EXISTENCE OF NONLINEAR NEUTRAL IMPULSIVE INTEGRODIFFERENTIAL EQUATIONS OF SOBOLEV TYPE WITH NONLOCAL CONDITION

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Abstract: In this paper, we prove the existence of solutions for a nonlinear impulsive integrodifferential equation of Sobolov type with nonlocal conditions. The results are obtained by using measure of non-compactness and the Monch's fixed point theorem. An application of the same problem is discussed.

Keywords: Existence, neutral differential equation, impulsive differential equation, measure of non compactness, fixed point theorem.

Introduction: The study of Differential Equations (DEs) is arguably the area of mathematics which has more applications to the real world. Differential equations are relations among quantities that change over time and/or space and therefore are relevant to the study of the evolving universe, the weather, stock market, etc. In general functional differential equations or evolution equations serve as abstract formulations of many partial differential equations which arise in problems connected with heat-flow in materials with memory, viscoelasticity and many other physical phenomena. Using the method of semigroups, various solutions of nonlinear and semilinear evolution equations have been discussed by Pazy [11] and the nonlocal problem for the same equations has been first studied by Byszewskii [6].

Integrodifferential equations form a very rich class of equations. The study of integrodifferential equations is relatively a new area in mathematics full of open problems that attracts an increasing level of interest. Differential and integrodifferential equations, especially nonlinear, present the most effective way for describing complex processes. The delays in many engineering systems such as power systems are often time-varying and sometimes vary violently with time. Time delays are frequently encountered in various engineering systems such as aircraft, long transmission lines in pneumatic models and chemical or process control systems. These delays may be the source of instability and lead to serious deterioration in the performance of closed loop systems. Neutral differential equations arise in many areas of applied mathematics and for this reason these equations have received much attention during the last few decades. A good guide to the literature for neutral functional differential equations is the book by Hale and Verduyn Lunel [9] and the references therein.

The theory of impulsive differential equation [13] is much richer than the corresponding theory of differential equations without impulsive effects. The impulsive condition $\Delta u(t_i) = u(t_i^+) - u(t_i^-) = I_i(u(t_i^-)),\ i = 1,2,\dots,m$ is a combination of traditional initial value problems and short-term perturbations whose duration is negligible in comparison with the duration of the process. Sobolev-type equation appears in a variety of physical problems such as flow of fluid through fissured rocks [4], thermodynamics, propagation of long waves of small amplitude and shear in second order fluids and so on. Brill [5] and Showalter [14] established the existence of solutions of semilinear Sobolev type evolution equations in Banach space. Controllability of Sobolev type integrodifferential systems in Banach spaces have been discussed in [2].

Measures of noncompactness are a very useful tool in many branches of mathematics. They are used in the fixed point theory, linear operator's theory, theory of differential and integral equations and others [1]. The Hausdorff measure of noncompactness X defined as infimum of numbers $r > 0$ such that X can be covered with a finite number of balls of radii smaller than r. The notion of a measure of weak compactness was introduced by De Blasi [7] and was subsequently used in numerous branches of functional analysis and the theory of differential and integral equations.

The purpose of this paper is to prove the existence of mild solutions for a nonlinear impulsive neutral delay integrodifferential equation of Sobolev type with nonlocal conditions using semigroup theory and the fixed point approach.

Preliminaries: Consider the following class of Sobolev-type neutral impulsive delay integrodifferential equation with nonlocal conditions

$$\frac{d}{dt}[Bx(t) + e(t, x(t))] + Ax(t) = f(t, x(\sigma_1(t))) + \int_0^t k(t,s)h(s, x(\sigma_2(s)))ds,\ t \in [0,a], \quad (2.1)$$

$$x(0) + g(x) = x_0$$

$$\Delta x(t_k) = I_k(x_{tk}),\ k = 1,2,\dots,m,$$

where A,B are two closed, linear operators such that $-AE^{-1}$ generates a semigroup of bounded linear operators S(t) in a Banach space X and the nonlinear operators

$f:[0,a]\times X\to X, k:[0,a]\times[0,a]\to R.$

$h:[0,a]\times X\to X, e:[0,a]\times X\to X,$

$g:PC([0,a],X)\to D(E)$

and the delay $\sigma_i(t)\le t, i=1,2$ are given appropriate functions; $x(t_k^+)$ *and* $x(t_k^-)$ represent the right and left limits of x(t) at $t = t_k$, for $0=t_0<t_1<\ldots<t_k<t_{k+1}=a$.

