AN ALGORITHM FOR A FINITE FAMILY OF NONEXPANSIVE MAPS IN HYPERBOLIC SPACES

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Abstract: In 2012, Khan et. al. [7] analyzed an implicit algorithm for two finite families of non-expansive maps in hyperbolic spaces. Further in 2013, Fukhar-ud-din et. al. [1] generalized a one-step implicit algorithm for two finite families of non-expansive mappings in hyperbolic spaces. Motivated by these results, we study the convergence results of Noor implicit algorithm for non-expansive mappings in hyperbolic spaces.

Keywords: Hyperbolic spaces, Noor implicit algorithm, nonexpansive maps and Δ-convergence.

Introduction: Hyperbolic spaces are general in nature and provide rich geometrical structures for different results with applications in topology, graph theory, multivalued analysis and metric fixed point theory. The class of Hyperbolic spaces includes normed spaces, CAT(0) spaces, Hadamard manifolds, R-trees and Hilbert balls equipped with hyperbolic metric[3]. In 1970, Takahashi [19] introduced the convex metric space. A subset K of a hyperbolic space X is convex if $W(x, y, \alpha) \in K$ for all $x, y \in K$ and $\alpha \in [0, 1]$.

In 2005, Kohlenbach [12] introduced a convex structure in hyperbolic space as follows:

A hyperbolic space (X, d, W) is a metric space (X, d) together with a map $W: X^2 \times [0, 1] \to X$ satisfying:

(1) $d(u,W(x, y, \alpha)) \le (1 - \alpha)d(u, x) + \alpha d(u, y)$

(2) $d(W(x, y, \alpha), W(x, y, \beta)) = |\alpha - \beta| d(x, y)$

(3) $W(x, y, \alpha) = W(y, x, (1 - \alpha))$

(4) $d(W(x, z, \alpha), W(y, w, \alpha)) \le (1 - \alpha)d(x, y) + \alpha\, d(z, w)$

for all $x, y, z, w \in X$ and $a, b \in [0, 1]$.

A hyperbolic space (X, d, W) is said to be:

(i) strictly convex [19] if for any $x, y \in X$ and $\lambda \in [0, 1]$, there exists a unique element $z \in X$ such that $d(z, x) = \lambda d(x, y)$ and $d(z, y) = (1-\lambda)d(x, y)$.

(ii) uniformly convex [18] if for all $u, x, y \in X$, $r > 0$ and $\varepsilon \in (0, 2]$, there exists a $\delta \in (0, 1]$ such that $d(x, u) \le r$, $d(y, u) \le r$, $d(x, y) \le \varepsilon r$ $\Rightarrow d(W(x, y, 1/2), u) \le (1- \delta)r$.

A map $\eta: (0, \infty) \times (0, 2] \to (0, 1]$ which provides such a $\delta = \eta(r, \varepsilon)$ for given $r > 0$ and $\varepsilon \in (0, 2]$, is called modulus of uniform convexity. We call η monotone if it decreases with r (for a fixed ε). A uniformly convex hyperbolic space is strictly convex [13]. In our paper, we consider the hyperbolic space introduced by Kohlenbach [12] which is more restricted than the hyperbolic type introduced by Goebel and Kirk [4].

Let K be a nonempty subset of a metric space (X, d), and let T be a self-mapping on K. Denote by $F(T) = \{x \in K : T(x) = x\}$ the set of fixed points of T. A self-mapping T on K is said to be

- Nonexpansive if $d(Tx, Ty) \le d(x, y)$ for $x, y \in K$.
- Quasi-nonexpansive if $d(Tx, p) \le d(x, p)$ for $x \in K$ and for $p \in F(T)$
- Asymptotically nonexpansive if there exists a sequence $k_n \subset [0,\infty)$ and $\lim_{n\to\infty} k_n = 0$ and $d(T^n x,T^n y) \le (1+k_n)d(x, y)$ for $x, y \in K$ *and* $n \ge 1$.

