A COMPARATIVE STUDY ON VARIATIONAL ITERATION METHOD AND MODIFIED VARIATIONAL ITERATION METHOD

RAJAN ARORA, ANKITA SHARMA

Abstract: The purpose of this paper is to study a comparison between Variational Iteration Method (VIM) and Modified Variational Iteration Method (MVIM). This comparison of VIM with its modified form shows that the numerical solutions obtained by MVIM converge to exact solution more rapidly than that obtained by VIM. This study also exhibits the capability of MVIM of reducing the size of calculations, maintaining the high accuracy of the numerical solutions at the same time. In this paper, we aim to show the comparison by solving a three-dimensional nonlinear partial differential equation. The obtained results, when compared with the known exact solution, are found to be in good agreement with it.

Keywords: Modified Variational Iteration Method (MVIM), Nonlinear Partial Differential Equation, Variational Iteration Method (VIM).

Introduction: Ordinary differential equations, partial differential equations, integral equations, functional equations and other equations are encountered in various fields of mathematics, physics, mechanics, chemistry, biology, economics, and numerous applications. The main challenging problem is to deal with the nonlinear partial differential equations. These equations are generally difficult to solve and their exact solutions are difficult to obtain, therefore, various approximation methods have been developed such as the homotopy perturbation method, the homotopy analysis method [1,2], the reduced differential transform method [3,4], the variational iteration method [5], etc., to solve linear and nonlinear differential equations.

The variational iteration method was first proposed by Ji-Huan He [6] to find the solution of differential equations [7]. The idea of the VIM is to construct an iteration method based on a correction functional that includes a generalized Lagrange multiplier. The value of the multiplier is chosen using the variational theory so that each iteration improves the accuracy of the solution.

The initial approximation (trial function) usually includes unknown coefficients which can be determined by using the initial or/and boundary conditions. Many researchers, in a variety of scientific fields, have applied this method and showed that the VIM has many merits and is reliable for a variety of scientific applications, linear and nonlinear ODEs and PDEs [8,9].

The solution procedure of the VIM shows some disadvantages, namely, repeated computation of redundant terms, which wastes time and effort. Abassy *et al.* [10], proposed the modified variational iteration method and used it to give an approximate power series solutions for some well-known nonlinear problems.

The modified variational iteration method (MVIM) facilitates the computational work and minimizes it. This method can effectively improve the speed of convergence. Abassy *et al.* [10] also proposed further treatments on MVIM results by using Padé approximants and Laplace transform. The treatment improves the convergence and gives the closed form solution in some cases, for more details see [11,12].

In this work, we aim to demonstrate the power of MVIM over VIM by solving the three-dimensional nonlinear non-homogeneous initial value problem by both methods, and then, comparing the obtained results with the known exact solution by showing the absolute error. We find that the solutions which are obtained by the MVIM tend to converge the exact solution more rapidly than that by the VIM.

This paper is arranged in the following manner: In Section 2 and 3, we present the analysis of VIM and MVIM, respectively. In Section 4, the application of VIM and MVIM on the 3-dimensional nonlinear partial differential equation is presented. In the end, the conclusion is presented in Section 5.

To illustrate the basic concepts of VIM and MVIM, we consider the following general non-linear initial value problem:

$$
\begin{aligned}
&Av(x,t)+Bv(x,t)+Cv(x,t)=f(x,t),\\
&v(x,0)=g_0(x),\\
&\frac{\partial v(x,t)}{\partial t}\Big|_{t=0}=g_1(x),\\
&\vdots\\
&\frac{\partial^{s-1} v(x,t)}{\partial t^{s-1}}\Big|_{t=0}=g_{s-1}(x), \qquad \ldots(1)
\end{aligned}
$$

where $A=\frac{\partial^s}{\partial t^s}$, $s=1,2,3,\ldots$ is the highest partial derivative with respect to t, B is a linear operator and $Cv(x,t)$ is the nonlinear term. $Bv(x,t)$ and $Cv(x,t)$ are free of partial derivatives with respect to t and $f(x,t)$ is the non-homogeneous term.

