive the corollary of the above theorem for three self maps as follows.

Corollary 2.1: *Let f, g and S be self mappings of a cone metric space (X, d) such that*
(i) $d(fx,gy) \leq \phi(\max\{d(Sx,Sy), d(Sx,gy), d(Sy,gy)\})$ *for all* $x,y \in X$.
(ii) (f,S) and (g,S) satisfies CLR(S) property.
(iii)(f,S) and (g,S) are weakly compatible.
Then f ,g and S have a unique common fixed point.

Proof: Follows from Theorem 2 by setting $S = T$.
The following two theorems are the generalization of Theorem 2.8 and 2.2 of [6] respectively.

Theorem 3: *Let (X, d) be a cone metric space and let f, g, S, T be four self maps on (X,d) with cone P having non empty interior such that*
(i) (f,S) and (g,T) satisfies CLR(S) and CLR(T) property respectively.
(ii) $d(fx,gy) \leq \mathrm{pd(Sx,Ty) + qd(fx,Sx) + rd(gy,Ty) + t[d(fx,Ty) + d(gy,Sx)]}$, $\forall x,y \in X$ *where* $p,q,r,t \in [0,1)$ *satisfying* $p+q+r+2t < 1$.
Then (f,S) and (g,T) have a unique point of coincidence in X. Moreovre, if (f,S) and (g,T) are weakly compatible, then f, g, S and T have a unique common fixed point.

Proof: Since f and S satisfies CLR(S) property, there exist a sequence $\{x_n\}$ in X such that $lim_{n\to\infty} fx_n = lim_{n\to\infty} Sx_n = Sx$ for some x in X.
Similarly, g and T satisfies CLR(T) property implies there exist a sequence $\{y_n\}$ in X such that $lim_{n\to\infty} gy_n = lim_{n\to\infty} Ty_n = Ty$ for some y in X.
Consider

$$d(x_n,gy_n) \leq \mathrm{pd}(Sx_n,Ty_n) + \mathrm{qd}(fx_n,Sx_n) + \mathrm{rd}(gy_n,Ty_n) + \mathrm{t}[\mathrm{d}(fx_n,Ty_n) + \mathrm{d}(gy_n,Sx_n)]$$

Letting $n\to\infty$, we obtain

$$[1-(p+2t)]d(Sx,Ty) \leq 0$$

which implies $Sx = Ty$.
Now consider

$$d(fx_n,gy) \leq \mathrm{pd}(Sx_n,Ty) + \mathrm{qd}(fx_n,Sx_n) + \mathrm{rd}(gy,Ty) + \mathrm{t}[\mathrm{d}(fx_n,Ty) + \mathrm{d}(gy,Sx_n)]$$

Letting $n \to \infty$ we obtain

$$[1-(r+t)]d(Ty,gy) \leq 0$$

which implies $Ty = gy$. Hence y is the coincidence point of T and g. Similarly, we can obtain $fx = Sx$ which means x is the coincidence point of f and S. Uniqueness of the coincidence point can be proved easily.
Since (f,S) and (g,T) are weakly compatible , we have $ffx = fSx = Sfx = SSx$ and $TTy = Tgy = gTy = ggy$.
Now we prove that $ffx = fx$.
Consider $d(ffx,fx) = d(ffx,gy) \leq pd(Sfx,Ty) + qd(ffx,Sfx) + rd(gy,Ty) + t[d(ffx,Ty) +$

$d(gy, Sfx)]$ which gives
$[1-(p+2t)]d(ffx, fx) \leq 0$ which in turn gives $ffx = fx = Sfx$. Thus fx is the common fixed point of f and S. Similarly, gy is the common fixed point of g and T. Since $fx = gy$, fx is the common fixed point of f,g,S and T. Uniqueness of the fixed point can be proved easily.

Theorem 4: *Let (X, d) be a cone metric space and let f, g, S and T be four self maps on (X,d) with cone P having non empty interior such that*

(i) (f,S) and (g,T) satisfies CLR(S) and CLR(T) property respectively.