Denote $J_0=[0,t_1], J_k=(t_k,t_{k+1}], k=1,2,\ldots,m]$, and define the following space:

Let $PC([0,a],x)=\{x:$ x is a function from [0, a] into X such that x(t) is continuous at $t\ne t_k$ and left continuous at $t=t_k$ and the right limit $x(t_k^+)$ exists for *i=1,2,...,m*. Similarly as in ([8]), we see that PC ([0, a], X) is a Banach space with norm $\|x\|_{PC}=\sup_{t\in[0,a]}\|x(t)\|$. Let us recall the following definition.

Definition 1 A continuous solution *x(t)* of the integral equation

$$x(t)=B^{-1}s(t)B[x_0-g(x)]+B^{-1}s(t)e(0,x(0))$$
$$-B^{-1}e(t,x(t))+\int_0^t AS(t-s)B^{-1}e(s,x(s))ds$$
$$+\int_0^t S(t-s)B^{-1}[f(s,x(\sigma_1(s)))+$$
$$\int_0^s k(s,\tau)h(\tau,x(\sigma_2(t)))d\tau]ds$$
$$+\sum_{0<t_i<t}B^{-1}S(t-t_k)I_kx(t_k) \qquad (2.4)$$

is said to be a mild solution of problem (2.1)-(2.3) on [0, a], where S(t) is the semigroup generated by -$B^{-1}A$. Next, we introduce the Hausdorff's measure of noncompactness ψ(.) defined on each bounded subset *E* of Banach space Y by $\psi(B)=\inf\{\varepsilon>0, B\ has\ a\ finite\ \varepsilon-net\ in\ Y\}$.

Lemma 2.2 [1] Let Y be a real Banach space and C, E ⊆ Y is bounded, with the following properties:

(i) C is pre-compact if and only if $\psi_Y(B)=0$.

(ii) $\psi_Y(C)=\psi_Y(\overline{C})=\psi_Y(conC)$, where $\overline{C}$ and *con* C mean the closure and convex hull of C respectively.

(iii) $\psi_Y(C)<\psi Y(E)$ $where\ c\subseteq E.$

(iv) $\psi_Y(C+E)=\psi_Y(C)+\psi_Y(E)$ $where\ C+E=\{x+y;\ x\in C, y\in E\}.$

(v) $\psi_Y(C\cup E)\le\max\{\psi_Y(C),\psi_Y(E)\}$

(vi) $\psi_Y(\lambda C)\le|\lambda|\psi_Y(C),\ for\ any\ \lambda\in R$.

(vii) If the map *F: D (F) ⊆Y →Z* is Lipschitz continuous with constant r, then $\psi_Z(FB)\le r\psi_Y(B)$ ≤ r Y (B), for any bounded subset B ⊆D(F), where Z be a Banach space.

Before we prove the existence results, we need the following Lemmas.

Lemma 2.3 If W ⊆ C([0, a],X) is bounded, then *ψ(W(t)) ≤ ψc(W)* for all t ∈ [0, a], where *W(t) ={u(t); u ∈ W}* ⊆ X. Furthermore if W is equicontinuous on [a, b], then ψ (W (t)) is continuous on [a, b] and ψc(W) = sup{ψ(W(t)), t ∈ [a, b]}.

Lemma 2.4 If $\{u_n\}_1^\infty\subset L^1([a,b],X)$ is uniformly integrable, then the function $\psi(\{u_n(t)\}_1^\infty)$ is measurable and $\psi\{(\int_0^t u_n(s)ds)_{n=1}^\infty\}\le 2\int_0^t\psi(u_n(s)\}_{n=1}^\infty)ds.$ (2.5) The following fixed point theorem, a nonlinear alternative of Monch type, plays a key role in our existence of mild solutions for nonlocal Cauchy problem (2.1) – (2.3).

Theorem 1 Let Y be a Banach space, U an open subset of Y and 0 ∈ U. Suppose that F : $\overline{U}$ →Y is a continuous map which satisfies Monch's condition (that is, if D ⊆ $\overline{U}$ is countable and D ⊆ $\overline{co}$(0 ∪ F(D)), then $\overline{D}$ is compact) and assume that x ≠ λF(x), for x ∈ ∂U and λ∈(0,1) (2.6) holds. Then F has a fixed point in $\overline{U}$.

3 Main Results: In order to prove our main theorem, we assume the following hypothesis:

($\mathbf{M_1}$) A and B are closed linear operators,

($\mathbf{M_2}$) D (B) ⊂ D (A) and B is bijective,

($\mathbf{M_3}$) B^{-1} : Y → D (B) is continuous.

