FUZZY GOAL PROGRAMMING FOR FRACTIONAL TRANSPORTATION PROBLEM

P. ANUKOKILA

Abstract. This paper presents fuzzy goal programming approach to determine an optimal solution for the multi-objective fractional transportation problem. In the proposed approach which is motivated by Mohamed [4] goal programming model for achievement of the highest membership value of each fuzzy goals defined for the fractional objectives is formulated. The fuzzy goal programming approach is used to achieve highest degree of each of membership goals by minimizing their deviational variable. Numerical example is given to illustrate the correctness of the proposed approach.

Introduction: The transportation problem is a special type of linear programming problem, which deals with shipping commodities from sources to destinations. The basic transportation problem was originally developed by Hitchcock [2]. In the beginning it was formulated for determining the optimal shipping pattern. So it is called transportation problem. Normally, existing multi-objective transportation models use a minimization of the total cost objectives as one of their objectives.

The problem of optimizing one or several ratios of functions are called a fractional programming has attracted the attention of many researchers in the past. Fractional programming deals with situation where a ratio between physical and/or economical functions, for example cost/time, cost/volume, cost/profit, or other quantities that measure the efficiency of a system, is minimized. Radhakrishnan and Anukokila [5] established the compensatory approach for the transportation problem in fuzzy environment. Transportation problem with fractional objective function are widely used as performance measures in many real life situations.

Goal programming is an approach used for solving a multi-objective optimization problem that balances trade-offs in conflicting objectives. In other words, it is an approach of deriving a best possible 'satisfactory' level of goal attainment. Optimization of multiple goals requires a different technique for decision making known as goal programming. In goal programming, the deviation between goals and achievable within the given set of constraints are to be minimized. Thus, the objective function contains mainly the deviational variables representing each goal or sub-goal. The deviational variable is represented in two dimensions (a positive and a negative deviation from each sub-goal and/or constraint) in the objective function Zimmermann [6] proposed fuzzy programming linear programming with several objective functions.

In this paper, we focus fractional transportation problem with membership function using fuzzy goal programming (FGP) approach. The FGP approach introduced by Mohamed [4] is extended to solve these problems. In this study, we propose a method that can also comply with problems including fractional goals by expanding Kono et al's. method. Lingo software package is used to solve optimization problem.

Problem Formulations

(i) Multi-objective Transportation Problem: A special type of linear programming problem in which constraints are of equality type and all the objectives are conflicting with each other, are called multi-objective transportation problem. Assuming there are m origins, n destinations, the MOTP can be formulated as follows:

$$\begin{cases} \max\ f^k(x) = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}{}^k x_{ij}, \quad k = 1,2,\ldots,K \\ sub\ to \\ \sum_{j=1}^{n} x_{ij} = A_i,\ i = 1,2,\ldots,m \\ \sum_{i=1}^{m} x_{ij} = B_j \quad j = 1,2,\ldots,n \qquad (2.1) \\ x_{ij} \geq 0, \quad \forall\ i,j \end{cases}$$

where A_i is the amount of homogeneous products for the i^{th} origin which are transported to n destinations. Bj is the demand of homogeneous products for the j^{th} destination. c_{ij} is the coefficients of the objective functions which are associated with transportation of an unit of the product from source i to destination j. x_{ij} is the unknown quantity to be transported from origin i to destination j.

(ii) Goal Programming: To formulate FGP models of the problems, the fuzzy goal level of the objectives as well as decision vectors controlled by the higher level decision makings are determined by determining individual optimal solution. The purpose of goal programming is to minimize the deviations between the achievement of goals $f^k(x)$, and these acceptable aspiration levels, Gk (k=1, 2. . . n). A mathematical formulation of goal programming is given below:

$$\min\ f^k(x)=\sum_{i=1}^{m}\sum_{j=1}^{n}|c_{ij}^k x_{ij}-G_k|$$

$$sub\ \ to$$

$$\sum_{j=1}^{n}x_{ij}=A_i,\ i=1,2,\ldots,m$$

$$\sum_{i=1}^{m}x_{ij}=B_j\quad j=1,2,\ldots,n\qquad(2.2)$$

$$x_{ij}\geq 0,\quad\forall\ i,j$$

$$x\in F,\ (F\ \ is\ \ a\ \ feasible\ \ set)$$

where $f^k(x)$ is the linear function of the kth goal, and G_k is the aspiration level of the k goal. In order to solve goal programming, we let the function $f^k(x)=D_k^+-D_k^-+G_k$, then