(ii) $d(fx, gy) \leq hu(x,y)$, *where* $h \in (0,1)$ *and* $\forall x, y \in X$, $u(x,y) \in \left\{ \begin{matrix} d(Sx,Ty), d(fx,Sx), d(gy,Ty), \\ \frac{d(fx,Ty)+d(gy,Sx)}{2} \end{matrix} \right\}$

(iii)(f,S) and (g,T) are weakly compatible.

Then f ,g,S and T have a unique common fixed point.

Proof: Since f and S satisfies CLR(S) property, there exist a sequence $\{x_n\}$ in X such that $lim_{n\to\infty} fx_n = lim_{n\to\infty} Sx_n = Sx$ for some x in X.
Similarly, g and T satisfies CLR(T) property implies there exist a sequence $\{y_n\}$ in X such that $lim_{n\to\infty} gy_n = lim_{n\to\infty} Ty_n = Ty$ for some y in X.
Consider $d(fx_n, gy_n) \leq hu(x_n, y_n)$,
Letting $n \to \infty$ we obtain
$d(Sx,Ty) \leq hu(x_n,y_n)$, where $u(x_n, y_n) \in \{d(Sx,Ty), d(Sx,Sx), d(Ty,Ty), \frac{d(Sx,Ty)+d(Ty,Sx)}{2}\} = d(Sx,Ty)$
Thus $d(Sx,Ty) \leq hd(Sx,Ty)$ which implies $Sx = Ty$.
Now we show that $fx = Sx$. Consider,

$$d(fx, gy_n) \leq hu(x, y_n)$$

Letting $n \to \infty$ we obtain
$d(fx, Sx) \leq hu(x, y_n)$, where

$$u(x,y_n) \in \left\{ \begin{matrix} d(Sx,Ty), d(fx,Sx), d(Ty,Ty), \\ \frac{d(fx,Ty)+d(Ty,Sx)}{2} \end{matrix} \right\} = \left\{ d(fx,Sx), \frac{d(fx,Sx)}{2} \right\}.$$

If $u(x,y_n) = d(fx,Sx)$, then $d(fx,Sx) \leq hd(fx,Sx)$, which implies $fx = Sx$.
If $u(x,y_n) = \frac{d(fx,Sx)}{2}$, then $d(fx,Sx) \leq \frac{hd(fx,Sx)}{2}$, which also implies $fx = Sx$. Therefore in both the cases $fx = Sx$. Similarly $gy = Ty$. Since (f,S) and (g,T) are weakly compatible , we have $ffx = fSx = Sfx = SSx$ and $TTy = Tgy = gTy = ggy$.
Now we prove that $ffx = fx$.
Consider $d(ffx, fx) = d(ffx, gy) \leq h\,u(fx, y)$, where

$$u(fx,y) \in \left\{ \begin{matrix} d(Sfx,Ty), d(ffx,Sfx), d(gy,Ty) \\ , \frac{d(ffx,Ty)+d(gy,Sfx)}{2} \end{matrix} \right\} = d(ffx, fx)$$

Hence $d(ffx,fx) \leq h\,d(ffx,fx)$ which gives $ffx = fx = Sfx$. Therefore fx is the common fixed point of f and S. Similarly, gy is the common fixed point of g and T. Since $fx = gy$, fx is the common fixed point of f,g,S and T. Uniqueness of the fixed point can be proved easily.
The following result is the common fixed point theorem for expansive type of contractions, which is a generalization of Theorem 2.1 of [7] and Theorem 3.1 of [8].

Theorem 5: *Let f and g be selfmappings of a cone metric space (X, d) with a solid cone satisfying*

(i) CLRg property.

(ii) $d(gx,gy) \geq ad(fx,fy) + bd(fx,gx) + cd(fy,gy)$, *for all* $x, y \in X$ *where* $a,b,c \geq 0$ *with* $a+b+c > 1$.