Variational Iteration Method: Using variational iteration method [6] to solve the nonlinear partial differential equation (1), the following variational iteration formula can be obtained:

$$V_{n+1}(x,t)=V_n(x,t)+\int_0^t \lambda(\zeta)\{AV_n+BV_n+CV_n-f(x,\zeta)\}d\zeta, \quad \ldots(2)$$

where λ is called as general Lagrange multiplier which can be identified optimally via variational theory, BV_n and CV_n are considered as restricted variations, i.e. $\delta(BV_n)=0,\ \delta(CV_n)=0$.

Calculating variation with respect to V_n, we obtain

$$\delta V_{n+1}(x,t)=\delta V_n(x,t)+\delta\int_0^t \lambda(\zeta)\{AV_n+BV_n+CV_n\}d\zeta, \quad \ldots(3)$$

which yields

$$\delta V_{n+1}(x,t)=\delta V_n(x,t)+\delta\int_0^t \lambda(\zeta)AV_n(x,\zeta)d\zeta. \quad \ldots(4)$$

The Lagrange multiplier, therefore, can be identified as:

$$\lambda(\zeta)=\frac{-(t-\zeta)^{(s-1)}}{(s-1)!}. \quad \ldots(5)$$

Substituting the identified multiplier into (2) results in the following iteration formula:

$$V_{n+1}(x,t)=V_n(x,t)-\int_0^t \frac{(t-\zeta)^{(s-1)}}{(s-1)!}\{AV_n(x,\zeta)+B(V_n)+C(V_n)-f(x,\zeta)\}d\zeta. \quad \ldots(6)$$

The second term on the right side of eq. (6) is called the correction term. The equation (6) can be solved iteratively using

$$V_0(x,t)=g_0(x)+g_1(x)t+\cdots+\frac{g_{s-1}(x)}{(s-1)!}t^{s-1}$$

as an initial approximation.

3. Modified Variational Iteration Method: In this method, the following iteration formula is used for solving (1).

$$V_{n+1}(x,t)=V_n(x,t)+\int_0^t \lambda(\zeta)\{B(V_n-V_{n-1})+(F_n-F_{n-1})-\left(f_{ns}(x)\zeta^{ns}+f_{ns+1}(x)\zeta^{ns+1}+\cdots+f_{s(n+1)-1}(x)\zeta^{s(n+1)-1}\right)\}d\zeta, \quad \ldots(7)$$

where λ is given by (5), $F_n(x,t)$ is a polynomial of degree $(s(n+1)-1)$ and is obtained from $CV_n(x,t)=F_n(x,t)+O\left(t^{s(n+1)}\right)$ and $f_n(x)$ is obtained by Taylor's series expansion of $f(x,t)$ where $f(x,t)=\sum_{n=0}^{\infty} f_n(x)t^n$.

The iteration formula (7) can be solved iteratively using

$$V_{-1}=0,$$

$$V_0=g_0(x)+g_1(x)t+\cdots+\frac{g_{s-1}(x)}{(s-1)!}t^{(s-1)},$$

to obtain an approximate power series solution for (1).

4. Application: We consider the following three-dimensional nonlinear initial value problem

$$u_{tt}=\left(2-t^2\right)+u-\left(e^{-x}u_{xx}^2+e^{-y}u_{yy}^2+e^{-z}u_{zz}^2\right), \quad \ldots(8)$$

with the initial conditions

$u(x,y,z,0)=e^x+e^y+e^z$, and $u_t(x,y,z,0)=0$.