The Noor iteration [20] is defined as

$$x_{n+1} = (1-\alpha_n)\, x_n + \alpha_n\, Ty_n$$
$$y_n = (1-\beta_n)\, x_n + \beta_n Tz_n$$
$$z_n = (1-\gamma_n)\, x_n + \gamma_n\, Tx_n \qquad (1.1)$$

for all $n \ge 1$, where $\{\alpha_n\}$, $\{ \beta_n \}$, $\{\gamma_n\}$ are sequences in $[0, 1]$. If we take $\beta_n = \gamma_n = 0$ for all n, (1.1) reduces to the Mann iteration [16] and we take $\gamma_n = 0$ for all n, (1.1) reduces to the Ishikawa iteration [5].

Inspired and motivated by the work of Kirk [11], Xu and Ori [21] , Khan et al. [7], we investigate Δ-convergence through a three-step implicit algorithm for a finite family of nonexpansive maps in hyperbolic spaces. In a hyperbolic space, the three- step algorithm (1.1) can be defined as:

$$x_n = W(x_{n-1}, T_n y_n, \alpha_n),$$
$$y_n = W(x_n, T_n z_n, \beta_n),$$
$$z_n = W(x_n, T_n x_n, \gamma_n),\ n \ge 1 \qquad (1.2)$$

where $T_n = T_{n (mod\, N)}$

In 1976, Lim [15] introduced the notion of asymptotic center and introduced the concept of Δ-convergence in a general setting of a metric space. In 2008, Kirk and Panyanak [10] investigated Δ-convergence in CAT(0) spaces and showed that Δ-convergence coincides with the usual weak convergence in Banach spaces. Moreover, both concepts share many useful properties in uniformly convex spaces. Many authors have studied the Δ-convergence of various iterative schemes with different mappings in hyperbolic spaces and CAT(0) spaces [2, 7, 8, 9, 17].

Let $\{x_n\}$ be a bounded sequence in a hyperbolic space X. For $x \in X$, define a continuous functional $r(\cdot, \{x_n\})$: $X \to [0,\infty)$ by $r(x, \{x_n\}) = \lim_{n\to\infty} \sup d(x, x_n)$. The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by $r(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in X\}$.

The asymptotic center of a bounded sequence $\{x_n\}$ with respect to a subset K of X is defined as follows:

$A_K(\{x_n\}) = \{x \in X : r(x, \{x_n\}) \le r(y, \{x_n\})$ for any $y \in K\}$.

This is the set of minimizers of the functional $r(\cdot, \{x_n\})$. If the asymptotic center is taken with respect to X, then it is simply denoted by $A(\{x_n\})$. It is known that uniformly convex Banach spaces and even CAT(0) spaces enjoy the property that 'bounded sequences have unique asymptotic centers with respect to closed convex subsets.' The following lemma is due to Leustean [14] and ensures that this property also holds in a complete uniformly convex hyperbolic space.

Lemma 1.1 [14] Let (X, d, W) be a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity η. Then every bounded sequence $\{x_n\}$ in X has a unique asymptotic center with respect to any nonempty closed convex subset K of X.

Lemma 1.2 [7] Let (X, d, W) be a uniformly convex hyperbolic space with monotone modulus of uniform convexity η. Let $x \in X$ and $\{a_n\}$ be a sequence in $[b, c]$ for some $b, c \in (0, 1)$. If $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $\limsup_{n\to\infty} d(x_n, x) \le r$, $\limsup_{n\to\infty} d(y_n, x) \le r$ and $\lim_{n\to\infty} d(W(x_n, y_n, \alpha_n), x) = r$ for some $r \ge 0$, then $\lim_{n\to\infty} d(x_n, y_n) = 0$.

