ON IDEAL BITOPOLOGICAL VIEW OF CONTINUITY

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Abstract : In this paper a new concept of Continuity in Ideal Bitopological space is introduced and some of the basic properties are discussed.The relationship between them and other existing sets are derived. As an application some new results are derived via Continuity of Ideal Bitopological Space.

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Introduction: Kelly [8] introduced Bitopology by considering two topologies on a Topological spaces.The concepts of g-closed sets introduced by Levine in Topological spaces are defined in Bitopological spaces by Fukutake. Thereafter several authors turned their attention towards generalizations of various concepts of Topology by considering Bitopological spaces. LellisThivagar defined a new type of open sets in Bitopological spaces namely (1,2)α-open sets and (1,2) semi-open sets. Various types of generalized closed sets have been defined in terms of (1,2)open sets. Ideals play an important role in Topology. Jankovic and Hamlet have introduced the notion of I-open sets in Topological spaces. In this paper we have introduced new type of continuity using Ideal open sets in Bitopological spaces. Also the relationship between them and other existing sets are derived. Some properties of sets are discussed. The most of the results in this paper can be extended to Fuzzy Topology and Digital Topology.

Preliminaries

A nonempty collection I of subsets of X is called an ideal on X if it has the following properties:

(i). If $A \in I$ and $B \subseteq A$ then, $B \in I$ (heredity)

(ii). If $A \in I$ and $B \in I$ then, $A \cup B \in I$ (finite additivity).

The triple (X, τ, I) denotes a set X with a topology τand an ideal I on X according to Rose and Hamlet [12]. The closure, interior and complement of a subset A of X are denoted by cl(A), int(A) and A^c. For a subset A of X, $A^*=\{x \in X: U\cap A \notin I$, for every $U\in\tau$containing x} is called the local function of A with respect to I and τ[4]. The simplest ideals are {ϕ} and $\wp(X) =\{A : A \subseteq X\}$. It was observed that $A^*(\{\phi\}) = cl(A)$ and $A^*(\wp(X))=\phi$.

We shall give some definitions and theorems which are used in the following sections.

Definition 2.1 Given a space (X,τ,I) and $A \subseteq X$,A is said to be I-open [7] if $A \subseteq int(A^*)$.

The family of all I-open sets in X is denoted by IO(X,τ). The following theorem describes many basic facts on the local function.

Theorem 2.2 Let (X, τ, I) be a space and A, B be subsets of X. Then

(i). If $A\subseteq B$, then $A^* \subseteq B^*$.

(ii). $A^* = cl(A^*)\subseteq cl(A)$ (A^* is a closed subset of cl(A)).

(iii). $(A\cup B)^* = A^*\cup B^*$.

(iv). $(A^*)^* \subseteq A^*$.

(v). If$U\in\tau$, then $U \cap A^* \subseteq (U \cap A)^*$.

Definition 2.3A subset $A \subset (X,\tau,I)$ is said to be

(i). *- dens e-in-itself [5]iff $A \subset A^*$.

(ii). * -perfect [5]iff $A = A^*$.

Definition 2.4A subset A of a bitopological space (X, τ_1,τ_2) is said to be

(i). (i,j)-preopen [11] if $A \subseteq \tau_i$-int(τ_j-cl(A)) where i,j=1,2 and i≠ j.

(ii). (i,j)-semi open [11] if $A \subseteq \tau_j$-cl(τ_i-int(A)).

(iii). (i,j)-semi-preopen[11] if $A\subseteq\tau_j$-cl(τ_i-int(τ_j-cl(A))).

(iv). (i,j)-γ-open [6] if A∩B is (i,j)-preopen for every (i,j)-preopen set B in X .

The complement of a (i,j)-semi open (resp. (i,j)-preopen, (i,j)-semi-preopen and (i,j)-γ-open) set in X is (i,j)semi-closed (resp. (i,j)-preclosed, (i,j)semi-preclosed and (i,j)-γ-closed). The family of (i,j)-preopen (resp. (i, j)-γ-open) sets in X is denoted by (i,j)-PO(X)(resp. (i,j)-γO(X)).