Hypotheses ($\mathbf{M_1}$) – ($\mathbf{M_3}$) and the closed graph theorem imply the boundedness of the linear operator AB^{-1}*: X → X* and $-AB^{-1}$ generates a uniformly continuous semigroup S(t), t ≥ 0, of bounded linear operators on Banach space X.

($\mathbf{M_4}$) (**i**) The nonlinear function *e: [0, a] × X → X,* for a.e *t∈[0, a]*, the function *e (•, x)* is continuous and for all x∈X, the function e(•, x) : [0, a] → X is measurable, for all x∈X.

(**ii**) There exists a function, $m_0, m_1, m_2\in L^1([0,a],R^+)$ and a nondecreasing continuous function $\Omega_e, \Omega_{Ae}: R^+\to R^+$ such that for every x∈X, we have

$\| e(t,x)\| \leq m_1 \Omega_e \|x\| \quad a.e. \ t \in [0,a]$

$\| Ae(t,x)\| \leq m_2 \Omega_{Ae} \|x\| \ a.e. t \in [0,a]$

$\| e(0,x)\| \leq m_0 \quad a.e. \ t \in [0,a].$

(M_5) **(i)** The nonlinear function *f*: [o; a]× X→X, for a.e t∈ [o, a]; the function *f* (.; x) is continuous and for all x∈X, the function f(.; x):[o, a]→X is measurable for all x∈X.

(ii) There exists a function, $m_3 \in (L^1[0,a], R^+)$ and a nondecreasing continuous function *f*: $R^+ \to R^+$ such that for every x∈X; we have

$\| f(t,x) \| \leq m_3(t) \Omega_f \| x \|, \ a,e, \ t \in [0,a]$.

(iii) There exists a function, $\gamma_f \in (L^1[0,a], R^+)$ such that, for every bounded D ⊂ X, we have

$\psi(f(t,D)) \leq \gamma_f(t)\psi(D), a,e,t \in [0,a].$

(M6) **(i)** The nonlinear function h: [o, a] ×X → X, for a.e. t∈[o, a], the function h(• , x) is continuous and for all x ∈ X, the function f(• , x) : [o, a] → X is strongly measurable for all x∈X.

(ii) There exists a function, $m_4 \in L^1([0,b], R^+)$ and a nondecreasing continuous function $\Omega_h : R^+ \to R^+$ such that for every x∈X, we have

$\| h(t,x) \| \leq m_4(t) \Omega_e \| x \|, a,e,t \in [0,a]$.

(iii) There exists a function, $\gamma_h \in L^1([0,a], R^{+)}$ such that for every bounded D ⊂ X, we have

$\psi(h(t,D)) \leq \gamma_h(t)\psi(D), a,e,t \in [0,a].$

(M7) The function k: [o, a] × [o, a] → R is measurable function such that there exist a constant *k* such that

$\| k(t,s) \| \leq \gamma_k \quad for \ \ s,t \in I.$

(M8) **(i)** I_k : X → X is continuous. There exists a nondecreasing continuous function $\Omega_I : R^+ \to R^+$ such that for every x ∈ X, we have

$\| I_k(x(t_k)) \| \leq \Omega_I \| x \|, \ where \ k = 1,2,3,\ldots,m$

(ii) There exists a function, $\gamma_I \in L^1([0,A], R^+)$ such that, for every bounded D ⊂ X, we have

$\psi(I_k(D)) \leq \gamma_I(t)\psi(D), \ \ k = 1,2,\ldots,m.$

(M_9) The function g: PC ([o, a], X) → D(B) is continuous compact map such that $\|g(x)\| \leq c\|x\|+d$, for all x∈ PC([o, a],X), for some positive constants c and d.

From **(M1)** – **(M3)** and Banach inverse operator theorem, we know that B is a bounded operator and denote $\alpha = \| B^{-1} \|, \beta = \| B \|$. Now, we give the existence results for (2.1) – (2.3).

Theorem 2. Assume that the conditions **(M_1)** – **(M_9)** are satisfied. Then, for every x_0∈D(B) the impulsive nonlocal problem (2.1)–(2.3) has at least one mild solution [o, a] provided that there exists a constant N > o with $\dfrac{(1-\alpha\beta M_c)N}{L_0} > 1$

where

$$L_0 = \alpha\beta M(d + \| x_0 \|) + \alpha M_1 \Omega_e(N) + \alpha M$$
$$[m_0 + m_2 \Omega Ae(N) + M_3 \Omega f(N) + K m_4 \Omega h(N)$$
$$+ \Omega I(N) \ \ and \ that$$

$$2\alpha \| re \| + M\{\| r_0 \| + \| rAe \| + \| rf \| + K \| rh \| + \| rI \|\} < 1 \tag{3.1}$$

$where \ \ M = \sup_{0 \leq t \leq a} \| S(t) \|$.