Lemma 1.3 [7] Let K be a nonempty closed convex subset of a uniformly convex hyperbolic space and $\{x_n\}$ a bounded sequence in K such that $A(\{x_n\}) = \{y\}$ and $r(\{x_n\}) = \rho$. If $\{y_m\}$ is another sequence in K such that $\lim_{m\to\infty} r(y_m, \{x_n\}) = \rho$, then $\lim_{m\to\infty} y_m = y$.

Main Result: In this section, we first prove a lemma useful for the main result.

Lemma2.1 Let K be a nonempty closed convex subset of a hyperbolic space X and let $\{T_i : i \in I\}$ be a finite family of non-expansive self map on K such that F is nonempty. Then for the sequence $\{x_n\}$ defined implicitly in (1.2), we have

(i) $\lim_{n\to\infty} d(x_n, p)$ exists for each $p \in F$.

(ii) $\lim_{n\to\infty} d(x_n, T_l x_n) = 0$, for each $l = 1, 2, 3, \ldots, N$.

Proof. **(i)** For any $p \in F$, it follows from (1.2) that

$$\begin{aligned} d(z_n, p) &= d(W(x_n, T_n x_n, \gamma_n), p) \\ &\le (1-\gamma_n) d(x_n, p) + \gamma_n d(T_n x_n, p) \\ &\le (1-\gamma_n) d(x_n, p) + \gamma_n d(x_n, p) \\ &= d(x_n, p) \qquad (2.1) \end{aligned}$$

(2.1)

now, using (2.1) we get

$$\begin{aligned} d(y_n, p) &= d(W(x_n, T_n z_n, \beta_n), p) \\ &\le (1-\beta_n) d(x_n, p) + \beta_n d(T_n z_n, p) \\ &\le (1-\beta_n) d(x_n, p) + \beta_n d(z_n, p) \\ &\le (1-\beta_n) d(x_n, p) + \beta_n d(x_n, p) \\ &= d(x_n, p) \end{aligned}$$

$$\Rightarrow d(y_n, p) \le d(x_n, p) \qquad (2.2)$$

(2.2)

Using (2.2), we have

$$\begin{aligned} d(x_n, p) &= d(W(x_{n-1}, T_n y_n, \alpha_n), p) \\ &\le (1-\alpha_n) d(x_{n-1}, p) + \alpha_n d(T_n y_n, p) \\ &\le (1-\alpha_n) d(x_{n-1}, p) + \alpha_n d(y_n, p) \\ &\le (1-\alpha_n) d(x_{n-1}, p) + \alpha_n d(x_n, p) \end{aligned}$$

$$\Rightarrow d(x_n, p) \le d(x_{n-1}, p) \qquad (2.3)$$

This shows that the sequence $\{d(x_n, p)\}$ is nonincreasing and bounded below, and so $\lim_{n\to\infty} d(x_n, p)$ exists for each for each $p \in F$. Hence, $\lim_{n\to\infty} d(x_n, F)$ exists. This completes the proof of part (i).

(ii). Assume that $\lim_{n\to\infty} d(x_n, p) = c$.

For $c = 0$, the case is trivial.

Now we discuss the case $c > 0$.

By (2.1) and (2.2), we have

$$d(z_n, p) \le d(x_n, p) \text{ and } d(y_n, p) \le d(x_n, p)$$

Now, taking lim sup on both sides of both inequalities, we get

$\limsup_{n\to\infty} d(z_n, p) \le c$ and

$\limsup_{n\to\infty} d(y_n, p) \le c$

By the nonexpansity of T_n, $\limsup_{n\to\infty} d(T_n z_n, p) \le c$, $\limsup_{n\to\infty} d(T_n y_n, p) \le c$

Also, $\limsup_{n\to\infty} d(x_{n-1}, p) \le c$

Now by Lemma 1.2, we have

$\lim_{n\to\infty} d(x_{n-1}, T_n y_n) = 0$ and

$$\lim_{n\to\infty} d(x_{n-1}, T_n z_n) = 0 \qquad (2.4)$$

Next,
$$\begin{aligned} d(x_n, x_{n-1}) &= d(W(x_{n-1}, T_n y_n, \alpha_n), x_{n-1}) \\ &\le \alpha_n d(x_{n-1}, T_n y_n) \end{aligned}$$

taking lim sup on both sides in the above inequality, we get $\limsup_{n\to\infty} d(x_n, x_{n-1}) \le 0$