Remark 2.5

(i). Every (i.j)-γ-open set is (i,j)-preopen but the converse is not true [6].

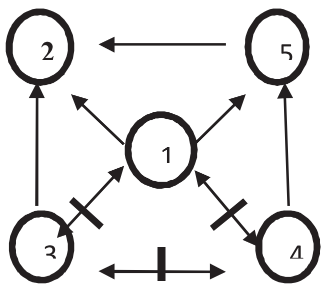
(ii). Every (i,j)-preopen set is (i,j)-semi-preopen [11] but the converse is not true.

(iii). (i,j)-semi open sets and (i,j)-γ-open sets are independent [6].

Open Sets in Ideal Bitopological Space: In this section we introduce the notion of I-open set in a bitopological space. Throughout this section X stands for a bitopological space with an ideal I on X, written as (X, τ_1,τ_2, I). We denote A^* with respect to $\tau_j$by $(A^*)_j$, the interior of A with respect to $\tau_i$by $int_i(A)$ and the closure of A with respect to the topology τ_j, by $cl_j(A)$.

Always i,j= 1, 2, and $i \neq j$.

Definition 3.1A subset $A \subseteq X$ is said to be (i,j)-I-openif $A \subseteq int_i((A^*)_j)$.
The family of all (i, j)-I-open sets in X is denoted by IO(X,(i,j)).
Example 3.2 Let X = {a, b, c}, τ_1= {ϕ, {b, c}, X}, τ_2 = {ϕ, {a}, X} and I ={ϕ,{c}}. Then IO(X, (1,2)) = {ϕ, {a}, {b}, {a, b}, {a, c}, {b, c},X}.
Definition 3.3A subset $A \subseteq X$ is said to be pairwise I-open (briefly p.I-open) if it is (1,2)-I-open and (2,1)-I-open.
Remark 3.4 It is easily observed that the (i,j)-I-open sets are independent to the (i,j)-semi-open sets.
Proposition 3.5 Every (i,j)-I-open set is (i.j)-preopen.
Proof. Let $A \subseteq X$ be (i, j)-I-open.Then$A \subseteq int_i((A^*)_j)$. Since $(A^*)_j$is a closed subset of $cl_j(A)$, $int_i((A^*)_j) \subseteq int_i(cl_j(A))$.Therefore,A is (i,j)-preopen.
Remark 3.6 The converse of the above proposition is not true, in general as shown in following example.
Example 3.7 Let X = {a,b,c}, τ_1= {ϕ, {a}, {b},{a, b}, X}, τ_2 = {ϕ, {c},X} and I = {ϕ, {c}}. Then the set {c} is (2,1)-preopen but not (2,1)-I-open.
Remark 3.8(i,j)-γ-open sets and (i.j)-preopen sets are independent. In Example 2.2, the set {c} is (1,2)-γ-open but not (1,2)-I-open. If we take the space X and $\tau_1$as in Example 3.7, τ_2 = {ϕ, {b},{a, c}, X} and I = {ϕ,{b}, {c}, {b, c}}, then the set {a} is (2,1)-I-open but it is not (2,1)-γ-open.
Proposition 3.9 Every (i,j)-I-open set is (i,j)-semi-preopen.
Proof. Let $A \subseteq X$ be (i,j)-I-open. Then $A \subseteq int_i((A^*)j)$. Since $(A^*)_j$-is a closed subset of $cl_j(A)$, $int_i((A^*)_j) \subseteq int_i(cl_j(A))$and therefore,$A \subseteq cl_j(int_i(cl_j(A)))$.
Remark 3.10 The converse of the above proposition is not true. In Example 3.2,{b} is (l,2)-semi-preopen but not (1,2)-I-open.
Remark 3.11 From the above discussions,we have obtained the following diagram.
1.(i,j)-I-open. 2.(i,j)-preopen 3.(i,j)-γ-open4. (i,j)-semi open5. (i,j)-semi-preopen
Moreover, $A \rightarrow B$ means A implies B but B does not imply A and $A \nleftrightarrow B$ means A and B are independent.