Goal programming can be formulated as the following achievement function.

$$\begin{cases}\min\quad (D_k^+-D_k^-)\\ sub\ \ to\\ c_{ij}x_{ij}-G_k\ =\ D_k^+-D_k^-,\ k=1,2,..,n\\ x\in F,\ \ F\ \ is\ \ a\ \ feasible\ \ set)\\ D_k^+-D_k^-\geq 0,\ k=\ 1,2,\ldots,n\end{cases}$$

3 Kono et al's Method: Most of the models are of the type which an optimum solution can be obtained by crisp values. However since the coefficients are given by vague numerical values, the solution should depend on the degree of vagueness. Accordingly it is natural that the solution itself is calculated as vague numerical values. As one of the models based upon such an idea, the method of Kono et al. is recommendable. Kono et al. [3] introduced a fuzzy solution by dividing the following fuzzy multi-goal programming problem composed of linear goals.

Model 1:

$$\sum_{i=1}^{m}\sum_{j=1}^{n}c_{ij}x_{ij}\geq G$$

Model 1 can then be rewritten as follows by utilizing an arbitrary α _ level set.

Model 2:

$$\begin{cases}\sum_{i=1}^{m}\sum_{j=1}^{n}c_{ij}^{\alpha}x_{ij}^{\alpha}\quad\geq G\\ \sum_{i=1}^{m}\sum_{j=1}^{n}c_{ij}^{\alpha}x_{ij}^{\alpha}\ -\ G^{\alpha}\ \geq 0.\end{cases}$$

4 Fractional Fuzzy Transportation Problem: If the numerator and denominator in the objective function as well as the constraint are linear, then it is called a linear fractional programming problem (LFPP). The general format of the multi objective fuzzy fractional transportation problem can be written as,

$$\begin{cases}\max\ f^k(x)=\dfrac{\alpha_{k0}+\sum_{i=1}^{m}\sum_{j=1}^{n}c_{ij}\,|x_{ij}|}{\beta_{k0}+\sum_{i=1}^{m}\sum_{j=1}^{n}d_{ij}\,|x_{ij}|}\\ \sum_{j=1}^{n}x_{ij}=A_i,\ i=1,2,\ldots,m\\ \sum_{i=1}^{m}x_{ij}=B_j\quad j=1,2,\ldots,n\qquad\qquad(4.1)\\ x_{ij}\geq 0,\quad\forall\ i,j\end{cases}$$

where $x_{ij}^T\varepsilon R^n$ is unrestricted; α_{k0} and β_{k0} are the scalars c_{ij}^k's and d_{ij}^k's are unconstrained in sign. Without loss of generality it is assume that $\beta_{k0}+\sum_{i=1}^{m}\sum_{j=1}^{n}d_{ij}^k\,|x_{ij}|>0$.

A major difficulty of solving fractional programming is that it is a highly non-linear fractional programming problem. In order to solve the problem, the membership function μ_k for the k^{th} fuzzy goal $f^k(x)\geq g^k$ can be expressed as follows:

$$\mu_k(f^k(x))=\begin{cases}1\quad if\ \ f^k(x)\geq g_k\\ \dfrac{f^k(x)-L_k}{g_k-L_k}\ \ if\ \ L_k\leq f^k(x)\leq g_k\\ 0\quad if\ \ f^k(x)\leq L_k\end{cases}$$ where

