SOLUTION OF THE NAGUMO'S EQUATION BY REDUCED DIFFERENTIAL TRANSFORM METHOD (RDTM)

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Abstract: Using the Reduced Differential Transform Method (RDTM), it is possible to find the exact solutions or better approximate solutions of wide classes of problems in mathematical physics. This technique doesn't require any discretization, linearization or small perturbations and therefore it reduces significantly the numerical computation. The numerical solutions obtained by the RDTM are compared with the known exact solutions by fixing the arbitrary constants. In this paper, this method is used for solving the Nagumo's equation, with given initial condition containing arbitrary constants. The obtained result shows that the solution obtained by the RDTM is in good agreement with the known exact solution.

Keywords: Reduced differential transform method (RDTM), the Nagumo's equation, Analytical solution.

Introduction: Partial differential equations (PDEs) have numerous essential applications in various fields of science and engineering such as fluid mechanics, thermodynamics, heat transfer and applied physics [1]. One of the most attractive and surprising wave phenomena is the creation of solitary waves or solitons. Most of these equations are nonlinear partial differential equations. It is difficult to handle nonlinear part of these equations. Although, most of scientists applied numerical methods to obtain the solution of these equations; solving such equations analytically is of fundamental importance since the existing numerical methods which approximate the solution of partial differential equations do not result in such an exact and analytical solution which is obtained by analytical methods. Hirota's bilinear method [2], the balance method [3], inverse scattering transform method [4], sine–cosine method [5], the homotopy analysis method [6, 7], the homotopy perturbation method (HPM) [8, 9], the differential transform method (DTM) [10], an adaptive method of lines [11], the variational iteration method (VIM) [12] and the Adomian's decomposition method (ADM) [13] are some examples of analytical methods. HPM, VIM and ADM methods can be used to solve the nonlinear partial differential equations with accurate approximations, but these approximations are acceptable only for a small range, because boundary conditions in one dimension are satisfied via these methods, consequently, this shows that most of the analytical techniques encounter the in-built deficiencies and involve huge computational work. Recently, Keskin and Oturanç [14, 15] introduced a reduced form of Differential Transform Method (DTM) as reduced DTM (RDTM), and applied it to obtain the solutions of non-linear PDEs and fractional PDEs. In this paper, we have applied the reduced differential transform method (RDTM) [14, 15] to the Nagumo's equation [16] with given initial condition containing arbitrary constants. The main advantage of this method is the fact that it provides its user with an analytical approximation, in many cases an exact solution, in a rapidly convergent sequence with elegantly computed terms. The structure of this paper is organized as follows: In section 2, we begin with some basic definitions and explain the reduced differential transformation method. In section 3, we apply this method to solve the Nagumo's nonlinear partial differential equation with given initial condition.

2. The RDT Method: The basic definitions in the Reduced Differential transform method [14, 15] are as follows:

Definition 2.1: If function $u(x, t)$ is analytic and k-times continuously differentiable with respect to time t and space x in the domain of interest, then let

$$U_k(x) = \left(\frac{1}{k!}\right)\left(\frac{\partial^k u(x,t)}{\partial t^k}\right)_{t=0}, \qquad (1)$$

where the function $U_k(x)$ is the transformed function of the function $u(x,t)$. The differential inverse transform of $U_k(x)$ is defined as

$$u(x,t) = \sum_{k=0}^{\infty} U_k(x) t^k. \qquad (2)$$

Then combining (1) and (2), we write

$$u(x,t) = \sum_{k=0}^{\infty}\left(\frac{1}{k!}\right)\left(\frac{\partial^k u(x,t)}{\partial t^k}\right)_{t=0} t^k. \qquad (3)$$