Hence, $\limsup_{n\to\infty} d(x_n, x_{n-1}) = 0$. (2.5)

Also
$$d(x_n, x_{n+l}) \le d(x_n, x_{n+1}) + d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ldots\ldots\ldots + d(x_{n+l-1}, x_{n+l}).$$

Taking lim sup on both sides in the above inequality and using (2.5) we get $\lim_{n\to\infty} d(x_n, x_{n+l}) = 0$ for $l < N$.

On the other hand,

$$\begin{aligned} d(x_n, p) &\le (1-\alpha_n) d(x_{n-1}, p) + \alpha_n d(T_n y_n, p) \\ &\le (1-\alpha_n) d(x_{n-1}, T_n y_n) + (1-\alpha_n) d(T_n y_n, p) + \alpha_n d(y_n, p) \\ &\le (1-\alpha_n) d(x_{n-1}, T_n y_n) + (1-\alpha_n) d(y_n, p) + \alpha_n d(y_n, p) \\ &\le (1-\alpha_n) d(x_{n-1}, T_n y_n) + d(y_n, p) \end{aligned}$$

Similarly, $d(y_n, p) \le (1-\beta_n) d(x_{n-1}, T_n z_n) + d(z_n, p)$

Now, using (2.4) and applying lim inf and lim sup on both sides in the above inequalities, we get

$$c \le \liminf_{n\to\infty} d(y_n, p) \le \limsup_{n\to\infty} d(y_n, p) \le c$$

and

$$c \le \liminf_{n\to\infty} d(z_n, p) \le \limsup_{n\to\infty} d(z_n, p) \le c$$

$\Rightarrow \lim_{n\to\infty} d(y_n, p) = c$ and $\lim_{n\to\infty} d(z_n, p) = c$.

$$\begin{aligned} d(x_n, T_n x_n) &\le d(x_n, T_n y_n) + d(T_n y_n, T_n x_{n-1}) + d(T_n x_{n-1}, T_n x_n) \\ &\le (1-\alpha_n) d(x_{n-1}, T_n y_n) + d(y_n, x_{n-1}) + d(x_{n-1}, x_n) \\ &\le (1-\alpha_n) d(x_{n-1}, T_n y_n) + (1-\beta_n) d(x_{n-1}, T_n z_n) + 2d(x_{n-1}, x_n) \end{aligned}$$

$\Rightarrow d(x_n, T_n x_n) = 0$.

For each $l \in I$, we have

$$\begin{aligned} d(x_n, T_{n+l} x_n) &\le d(x_n, x_{n+l}) + d(x_{n+l}, T_{n+l} x_{n+l}) + d(T_{n+l} x_{n+l}, T_{n+l} x_n) \\ &\le 2\, d(x_n, x_{n+l}) + d(x_{n+l}, T_{n+l} x_{n+l}) \end{aligned}$$

Thus the sequence $\{d(x_n, T_l x_n)\}$ is a subsequence of $\bigcup_{i=1}^{N} \{d(x_n, T_{n+l} x_n)\}$

and $\lim_{n\to\infty} d(x_n, T_{n+l}\, x_n) = 0$ for each $l \in I$.
Therefore, $\lim_{n\to\infty} d(x_n, T_l\, x_n)=0$ for each $l\in I$.
This completes the proof of part (ii). Now, we will prove the main result.

Theorem 2.2 Let M be a nonempty closed convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η and let $\{T_i : i \in I\}$be a finite family of nonexpansive selfmaps on M such that F is non-empty. Then the sequence $\{x_n\}$ defined implicitly in (1.2), Δ-converges to a common fixed point of $\{T_i: i \in I\}$.