L_k is the lower tolerance limit for the kth fuzzy goal $(g_k - L_k)$ is the tolerance (P_k) which is subjectively chosen. Again the membership function μ_k for the k^{th} fuzzy goal $f^k(x)\leq g_k$ can be expressed as follows:

$$\mu_k(f^k(x))=\begin{cases}1 \quad if \;\; f^k(x)\le g_k \\ \dfrac{U_k-f^k(x)}{U_k-g_k} \;\; if \;\; g_k\le f^k(x)\le U_k \\ 0 \quad if \;\; f^k(x)\le U_k\end{cases}$$

where U_k is the upper tolerance limit for the fuzzy goal and (U_k -g_k) is the tolerance which is subjectively chosen.

5 Existing Membership Goals: One of the major assumptions in solving fuzzy mathematical programming problem in the literature involves the use of linear membership functions for all fuzzy sets involved in a decision making process. A linear approximation is most commonly used because of its simplicity and is defined by fixing two points, the upper and lower level acceptability. The fuzzy goal programming approach of Mohamed [4] that solves single-level multi-objective transportation problem is considered, in this paper, to solve the upper level multiobjective transportation problem. The FGP model formulation of this approach can be stated as:

$$\min Z = \sum_{i=1}^{m}\sum_{j=1}^{n} W_{ij}^{+} D_k^{+}$$

$$sub \;\; to$$

$$\frac{U_k - \dfrac{\alpha_{k0} + \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij} \mid x_{ij} \mid}{\beta_{k0} + \sum_{i=1}^{m}\sum_{j=1}^{n} d_{ij} \mid x_{ij} \mid}}{U_k - g_{ij}} - D_k^{-} - D_k^{+} = 1$$

$$\sum_{j=1}^{n} x_{ij} = A_i, \;\; i = 1,2,\ldots,m$$

$$\sum_{i=1}^{m} x_{ij} = B_j \quad j = 1,2,\ldots,n \qquad (5.1)$$

$$x_{ij} \ge 0, \quad \forall \;\; i,j$$

where $D_k^{+}, D_k^{-}, k = 1,2$ represent the under and over deviations from the aspired levels, respectively. And the numerical weights $W_{ij}^{+}, W_k^{R} \;\; and \;\; W_k^{L}$ represent the relative importance of achieving the aspired levels of the respective fuzzy goals subject to the constraints set in the decision situation. To assess the relative importance of fuzzy goals properly, the weighting scheme suggested by Mohamed [4] can be used to assign the values of $W_{ij}^{+}, W_k^{R} \;\; and \;\; W_k^{L}$. In the present formulation, these values are determined as

$$\begin{cases} W_{ij}^{-} = \dfrac{1}{U_k - g_k}, \;\; i = 1,2,\ldots,,m, \;\; j = 1,2,\ldots,n \\ W_k^{L} = \dfrac{1}{t_k^{L}} \; and \; W_k^{R} = \dfrac{1}{t_k^{R}}, \;\; k = 1,2,\ldots,K \end{cases}$$

Let $t_k^{L} \;\; and \;\; t_k^{R}, \;\; k = 1,2,\ldots,n$ be the maximum acceptable negative and positive tolerance (relaxation) values on the decision vector considered by the upper level decision making. The tolerances $t_k^{L} \;\; and \;\; t_k^{R}, \;\; k = 1,2,\ldots,n$ are not necessarily same. The tolerances give the lower level decision making (DM) an extent feasible region to search for the satisfactory solution. In other words the upper level DMset negative and positive tolerances depend on the needs, desires and practical situations in the DM situation.