Then f and g have a coincidence point. If $a > 1$, *then the coincidence point is unique. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point.*

Proof: Since f and g satisfies CLRg property, there exist a sequence $\{x_n\}$ in X such that $lim_{n\to\infty} fx_n = lim_{n\to\infty} gx_n = gx$ for some x in X.
Consider
$d(gx_n, gx) \geq ad(fx_n, fx) + bd(fx_n, gx_n) + cd(fx, gx)$. As $n \to \infty$, we obtain, $d(gx,gx) \geq ad(gx,fx) + bd(gx,gx) + cd(fx,gx)$ which gives $(a+c)d(fx,gx) \leq 0$. Hence $fx = gx$. i.e.x is the coincidence point of f and g.
Uniqueness: let z and w be the points of coincidence of f and g. Then $z = fx = gx$ and $w = fy = gy$.
Consider $d(z,w) = d(gx,gy) \geq ad(fx,fy) + bd(fx,gx) + cd(fy,gy)$. This implies $z = w$, since $a > 1$. Therefore x is the unique point of coincidence of f and g.
Since f and g are weakly compatible and by Proposition 1, $z = fx$ is the unique common fixed point of f and g,

Corollary 5.1: *Let f and g be selfmappings of a cone metric space (X, d) with a solid cone satisfying*

(i) CLRg property

(ii) $d(gx,gy) \geq ad(fx,fy) + bd(fx,gx)$ *for all* $x, y \in X$ *where* $a, b \geq 0$ *with* $a+b > 1$.

Then f and g have a coincidence point. If $a > 1$, *then the coincidence point is unique. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point.*

Corollary 5.2: *Let f and g be selfmappings of a cone metric space (X, d) with a solid cone satisfying*

(i) CLRg property

(ii) $d(gx,gy) \geq ad(fx,fy)$ *for all* $x, y \in X$ *where* $a > 1$.

Then f and g have a unique coincidence point. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point.

References:

1. L.G.Huang, X.Zhang, "Cone metric spaces and fixed point theorems of contractive mappings", J.Math.Anal. Appl.,323(2007),1468-1476.
2. S.Rezapour, R.Hamlbarani, Some notes on the paper "cone metric spaces and fixed point theorems of contractive mappings", J.Math.Anal.Appl.,345 (2008), 719-724.
3. Jungck, G., "Common fixed points for non continuous non self maps on non metric spaces", Far East J. Math. Sci., 4(1996), 19-215.
4. W.Sintunavarat, P.Kumam, "Common fixed point theorems for a pair of weakly compatible mappings in fuzzy metric spaces", Journal of Applied Mathematics(2011),Article ID 637958.
5. M.Aamri,D.El.Moutawakil, "Some new common fixed point theorems under strict contractive conditions", J.Math.Anal.Appl.,270(2002),181-188.
6. M.Abbas, B.E.Rhoades, T.Nazir, "Common fixed points for four maps in cone metric spaces", Appl.Math.Comput.(2010).
7. W.Shatanawi, F.Awawedh, "Some fixed and coincidence point theorems for expansive maps in cone metric spaces" , Fixed point theory and Applications(2012),2012:19.
8. Z.Kadelburg, P.P.Murthy, S.Radenovic, "Common fixed points for expansive mappings in cone metric spaces", Int.J.Math.Anal., Vol.5,2011, No.27,1309-1319.
9. J.G.Mehta, M.L.Joshi, "On complete cone metric space and fixed point theorems", J.Sci.Res.,3(2),303-309 (2011).
10. J.Mathkowski, "Fixed point theorems for mappings with a contractive iterate at a point", Proc. Amer. Math. Soc., 62 (1977), 344-348.
11. Manoj Kumar, Pankaj Kumar, Sanjay Kumar, "Some common fixed point theorems using CLRg property in cone metric spaces", Advances in Fixed point Theory,2(2012), No.3,340-356.
12. M.Imdad, S.Chauhan, "Employing common limit range property to prove unified metrical common fixed point theorems", Int.J.Anal.,Vol.2013, Article ID:763261.
13. M.Jain, S.Kumar, "A common fixed point theorem in fuzzy metric space using the property (CLRg)", Thai J.Math,Article, submitted for publication.

* * *

Associate Professor, Department of Mathematics, PES College of Science, Arts and Commerce, Mandya, Karnataka, India. Email: gswamypes@gmail.com

Assistant Professor, Department of Mathematics, Government First Grade College, Vijayanagar, Bangalore, Karnataka, India,maheshwari616@yahoo.com