Proof. Since the sequence $\{x_n\}$ is bounded by Lemma 2.1(i). Also by Lemma 1.1, $\{x_n\}$ has a unique asymptotic center, that is , $A(\{x_n\}) = \{x\}$. Let $\{s_n\}$ be any sequence of $\{x_n\}$, such that $A(\{s_n\}) = \{s\}$. Then by Lemma 2.1(ii), we have $\lim_{n\to\infty} d(s_n, T_l\, s_n)=0$ for each $l \in I$. We claim that s is the fixed point of $\{T_i: i \in I\}$.
Now, we define a sequence $\{t_m\}$ in M by $t_m = T_m s$ where $T_m = T_{m(mod\, N)}$.
Now, we can write

$$d(t_m, s_n) \le d(T_m s, T_m s_n) + d(T_m s_n, T_{m-1} s_n) + \ldots$$
$$\ldots + d(T s_n, s_n)$$
$$\le d(s, s_n) + \sum_{i=1}^{m-1} d(s_n, T_i s_n).$$

Therefore by above estimation, we have

$$r(t_m, \{s_n\}) = \lim_{n\to\infty}\sup d(t_m, s_n)$$
$$\le \lim_{n\to\infty}\sup d(s, s_n) = r(s, \{s_n\}).$$
$$\Rightarrow |r(t_m, \{s_n\}) - r(s, \{s_n\})| \to 0 \text{ as } m\to\infty.$$

Now by Lemma 1.3, $T_{m(mod\ N)}\, s = s$. Thus s is the common fixed point of $\{T_i: i \in I\}$.
Next, we claim the uniqueness of the asymptotic center's' for each subsequence $\{s_n\}$ of $\{x_n\}$. Let us assume that $x \neq s$.
By Lemma 2.1(i), $\lim_{n\to\infty} d(x_n, s)$ exists and by the uniqueness of asymptotic centers,

$$\lim_{n\to\infty}\sup d(s_n, s) < \lim_{n\to\infty}\sup d(s_n, x)$$
$$\le \lim_{n\to\infty}\sup d(x_n, x)$$
$$< \lim_{n\to\infty}\sup d(x_n, s)$$
$$= \lim_{n\to\infty}\sup d(s_n, s),$$

a contradiction. Hence $x = s$. Since $\{s_n\}$ is an arbitrary sequence of $\{x_n\}$, therefore $A(\{s_n\}) = \{s\}$ for all subsequences $\{s_n\}$ of $\{x_n\}$. This proves that $\{x_n\}$ Δ-converges to a common fixed point of $\{T_i: i \in I\}$.

We now establish strong convergence of the iteration (1.2) based on lemma 2.1.

Theorem: 2.3 Let M, X, $\{T_i: i \in I\}$ and $\{x_n\}$ be as in Theorem 2.2. Suppose that $T_m \in \{T_m\}^r_{m=1}$ is semi-compact. Then the sequence $\{x_n\}$ defined in (1.2) converges strongly to $s \in F$.

Proof: Suppose that T_m is semi-compact for some positive integers $1\le m \le r$. Then by lemma 2.1(ii), we have
$\lim_{n\to\infty} d(T_n x_n, x_n) = 0$. Since $\{x_n\}$ is bounded and T_m is semi-compact, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \to s$ as $j \to\infty$. By continuity of T_m and lemma 2.1(ii), we obtain
$d(T_m\, s, s) = \lim_{j\to\infty} d(T_m x_{n_j}, x_{n_j}) = 0$ for each m = 1, 2, 3, ……,r.
This implies that s is the common fixed point of $\{T_m\}^r_{m=1}$. The remaining proof is similar to Theorem 2.1.

Remark: From the above, we can state that the result is an extended result in the general setup of uniformly convex Hyperbolic spaces. This result can be extended to strong convergence and multi-step iteration with different classes of nonlinear mappings in different fields.

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