Using MVIM for solving (8) leads to the following iteration formula:

$$U_{n+1}=U_n-\int_0^t (t-\zeta)\{B(U_n-U_{n-1})+(F_n-F_{n-1})-\left(f_{2n}\zeta^{2n}+f_{2n+1}\zeta^{2n+1}\right)\}d\zeta, \quad \ldots(9)$$

where $U_{-1}=0$, $U_0=e^x+e^y+e^z$, $F_n(x,y,z,t)$ is a polynomial of degree $(2n+1)$, which is obtained from

$$\left(e^{-x}u_{xx}^2+e^{-y}u_{yy}^2+e^{-z}u_{zz}^2\right)=F_n(x,y,z,t)+O\left(t^{2(n+1)}\right),$$

and $f_n(x,y,z)$ is given by the Taylor's series expansion of $\left(2-t^2\right)$ around $t = o$

$$\left(2-t^2\right)=\sum_{n=0}^{\infty} f_n(x,y,z)t^n.$$

The following MVIM results are obtained:

$$U_1 = e^x + e^y + e^z + t^2,$$
$$U_2 = e^x + e^y + e^z + t^2, \qquad \text{...(10)}$$
$$U_3 = e^x + e^y + e^z + t^2,$$
$$\vdots$$

This is same as the closed form solution of (8), *viz.*, $u(x, y, z, t) = e^x + e^y + e^z + t^2$.

Now, using VIM for solving (8) leads to the following iteration formula:

$$U_{n+1} = U_n - \int_0^t (t-\zeta)\{(U_n)_{\zeta\zeta} + B(U_n) + C(U_n) - 2 + \zeta^2\} d\zeta, \qquad \text{...(11)}$$

where $B(U_n) = -U_n$ and

$$C(U_n) = e^{-x}(U_n)_{xx}^2 + e^{-y}(U_n)_{yy}^2 + e^{-z}(U_n)_{zz}^2 .$$

The following VIM results are obtained:

$$U_1 = e^x + e^y + e^z + t^2 - \frac{t^4}{2},$$
$$U_2 = e^x + e^y + e^z + t^2 - \frac{5t^4}{12} - \frac{t^6}{60},$$
$$U_3 = e^x + e^y + e^z + t^2 - \frac{5t^4}{12} - \frac{t^6}{72} - \frac{t^8}{3360},$$
$$\vdots \qquad \text{...(12)}$$

The main comparison results are summarized in Table 1 and Figure 1. Table 1 shows the absolute errors between the closed form solution and the tenth iterative values obtained by both VIM and MVIM. But the $U_{10}(x, y, z)$ value, obtained by VIM, is not shown here since its expression is very large. In the similar way, other components may also be computed.

Table I: The absolute error between the closed form solution of (8) and $U_{10\ VIM}$ and $U_{10\ MVIM}$ values.

t	(x,y,z)	VIM	MVIM
0.05	(5,5,5)	2.60438367E-06	0.00000000
	(10,10,10)	2.60438537E-06	0.00000000
	(15,0,0)	2.60444358E-06	0.00000000
0.08	(5,5,5)	1.70703080E-05	0.00000000
	(10,10,10)	1.70703133E-05	0.00000000
	(15,0,0)	1.70702115E-05	0.00000000
0.11	(5,5,5)	6.10287770E-05	0.00000000
	(10,10,10)	6.10287680E-05	0.00000000
	(15,0,0)	6.10286370E-05	0.00000000

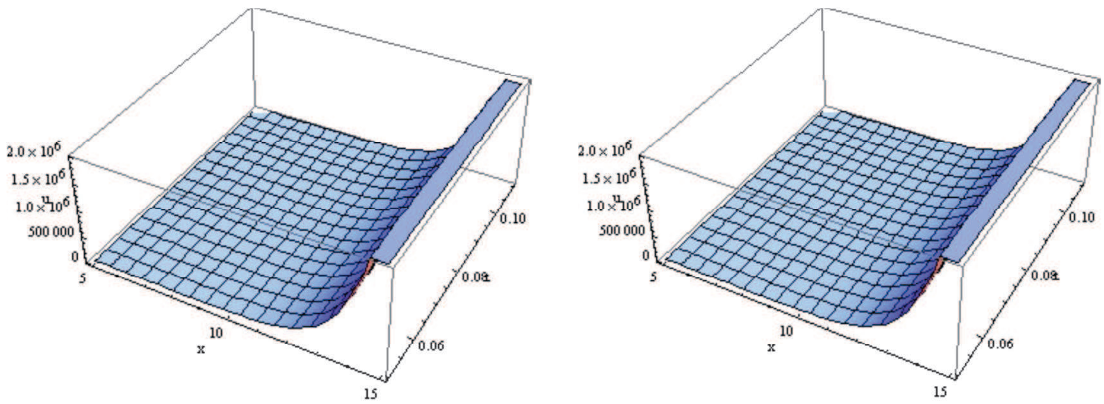

Figure 1. The closed form solution *u* and $U_{10\ MVIM}$; and $U_{10\ VIM}$ of equation (8) are shown in figures (a) and (b) respectively.