Proof: We consider the operator T:PC([o,a],X) defined (T x)(t)=T($(T_1 x)(t) + (T_2 x))(t)$ with

$$(T_2 x)(t) = B^{-1} S(t) B[x_0 - g(x)]$$
$$+ \int_0^t As(t-s) B^{-1} e(s,x(s)) ds + \int_0^t S(t-s) B^{-1}$$
$$[f(s,x(\sigma_1(s))] + \int_0^s k(s,\tau) h(\tau, x(\sigma_2(\tau)) d\tau] ds$$
$$+ \sum_{0<t_i<t} B^{-1}(s(t-t_k) I_k x(t_k), \ \ \forall t \in [0,1]. \tag{3.2}$$

It is easy to see that the fixed point of T is the mild solutions of impulsive nonlocal problem (2.1) – (2.3). Subsequently, we will prove that T has a fixed point by using Theorem 2.2.

First, we claim that the operator T is continuous on PC ([o, a], X). For this purpose, we assume that $x_n \to x$ in PC ([o, a], X). Then by (**M_4** – ii) we get that

$e(t,x_n(t)) \to e(t,x(t)) \quad a.e. \ \ t \in [0,a]$

$Ae(s,x_n(\sigma_1(s))) \to Ae(s,x(s)) \ \ a.e. \ \ \ s \in [0,1]$ By the same reason (**M_5 – ii**) and (**M_6–ii**) we get

$f(s,x_n(\sigma_1(s))) \to f(s,x(\sigma_1(s))) \ \ a.e. \ \ s \in [0,1]$

$h(T,x_n(\sigma_2(T))) \to h(T,x(\sigma_2(T))) \ a.e. T \in [0,1]$

Since (**M_6 – ii**), (**M_7**) hold, by the dominated convergence theorem, for every s ∈ [o, a] we have

$$\int_0^s k(s,T) h(T, x_n(\sigma_2(T))) dT \to$$
$$\int_0^s k(s,T) h(t, x(\sigma_2(T)) dT, \ \ n \to +\infty, \ \ thus$$

$$\| Rx_n - Rx \| \to 0 \ \ as \ \ n \to \infty. \tag{3.3}$$

That is *R* is continuous.

Next, we claim that the Monch's condition holds. Suppose that D ⊆ Br is countable and

$D \subseteq \overline{co}(0 \cup R(D))$, we show that $\psi(D) = 0$, where B_r is the open ball of the radius r centered at the zero in *C ([o, a]; X)*. Without loss of generality, we may suppose that $D = \{x_n\}_{n=1}^{\infty}$. By using the condition (**M₁**) – (**M₉**), we can easily verify that $\{Tx_n\}_{n=1}^{=\infty}$ is equicontinuous. So, $D \subseteq \overline{co}(0 \cup T(D))$ is also equicontinuous.

Now, from the Lemma 2.2–2.4 and the continuity of $B^{-1}S(t)B$, it follows that

$$\psi\{Tx_x\}_{n=1}^{=\infty} \le 2\alpha\,[\| re \| + M \| r_0 \| + \| rAe \| + \| rf \| + K \| rh \| + \| rI \|\}]\psi\{x_n\}_{n=1}^{=\infty}.$$

Thus we get that

$$\psi(D) \le \psi(\overline{co}(0 \cup R(D))) = \psi(T(D)) \le 2\alpha\,[\| re \| + M\{\| r_0 \| + \| rAe \| + \| rf \| + K \| rh \| + \| rI \|\}]\psi(D)$$

which implies that (D) = o, since the condition (3.2) holds. Now let λ ∈ (o, 1) and x = $\lambda T(x)$. Then, for t ∈ [o, a]

$$x(t) = \lambda B^{-1}S(t)BE[x_0 - g(x)] + \lambda B^{-1}s(t)e(0.x(0)) - \lambda B^{-1}(e(t,x(t)) + \lambda\int_0^t AS(t-s)B^{-1}e(s,x(s))ds + \lambda\int_0^t S(t-s)B^{-1} [f(s,x(\sigma_1(s))] + \int_0^s k(s,\tau)h(\tau,x(\sigma_2(\tau))d\tau]ds + \lambda\sum_{0<t_i<t} B^{-1}(s(t-t_k)I_k x(t_k)),\ \forall t \in [0,1]$$

and one has

$$\| x(t) \| \le \alpha\beta\, M(\| x_0 \| + c \| x \| + d) + \alpha \| m_1 \| L^1\Omega_e \| x \| + \alpha M[\| x_0 \| + \|m_2\| L^1\Omega A_e \| x \| + \| m_3 | | L^1\Omega_f \| x \| + K \| \phi_4 \| L^1\Omega_h \| x \| + \Omega_I \| x \|]$$