The linear membership functions for each of n components of decision vector controlled by the upper level decision making can be formulated as,

$$\mu_k(x_k) = \begin{cases} \dfrac{x_k - (x_k^{*} - t_k^{L})}{t_k^{L}} \quad if \quad x_k^{*} - t_k^{L} \le x_k \le x_k^{*} \\ \dfrac{(x_k^{*} - t_k^{R}) - x_k}{t_k^{R}} \quad if \quad x_k^{*} \le x_k \le x_k^{*} + t_k^{R} \\ 0 \qquad\qquad if \quad Otherwise \end{cases}$$

Now in fuzzy decision environment, the achievement of the fuzzy goals, the fuzzy goals of the DM objective functions at both levels and the vector of fuzzy goals of the decision variables controlled by upper level DM to their aspired levels to the extent possible is actually represented by the possible achievement of their respective membership values to the highest degree In fuzzy programming technique the highest degree of membership is equal to 1. As discussed by Mohamed [4] we can write the following goal constraints for the membership functions:

$$\begin{cases} \dfrac{f^k(x) - L_k}{g_k - L_k} + D_k^{-} - D_k^{+} = 1 \\ \dfrac{U_k - f^k(x)}{U_k - g_k} + D_k^{-} - D_k^{+} = 1 \end{cases}$$

In order to solve the problem, we assume that $L_k \le f^k(x) \le U_k$ where L_k and U_k are respectively, upper and lower bounded of f $f^k(x)$. Introducing FGP technique the achievement of highest membership value of goal in (5.1) can be represented as follows:

$$\mu_k(f^k(x))=\begin{cases}1 & if\ f^k(x)\geq g_k\\ \dfrac{f^k(x)-L_k}{g_k-L_k} & if\ L_k\leq f^k(x)\leq g_k\\ 0 & if\ f^k(x)\leq L_k\end{cases} \quad (5.4)$$

Equation (5.4) can be expressed as following FGP problem.

$$\begin{cases}\min\ D_k^+-D_k^-\\ subto\\ \dfrac{f^k(x)-L_k}{U_k-L_k}-D_k^--D_k^+=1\\ \sum_{j=1}^{n}x_{ij}=A_i,\ i=1,2,\dots,m\\ \sum_{i=1}^{m}x_{ij}=B_j\quad j=1,2,\dots n \qquad (5.5)\\ x_{ij}\geq 0,\ \ \forall\ i,j\end{cases}$$

where D_k^- and D_k^+ are respectively.

5.1 Linearization of Membership Goals

Equation (5.5) can be expressed as follows:

$$\begin{cases}\dfrac{f^k(x)-L_k}{g_k-L_k}+D_k^--D_k^+=1,\\ where\, H_k=\dfrac{1}{g_k-L_k}\\ (f^k(x)-L_k)+D_k^-(g_k-L_k)-D_k^+(g_k-L_k)=(g_k-L_k)\\ H_k(f^k(x)-L_k)+D_k^--D_k^+=1\\ H_k\dfrac{\alpha_{k0}+\sum_{i=1}^{m}\sum_{j=1}^{n}c_{ij}\,|\,x_{ij}\,|}{\beta_{k0}+\sum_{i=1}^{m}\sum_{j=1}^{n}d_{ij}\,|\,x_{ij}\,|}-H_kL_k+D_k^--D_k^+=1\\ H_k(\alpha_{k0}+\sum_{i=1}^{m}\sum_{j=1}^{n}c_{ij}\,|\,x_{ij}\,|)+D_k^-(\beta_{k0}+\sum_{i=1}^{m}\sum_{j=1}^{n}d_{ij}\,|\,x_{ij}\,|)\\ \qquad -D_k^+(\beta_{k0}+\sum_{i=1}^{m}\sum_{j=1}^{n}d_{ij}\,|\,x_{ij}\,|)\\ =(H_kL_k+1)(\beta_{k0}+\sum_{i=1}^{m}\sum_{j=1}^{n}d_{ij}\,|\,x_{ij}\,|),\\ where\, H_k^{'}=H_kL_k+1\\ c_k\sum_{i=1}^{m}\sum_{j=1}^{n}|\,x_{ij}\,|+D_k^-(\beta_{k0}+\sum_{i=1}^{m}\sum_{j=1}^{n}d_{ij}\,|\,x_{ij}\,|)+\\ D_k^+(\beta_{k0}+\sum_{i=1}^{m}\sum_{j=1}^{n}d_{ij}\,|\,x_{ij}\,|)=G_k\\ where\ c_k=H_kc_{ij}-H_k^{'}d_{ij},\ G_k=H_k\alpha_{k0}-H_k^{'}\beta_{k0}\\ c_k\sum_{i=1}^{m}\sum_{j=1}^{n}|\,x_{ij}\,|+D_k^--D_k^+=G_k\end{cases}$$