Result & Conclusion: In this paper, we have shown the comparison between the solutions obtained by the VIM and its modified form, MVIM. We have found that unlike VIM, the MVIM takes less time in calculations, and is simple, efficient and easy to use. MVIM also reduces the size of calculations to a great extent and sometimes can also give the closed form solution of a differential equation. It can be observed that, here, we have obtained the closed form solution of the given differential equation by MVIM whereas by VIM, some extra terms along with the closed form solution are also obtained. Thus, MVIM numerical results converge faster than VIM numerical results. The modified form introduces a change in the formulation of variational iteration relation and provides a qualitative improvement over standard VIM. The method has features in common with many other methods, but it is distinctly different on close examination.

Acknowledgment: One of the authors, Ms. Ankita Sharma is highly thankful to the University Grants Commission, New Delhi, India for the financial research grant for pursuing the Ph.D. work.

References:

1. R. Arora and A. Kumar, "Solution of the Coupled Drinfeld's–Sokolov–Wilson (DSW) System by HAM", Adv. Sci. Eng. and Med., 5, 2013, pp. 1-7
2. R. Arora, A. Tomar and V. P. Singh, "Numerical Simulation of Ito Coupled System by HAM", Adv. Sci. Eng. and Med., 4, 2012, pp. 522–529.
3. A. Kumar and R. Arora, "Solutions of the coupled system of Burgers' equations and coupled Klein-Gordon equation by RDTM", Int. J. of Adv. in Appl. Mathe.and Mech., 1(2), 2013, pp. 133-145.
4. R. Arora, M. J. Siddiqui and V. P. Singh, "Solutions of Inviscid Burgers' and Equal Width Wave Equations by RDTM", Int. J. of Appl. Phys. and Mathe., 2(3), 2012, pp. 212-214.
5. J.H. He, "Some asymptotic methods for strongly nonlinear equation", Int. J. Modern Phys., B 20(10), 2006, pp. 1141-1199.
6. J.H. He, "Approximate analytical solution for seepage flow with fractional derivatives in porous media", Comput. Appl. Mech. Eng., 167, 1998, pp. 57-68.
7. J.H. He, "VIM-some recent results and new interpretations", J. of Comput. and Appl. Mathe., 207(1), 2007, pp. 3-17.
8. A.M. Wazwaz, "The VIM for analytic treatment for linear and nonlinear ODEs", Appl. Mathe. and Comp., 212, 2009, pp. 120-134.
9. R. Arora and A. Kumar, "Solution of Linear and Nonlinear PDEs by the He's VIM", Recent Adv. in Intelligent Control, Modelling and Comput. Sci., ISBN: 978-960-474-319-3, 2013, pp. 15-19.
10. T.A. Abassy, M.A. El-Tawil and H. El-Zoheiry, "Toward a modified variational iteration method", J. of comput. and Appl. Mathe., 207(1), 2007, pp. 137-147.
11. T.A. Abassy, "Modified variational iteration method (nonlinear homog. initial value problem)", Comput. Math. Appl., 59 (2), 2010, pp. 912–918.
12. T.A. Abassy, "New Applications of the Modified Variational Iteration Method", Studies in Nonlinear Sciences, 3 (2), 2012, pp. 49-61.

***

Research Scholar, Assistant Professor, rajan_a100@yahoo.com.
Department of Applied Science and Engineering, IIT Roorkee, Saharanpur Campus,
Saharanpur,U.P.-247001, India, Email- ankitasharma.iitr@gmail.com.