Remark 3.12 The intersection of two (i,j)-I-open sets is not (i,j)-I-open, in general. In Example 3.2, The sets {a,b}and {b.c} are (1,2)-I-open but {b} is not (1,2)-I-open.
Theorem 3.13 For any subset A of X,
(i). if I = {ϕ}, then $((A^*)_i)$ = $cl_i(A)$, i= 1,2 and hence each (i,j)-preopen set is (i,j)-I-open.
(ii). if I = $\wp(X)$, then $((A^*)_i) = \phi$for i = 1,2 and hence A is (i,j)-I-open if and only if A = ϕ.
Theorem 3.14 For any (i,j)-I-open set A of (X, τ_1,τ_2, I), $(A^*)_j = ((int_i(((A^*)_j))^*)_j$.
Proof. If A is (i,j)-I-open, $A \subseteq int_i((A^*)_j)$.Therefore, $(A^*)_j \subseteq ((int_i((A^*)_j))^*)_j$. Since $int_i((A^*)_j) \subseteq (A^*)_j$, $((int_i((A^*)_j))^*)_j \subseteq (((A^*)_j)^*)_j \subseteq (A^*)_j$. Hence $(A^*)_j$, = $((int_i(((A^*)_j))^*)_j$.
Definition 3.15A subset $A \subseteq X$ is said to be (i,j)-I -closed if its complement in X is (i,j)-I-open.
Theorem 3.16 For $A \subseteq X$,$(int_i((A^*)_j))^c \neq int_i(((A^c)^*)_j)$, in general.
Theorem 3.17 If A is (i,j)-I-closed, then $(int_i((A^*)_j) \subset A$.
Proof.Follows from the definition.
Theorem 3.18 Let $A \subseteq X$ and $X \setminus (int_i((A^*)_j)$ = $int_i((X \setminus A)^*)_j)$.
Then A is (i,j)-I-closed if and only if $int_i((A^*)_j) \subset A$.
Theorem 3.19 Let (X,τ_1,τ_2,I) be a space and A, $B \subseteq X$. Then
(i). $\{U_\alpha : \alpha \in \nabla\} \subseteq IO(X, (i,j))$, then U $\{U_\alpha: \alpha \in \nabla\} \in IO(X, (i, j))$.
(ii). If $A \in IO(X, (i, j))$ and $B \in \tau_i$, then $A \cap B \in IO(X, (i,j))$.
(iii). If $A \in IO(X,(i,j))$ and B is an α-set with respect to τ_i, then $A \cap B \in$ (i,j)-PO(X).
Proof.(i). Since $\{U_\alpha: \alpha \in \nabla\} \subseteq IO(X,(i,j))$, $U_\alpha \subseteq int_i(((U_\alpha)^*)_j)$ for every $\alpha \in \nabla$. Therefore, $\cup U_\alpha \subseteq U int_i(((U_\alpha)^*)_j) \subseteq int_i(\cup(((U_\alpha)^*)_j))$ for every $\alpha \in \nabla$.
(ii). If $A \in IO(X,(i,j))$ and $B \in \tau_i$, then $A \cap B \subseteq int_i((A^*)_j) \cap B = int_i((A^*)_j \cap B)$,since $B \in \tau_i \subseteq int_i((((A \cap B))^*)_j)$.
(iii).Obvious. Since $(A^*)_j \subseteq cl_j(A)$.
Theorem 3.20
(i). The union of (i,j)-I-closed set and τ_i-closed set is (i,j)-I-closed.
(ii). The union of (i,j)-I-closed set and α-closed set with respect to τ_i is (i,j)-preclosed.
Theorem 3.21 IfA $\subseteq X$ is (i,j)-I-open and (i,j)-semi closed, thenA = $int_i((A^*)_j)$.
Proof. If A is (i,j)-I-open, then $A \subseteq int_i((A^*)_j)$. Since A is (i,j)-semi closed, $A \supseteq int_i(cl_j(A)) \supseteq int_i((A^*)_j)$. Therefore A = $int_i((A^*)_j)$.
Theorem 3.22 If $A \in \tau_i$and $B \in IO(X,(i,j)$, then there exists a τ_i-open set G of X such that $A \cap G = \phi$, implies that $A \cap B = \phi$.
Proof. Since $B \in IO(X,(i,j)$, $B \in int_i((A^*)_j)$. LetG =

$int_i((A^*)_j)$. Then G is τ_i-open such that $B \subset G$. If $A \cap G = \phi$, then $G \subseteq X \setminus A$ implies that $cl_i(G\} \subseteq X \setminus A$. Hence $B \subseteq X \setminus A$.