which is actually linearized form of the goal expression

$$H_k(\alpha_{k0}-\sum_{i=1}^{m}\sum_{j=1}^{n}c_{ij}\,|\,x_{ij}\,|+D_k^-(\beta_{k0}+\sum_{i=1}^{m}\sum_{j=1}^{n}d_{ij}\,|\,x_{ij}\,|)$$

$$-D_k^+(\beta_{k0}+\sum_{i=1}^{m}\sum_{j=1}^{n}d_{ij}\,|\,x_{ij}\,|).$$

This expression can be written as

$$H_k\frac{\alpha_{k0}+\sum_{i=1}^{m}\sum_{j=1}^{n}c_{ij}\,|\,x_{ij}\,|}{\beta_{k0}+\sum_{i=1}^{m}\sum_{j=1}^{n}d_{ij}\,|\,x_{ij}\,|}+D_k^--D_k^+=H_k^{'}$$

Where $H_k=\dfrac{1}{g_k-L_k}$ and $H_k^{'}=H_kL_k+1$.

FGP solution Approach

$$\begin{cases} \min \quad f^k(x) = \sum_{i=1}^{m}\sum_{j=1}^{n} W_{ij}^{+} D_k^{+} + \sum_{k=1}^{n} W_k^{L}(D_k^{L+} + D_k^{L-}) \\ \sum_{k=1}^{n} W_k^{R}(D_k^{R+} + D_k^{R-}) \\ sub \;\; to \\ c_k \sum_{i=1}^{m}\sum_{j=1}^{n} |x_{ij}| + D_k^{-} D_k^{+} = G_k \\ D^{-} \sum_{i=1}^{m}\sum_{j=1}^{n} d_{ij} |x_{ij}| \le \beta_{k0} \\ \frac{x_k - (x_k^{*} - t_k^{L})}{t_k^{L}} \;\; if \;\; x_k^{*} - t_k^{L} \le x_k \le x_k^{*} \\ \frac{(x_k^{*} - t_k^{R}) - x_k}{t_k^{R}} \;\; if \;\; x_k^{*} \le x_k \le x_k^{*} + t_k^{R} \\ \sum_{j=1}^{n} x_{ij} = A_i, \; i = 1,2,\dots,m \\ \sum_{i=1}^{m} x_{ij} = B_j \quad j = 1,2,\dots,n \qquad (5.6) \\ b_{ij} x_{ij} \ge 0, \quad \forall \;\; x_{ij} \ge 0 \\ D_k^{L-}, D_k^{L+} \ge 0 \;\; with \;\; D_k^{L-} \times D_k^{L+} = 0, \; k = 1,2 \\ D_k^{R-}, D_k^{R+} \ge 0 \;\; with \;\; D_k^{R-} \times D_k^{R+} = 0, \; k = 1,2 \end{cases}$$

6 Example: To illustrate the proposed method, we consider the following fractional multi objective transportation problem as:

$$\begin{cases} \max \;\; f^k(x) = \frac{4 + 3|x_{11}| - 2|x_{12}| + 2|x_{21}| + |x_{22}|}{6 + |x_{11}| - 2|x_{12}| + 10|x_{21}| + |x_{22}|} \\ \text{sub to} \\ 2x_{11} + x_{12} + x_{21} - x_{22} = 2 \\ 2x_{11} - x_{12} + 4x_{21} + 5x_{22} = 2.9 \\ x_{11}, x_{12}, x_{21}, x_{22} \ge 0. \end{cases}$$

Table 1. summarizes the coefficients α, β, c_{ij} and d_{ij} for all multiobjective transportation problem. The decided aspiration levels and upper tolerance limits to the objective functions are also mentioned. The values H_k, H_k^1, c_k, g_k and the weights W_{ij} are calculated in the following table.