From the above definitions, it can be found that the concept of the Reduced Differential transform method is derived from the Taylor's series expansion. We write the gas dynamics equation in standard form

$$L(u) + R(u) + N(u) = 0, \qquad (4)$$

with initial condition: $u(x,0) = f(x)$, (5) where $L(u) = u_t(x,t)$, $R \equiv \frac{\partial^3}{\partial x^2 \partial t}$ is a linear operator which has mixed partial derivatives and $N(u)$ is a nonlinear term. According to the RDTM, the iteration formula is

$$(k+1)U_{k+1}(x) = -N(U_k(x)) - \frac{\partial^3}{\partial x^2 \partial t}\overline{u_k}(x,t),$$
$$(k = 0,1,2,...) \qquad (6)$$

where $\frac{\partial^3}{\partial x^2 \partial t}\overline{u_k}(x,t)$ and $N(U_k(x))$ are the reduced differential transformations of the functions $R(u(x,t))$ and $N(u(x,t))$, respectively. Definition 2.1 implies that the initial approximation $U_0(x)$ is given by the initial condition, that is

$$U_0(x) = u(x,0). \qquad (7)$$

Substituting $U_0(x)$ in the iteration formula (6), we obtain the values of $U_k(x)$, then the differential inverse transformation of the set of values $[U_k(x)]_{k=0}^{n}$ gives approximation solution as

$$\overline{u_k}(x,t) = \sum_{k=0}^{\infty}\left(\frac{1}{k!}\right)\left(\frac{\partial^k u(x,t)}{\partial t^k}\right)_{t=0} t^k. \qquad (8)$$

Therefore, the differential inverse transform of $U_k(x)$ is given by

$$u(x,t) = \lim_{k\to\infty}\overline{u_k}(x,t). \qquad (9)$$

Table 1: Reduced Differential Transformation

Function	**Reduced Differential Transform**
$u(x,t)$	$U_k(x) = \left(\frac{1}{k!}\right)\left(\frac{\partial^k u(x,t)}{\partial t^k}\right)_{t=0}$
$w(x,t) = u(x,t) \pm v(x,t)$	$W_k(x) = U_k(x,t) \pm V_k(x,t)$
$w(x,t) = \alpha\, u(x,t)$	$W_k(x) = \alpha U_k(x)$ (α is constant)
$w(x,t) = \frac{\partial}{\partial x}u(x,t)$	$W(x) = \frac{\partial}{\partial x}(U_k(x))$
$w(x,t) = u(x,t)v(x,t)$	$W(x) = \sum_{r=0}^{k} V_r(x)U_{k-r}(x) = \sum_{r=0}^{k} U_r(x)V_{k-r}(x)$
$w(x,t) = \frac{\partial^r}{\partial t^r}u(x,t)$	$W_k(x) = (k+1)...(k+r)U_{(k+r)}(x) = \frac{(k+r)!}{k!}U_{(k+r)}(x)$
$w(x,t) = x^m t^n\, u(x,t)$	$W(x) = x^m U_{k-n}(x)$

3. Applications:

3.1. The Nagumo's equation [16]:

The Nagumo's equation is

$$u_t = u_{xx} + u - u^3, \qquad (10)$$

where $u_t = \frac{\partial u}{\partial t}$ subject to the following initial condition:

$$u(x,0) = \left[\frac{1}{2} + \frac{1}{2}\tanh\left[\left(-\frac{x}{2\sqrt{2}}\right) + \frac{b}{2}\right]\right], \qquad (11)$$

where b is arbitrary constant and here $u = u(x,t)$ is the solution of equation (10) with initial condition (11). The true solution for equation (10) obtained by Ablowitz and Zeppetella in 1979 [17] is given by

$$u(x,t) = \left[\frac{1}{2} + \frac{1}{2}\tanh\left[-\frac{1}{2\sqrt{2}}\left(x - \frac{3t}{\sqrt{2}}\right) + \frac{b}{2}\right]\right].$$
(12)

Let us now solve the equation (10) by the RDT Method. Taking the reduced differential transformation of both sides of equation (10), we obtain the iterative scheme as follows:

$$(k+1)U_{k+1}(x) = \frac{\partial^2}{\partial x^2}(U_k(x)) + U_k(x) -$$
$$N_k(U_k(x)),\; k = 0,1,2..., \qquad (13)$$

where $N_k(U_k(x))$ is the reduced differential transformation of u^3. Using the initial conditions