Continuity in IdealBitopological Spaces.

Throughout this section $k \in \{1, 2\}$ and i, j = 1,2 and $i \neq j$.

Definition 4.1 A function $f : (X,\tau_1,\tau_2, I_x) \rightarrow (Y, \sigma_1, \sigma_2)$ is said to be (i, j)-I-σ_k-continuous if for every $V \in \sigma_k$-open set in Y, $f^{-1}(V)$ is (i,j)-I-open in X, k= 1,2.

Example 4.2 Let X be the space in Example 3.2 and Y be the space {p,q,r} with the topologies$\sigma_1 = \{\phi, \{p\},Y\}$ and $\sigma_2 = \{\phi, \{p,r\},Y\}$. Define a function $f : X \rightarrow Y$ as follows f(a) = p, f(b) = q and f(c) = r. Then f is (1,2)-σ_1-continuous and (1,2) -σ_2-continuous.

Theorem 4.3 For a function $f:(X, \tau_1.\tau_2,I_x) \rightarrow (Y,\sigma_1, \sigma_2)$, the following are equivalent.

(i). f is (i,j)-I-σ_k-continuous.

(ii). For each $x \in X$and each $V \in \sigma_k$containing f(x), there exists $W \in IO(X, (i, j))$ containing x such that $f(W) \subseteq V$.

(iii). For each $x \in X$and each $V \in \sigma_k$ containing f(x), $((f^{-1}(V))^*)_j$is a neighbourhood of x.

Proof.(i) ⇒(ii). Since $V \in \sigma_k$:containing f(x),by (i), $f^{-1}(V) IO(X,(i,j))$. Put $W=f^{-1}(V)$. Then $x \in W$ and therefore, $f(W) \subset V$.

(ii) ⇒(iii). Since $V \in \sigma_k$containing f(x), by (ii), there exists $W \in IO(X, (i,j))$ containing x such that $f(W) \subset V$. Hence $x \in W \subseteq int_i((W^*)_j) \subseteq int_i(((f^{-1}(V))^*)_j) \subseteq ((f^{-1}(V))^*)_j$.Therefore, $(f^{-1}(V^*)_j)$ is a neighbourhood of x.

(iii) ⇒(i). Obvious.

Theorem 4.4 For a function $f :(X,\tau_1,\tau_2, I_x) \rightarrow (Y,\sigma_1, \sigma_2)$ then the following are equivalent.

(i). f is (i, j)-I-σ_k-continuous.

(ii). The inverse image of each σ_k-closed set in Y is (i, j)-I-closed in X.

(iii). $((int_i(f^{-1}(V)))^*)_j \subset f^{-1}(V^*)$, for each $*_k$-dense-in-itself subset $V \subset Y$ where$*_k$-dense-in-itself means that *-dense-in-itself with respect to σ_k.

(iv). $f((int_i(U)Y)_j) \subset ((f(U))^*)_j$for each $U \subset X$ and for each $*_k$-perfect subset of Y where $*_k$-perfect means that *-perfect with respect to σ_k.

Proof. (i) ⇒ (ii). Let $F \subset Y$ be closed. Then $Y \setminus F$ is open in Y and by (i), $f^{-1}(Y \setminus F) = X \setminus f^{-1}(V)$ is (i,j)-I-open in X, $k \in \{1,2\}$. Thus $f^{-1}(V)$ is (i,j)-I-closed.

(ii) ⇒ (iii). Let $V \subset Y$ be $*_k$-dense-in-itself. Since $(V^*)_k$is closed with respect to σ_k by (ii), $f^{-1}((V^*)_k)$is (i,j)-I-closed in X. By Theorem 3.17,$f^{-1}((V^*)_k) \supset ((int_i(f^{-1}((V^*)_k)))^*)_k$. Since $(V^*)_k$ is $*_k$-dense-in-itself, $f^{-1}((V^*)_k) \supset ((int_i(f^{-1}((V)^*)_k))^*)_k \supset ((int_i(f^{-1}(V)))^*)_k$.