Similarly, following the above discussion and the linearization process, the upper level DM decide x^{*} = 1 with the negative and positive tolerances $t^R = t^L = 0.75$ with weights $W^L = W^R = \frac{1}{0.75} = 1.33$. Thus the fractional fuzzy goal programming multi- objective transportation problem is

	f_{11}	f_{12}
α	-4	1
β	6	1
c_{ij}	(3,-2,2,1)	(-7,-2,1,3)
d_{ij}	(1,-2,10,1)	(5,2,1,2)
H_k	0.56	0.5
H_k^1	0.2785	0.5
u_k	1.3	1
g_k	-0.5	-1
c_k	(-1.94,1.67,3.89,0.83)	(1.5,0,-1,-2.5)
G_k	0.83	1
w_{ij}	0.56	0.5

$$\begin{cases} \text{Min } Z = 0.55DD_{11}^{+} + 0.5D_{12}^{+} + 1.33[(D_k^{L+} + D_k^{L-}) \\ \qquad + (D_k^{R-} + D_k^{R+})] \\ sub\ to \\ -1.95x_{11} + 1.67x_{12} - 3.89x_{21} + 0.83x_{22} \\ \qquad + D_1^{-} - D_1^{+} = -2.256 \\ -x_{11} + 2x_{12} - 10x_{21} - x_{22} + D_1^{+} \le 1 \\ 1.5x_{11} + 0 * x_{12} - x_{21} - 2.5x_{22} + D_2^{-} - D_2^{+} = 1.2 \\ -5x_{11} - 2x_{12} - x_{21} - 2x_{22} + D_2^{+} \le 6 \\ 1.33x_{11} + D_1^{L-} - D_1^{L+} = 1.33 \\ 1.33x_{11} + D_1^{R-} - D_1^{R+} = 1.33 \\ 2x_{11} + x_{12} + x_{21} - x_{22} = 2 \\ 2x_{11} - x_{12} + 4x_{21} + 5x_{22} = 2.9 \\ x_1, x_2 \ge 0, D_1^{-}, D_1^{+} \ge 0, D_2^{-}, D_2^{+} \ge 0 \\ D_1^{L-}, D_1^{L+}, D_1^{R-}, D_1^{R+} \ge 0 \\ D_1^{L-} \times D_1^{L+} = 0, D_1^{R-} \times D_1^{R+} = 0. \end{cases}$$

The solutions of transportation problem are $x_{11} = 1, x_{12} = 0, x_{21} = 0.1, x_{22} = 0.1$, with objective function value is 5.409, and with membership function value is 2.2827.

References:

1. Farhad Hossein Zadeh Lotfi et.al., *AppliedMathematicalModeling*, A linear programming approach to test efficiency in Multi-objective linear fractional programming problems, **34** 4179-4183, (2010).

2. F.L. Hitchcock, *Journal of Mathematical Physics*, The distribution of a product from several sources to numerous localities, **20** 224-230, (1941).

3. Y. Kono and T. Yamaguchi, *Communications of Operations Research Society of Japan*, An interactive method for multi goal programming problems with fuzzy solutions, **37** 603-610, (1992).
4. R.H. Mohamed, *Fuzzy Sets and Systems*, The relationship between goal programming and fuzzy programming, **89** 215-222, (1997).
5. B. Radhakrishnan and P. Anukokila, *International Journal of Mathematics in Operational Research*, A Compensatory Approach to Fuzzy Fractional Transportation Problem, (to appear).
6. H.J. Zimmermann, Fuzzy set theory and its Applications,third ed., Kluwar - Nijhoff, Boston, (1996).

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Assistant Professor, Department of Mathematics
Kumaraguru College of Technology
Coimbatore-641049,Tamilnadu
E.mail:anuparaman@gmail.com