(11), we obtain

$$U_0(x) = u(x,0) = \left[\frac{1}{2} + \frac{1}{2}\tanh\left[\left(-\frac{x}{2\sqrt{2}}\right) + \frac{b}{2}\right]\right]. \quad (14)$$

Now, substituting $k = 0,1,2\ldots,$ in equations (13) and using (14), we obtain the following values successively

$$U_1(x) = \frac{1}{2} - \left(\frac{1}{2} + \frac{1}{2}\tanh\left[\frac{b}{2} - \frac{x}{2\sqrt{2}}\right]\right)^3 + \frac{1}{2}\tanh\left[\frac{b}{2} - \frac{x}{2\sqrt{2}}\right] + \frac{1}{8}\sec h\left[\frac{b}{2} - \frac{x}{2\sqrt{2}}\right]^2 \tanh\left[\frac{b}{2} - \frac{x}{2\sqrt{2}}\right].$$

Also, we have obtained $U_2(x)$ and $U_3(x)$ but these are not shown here since these expression are very large. In the similar way, other components may be computed.

Now, the differential inverse transformation of the set of values $\left[U_k(x)\right]_{k=0}^{3}$ gives the solution as

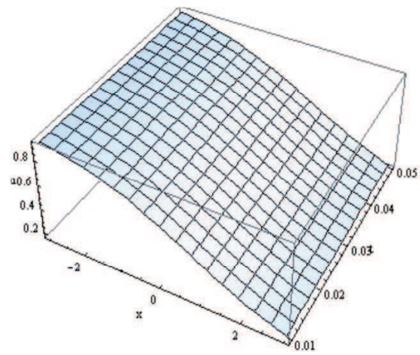
$$\overline{u_3}(x,t) = \sum_{k=0}^{3} U_k(x)t^k,$$ or

$$\overline{u_3}(x,t) = U_0(x) + U_1(x)t + U_2(x)t^2 + U_3(x)t^3.$$

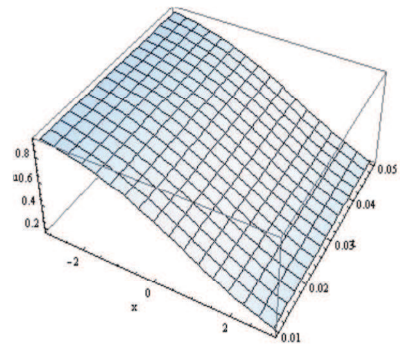
The comparisons of the present approximation solution with the exact solution (12) of the Nagumo's equation are made in the following table:

Table 2. Comparison of the RDTM approximate solution $\overline{u_3}(x,t)$ with the exact solution (12) of the Nagumo's equation for $b = 0.5$.

t	x	RDTM	Exact [16]	Absolute error
0.01	-1	0.77243409	0.77243409	0.00000000
	0	0.62597786	0.62597786	0.00000000
	1	0.45212046	0.45212046	0.00000000
0.03	-1	0.77766432	0.77766434	0.00000002
	0	0.63297476	0.63297478	0.00000002
	1	0.45956183	0.45956182	0.00000001
0.05	-1	0.78280804	0.78280818	0.00000014
	0	0.63991596	0.63991609	0.00000013
	1	0.46702132	0.46702125	0.00000007



$\overline{u_3}(x,t)$



$u(x,t)$

Figure 1. The numerical results for $\overline{u_3}(x,t)$ (given in (**a**)) in comparison with the exact analytical solutions $u(x,t)$ (given in (**b**)) for $b = 0.5$.

Conclusion: The main aim of this article is to construct analytical solution of the Nagumo's equation. We have achieved this goal by applying the reduced differential transform method.Its rapid convergence shows that the method is reliable and introduces a significant improvement over the existing numerical methods in solving the Nagumo's equation with given initial condition. As the method is usually tedious to use by hand, we have used the software package "MATHEMATICA" to calculate few terms of the series obtained from the RDTM. The numerical results are compared with the exact solution in Table 2. The graph of the approximate solution is also shown in Figure 1.

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