(iii) ⇒ (iv). Let $U \subset X$ and W = f(U). Then by (iii), $f^{-1}((W^*)_k \supset (int_i(f^{-1}(W))^*)_k \supset ((int_i(U))^*)_k$. Therefore,$f((int_i(U))^*)_j) \subset (W^*)_k = ((f(U))^*)_k$.

(iv) ⇒ (i). Let V be σ_k-open, $W = Y \setminus V$ and $U = f^{-1}(W)$. Then $f(U) \subset W$ and by (iv),$f((int_i(U))^*)_j) \subset ((f(U))^*)_k \subset (W^*)_k = W$ since W is $*_k$-perfect. Thus $f^{-1}(W) \supset ((int_i(U))^*)_k = ((int_i(f^{-1}(W)))^*)_k$ and therefore, $f^{-1}(W) = f^{-1}(Y \setminus V)$ is (i, j)-I-closed. Hence $f^{-1}(V)$ is (i, j)-I-open in X and so f is (i,j)-I-σ_k-continuous.

Applications: Applying the concept of Continuity of IdealBitopological Space in this section, we obtain the following results.

Theorem 5.1A function $f : (X, \tau_1,\tau_2,I_x) \rightarrow (Y, \sigma_1, \sigma_2)$ is (i,j)-I-σ_k-continuous if and only if the graph function $g : X \rightarrow Xx\ Y$ is (i, j)-I-ρ_k-continuous where $\rho_k = \tau_k x \sigma_k$, $k \in \{1,2\}$.

Proof.Necessity. Let f be (i,j)-I-σ_k-continuous. Let $x \in X$and let V be any ρ_k-open set in X x Y containing g(x) = (x, f(x)). Then there exists a basic open set U x W such that $g(x) \in U\ x\ W \subset V$. Since f is (i,j)-I-$\sigma_k$-continuous, there exists (i,j)-I-open set U_1in X such that $x \in U_1 \subset X$ and $f(U_1) \subset W$. Since $U_1 \cap U \subset U$, $g(U_1 \cap U) \subset U\ x\ W \subset V$. Therefore g is (i,j)-I-ρ_k-continuous.

Sufficiency Let $g : X \rightarrow XxY$ is (i,j)-I-ρ_k-continuous and let V be σ_k-open containing f(x). Then X xV is ρ_k-open in X x Y for some k. Sinceg is (i, j)-I-ρ_k-continuous there exists a (i,j)-I-open set W in X such that $g(W) \subset XxV$. This implies that $f(W) \subset V$. Therefore, f is (i,j)-I-σ_k-continuous.

Theorem 5.2 Let $f : (X,\tau_1,\tau_2,I_x) \rightarrow (Y, \sigma_1, \sigma_2)$ is (i,j)-I-σ_k-continuous and $U \in \tau_i$. Then the restriction $f \setminus U$ is (i, j)-I-σ_k-continuous.

Proof. Let V be σ_k-open. Then $f^{-1}(V) \subset int_i((f^{-1}(V))^*)$ and so $U \cap f^{-1}(V) \subseteq U \cap int_i((f^{-1}(V))^*)$. Hence $f \setminus U \subseteq U \cap int_i((f^{-1}(V))^*)$ since $U \in \tau_i$. Now $(f \setminus U)^{-1}(V) = U \cap int_i((f^{-1}(V))^*) \subseteq int_i((U \cap (f^{-1}(V)))^*) = int_i((((f|U)^{-1}(V))^*)$. Therefore, $f \setminus U$ is (i,j)-I-σ_k-continuous.

Theorem 5.3 If $f : (X. \tau_1, \tau_2, I_x) \rightarrow (Y, \sigma_1, \sigma_2)$, is (i, j)-I-$\sigma_k$-continuous and $g : (Y, \sigma_1, \sigma_2) \rightarrow (Z, \rho_1, \rho_2)$ is pairwise continuous, then g o f is (i, j)-I-ρ_k-continuous.

Proof. Obvious.

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