Consequently, $\dfrac{(1-\alpha\beta M_c)\| x \|}{L_1}$

$$| L1 = \alpha\,\beta\, M(d + \| x_0 \|) + \alpha m_1\Omega_e \| x \| \, | + \alpha M[m_0 + m_2 \| \Omega A_e \| x \| + m_3\Omega_f \| x \| + Km_4\Omega_h \| x \| + \Omega_I \| x \|]$$

Then by (3.1) there exists N such that $\| x \| \neq N$. Set $U = \{x \in PC[0,a], X : \| x \| < N$. From the choice of U there is no $x \in \partial U$ such that $x = \lambda T(x)$ for some $\lambda \in$ (o, 1). Thus we get a fixed point of T in U due to Theorem 2, which is a mild solution to (2.1) – (2.3). The proof is completed.

Now, we will give the existence for (2.1) – (2.3) when the nonlocal item g has no compactness. Assume the following holds:

(**M₁₀**) The function g: PC ([o, a], X) → D(B) is Lipschitz continuous with constant L.

Theorem 3. Assume that the conditions (**M₁**)–(**M₈**) and (**M₁₀**) are satisfied. Then for every x_0 ∈D(B) the impulsive nonlocal problem (2.1)–(2.3) has at least one mild solution [o, a] provided that there exists a constant N > o with $\dfrac{(1-\alpha\beta ML)N}{L2} >$ 1where

$$L_2 = \alpha\beta\, M(\| g(0) \| + \| x_0 \|) + \alpha m_1\Omega_e(N) + \alpha M[m_0 + m_2\Omega Ae(N) + m_3\Omega_f(N) + km_4\Omega h(N) + \Omega_I(N)]$$

and that

$$\alpha\beta ML + 2\alpha[\| re \| + M \| r_0 \| + \| rAe \| + \| rf \| + K \| rh \| + \| rI \|\}] < 1 \qquad (3.4)$$

where M equals to $\sup_{0 \le t \le a} \| s(t) \|$.

Proof. On account of Theorem 3, we can prove that operator R defined by (3.3) is continuous on PC ([o, a],X). We prove that T satisfies the Monch's condition holds.

For this purpose, Let $D \subseteq U_r$ is countable and $D \subseteq \overline{co}(0 \cup R(D))$ we show that $\psi(D) = 0$. Without loss of generality, we may suppose that D= $\{x_n\}_{n=1}^{\infty}$. By using the condition (**M₁**) – (**M₃**), we can easily verify that $\{T_2x_n\}_{n=1}^{\infty}$ is equicontinuous. Moreover, T_1: D → PC ([o, a], X) is Lipschitz continuous with constant_ML due to the condition (**M₁₀**). In fact, for x, y ∈ D, we have

$$\| R_1x - R_1y \| = \sup_{t \in [0,a]} \| B^{-1}S(t)Bg(x) - B^{-1}S(t)g(y) \| \le \alpha\,\beta\, ML \| x - y \|.$$

So, from (**M₄ – ii**), (**M₅ – ii**), (**M₆ – ii**), (**M₁₀**) and Lemma 2.2 – 2.4 it follows that

$$\psi(\{Rx_n\}_{n=1}^{\infty}) \le \psi(R_1x_n\}_{n=1}^{+\infty} + \psi(R_2x_n\}_{n=1}^{+\infty}$$

$$\psi(\{Rx_n\}_{n=1}^{\infty}) \le \alpha\beta ML + 2\alpha[\| re \| + M\{\| r_0 \| + \| rAe \| + \| rf \| + K \| rh \| + \| rI \|]\}\psi\{x_n\}_{n=1}^{+\infty}$$

Thus, we get that

$$\psi(D) \leq \psi(\overline{co}(0 \cup RD))$$
$$= \psi(RD) \leq \alpha\{\beta ML + 2[\| re \|$$
$$+ M\{\| r_0 \| + \| rAe \| + \| rf \| + K \| rh \|$$
$$+ \| rI \|]\}\psi(D)$$

which implies that (D) = 0, since the condition (3.2) holds. Now, with analogous arguments as in the proof of theorem 3, we can get an open ball U by the condition of (3.6), and there is no x $\in \partial U$ such that x = $\lambda T(x)$ for some $\lambda \in$ (0, 1). Thus we get a fixed point of T in U due to Theorem 2, which is a mild solution to (2.1) – (2.3). The proof is completed.

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