STABILITY RESULT FOR S-ITERATION IN CONVEX METRIC SPACES

PREETY

Abstrac: The aim of this paper is to prove the stability result for S-iteration in complete convex metric spaces for a selfmapping satisfying contractive conditions.

Keywords: Convex metric spaces, Fixed point, Iterative schemes, Stability.

Introduction And Preliminaaries: In 1970, Takahashi [11] introduced the concept of convexity in metric space (X,d) as follows:

Definition 1.1 [11] Let (X,d) be a metric space. A map $W: X^2 \times [0,1] \to X$ is a convex structure on X if $d(u,W(x,y,\lambda)) \le \lambda d(u,x) + (1-\lambda)d(u,y)$ for all x,y,u ∈ X and $\lambda \in [0,1]$. A metric space (X, d) together with a convex structure W is known as convex metric space and is denoted by (X,d,W) . A nonempty subset C of a convex metric space is convex if $W(x,y,\lambda) \in C$ for all x,y ∈ C and $\lambda \in [0,1]$. All normed spaces and their subsets are the examples of convex metric spaces.

Several stability results have been obtained by various authors using different contractive definitions. The concept of stability of a fixed point iteration scheme has been systematically studied by Harder in her thesis and published in the papers of Harder and Hicks ([2], [1]). Harder and Hicks [1] obtained interesting stability results for some iteration procedures using various contractive definitions. Olatinwo [5] gave excellent introduction and some interesting comments about several stability results established in metric spaces and normed linear spaces.

In complete metric space setting, Harder and Hicks [1] defined the concept of stability of iterative schemes as follows:

Definition 1.2 [1] Let (X,d) be complete metric space and T: X→ X a selfmapping.

Let $\{x_n\}_{n=0}^{\infty} \subset X$ be the sequence generated by an iteration scheme involving T which is defined by

$x_{n+1} = f(T, x_n)$, n = 0, 1, 2... (1.1)

where $x_o \in X$ is the initial approximation and f is some function. Suppose $\{x_n\}$ converges to a fixed point p of T. Let $\{y_n\}_{n=o}^{\infty} \subset X$ and set $\epsilon_n = d(y_{n+1}, f(T, y_n))$, n = 0, 1, 2..........., Then, the iteration scheme (1.1) is said to be T-stable or stable with respect to T if and only if $\lim_{n\to\infty} \varepsilon_n = 0$ implies $\lim_{n\to\infty} y_n = p.$

Now in the literature of fixed point theory some iterative schemes are as follows:

Let (X, d) be a complete metric space and T : X → X be a selfmap of X. Suppose that $F_T = \{p \in X, Tp = p\}$ is the set of fixed points of T. Let $x_o \in X$ as initial approximating point of the iterative schemes under consideration.

The Picard, Mann [12], Ishikawa [8] iterative schemes are defined by the sequence $\{x_n\}$ as:

$x_{n+1} = Tx_n$, n = 0, 1, 2.... (1.2)

$x_{n+1} = (1-\alpha_n)x_n + \alpha_n Tx_n$, n = 0, 1, 2... (1.3)

$x_{n+1} = (1-\alpha_n)x_n + \alpha_n Ty_n$,

$y_n = (1-\beta_n)x_n + \beta_n Tx_n$, n = 0, 1, 2... (1.4)

For $x_o \in X$ Noor [6] introduced the Noor three step iteration scheme defined as:

$x_{n+1} = (1-\alpha_n)x_n + \alpha_n Ty_n$,

$y_n = (1-\beta_n)x_n + \beta_n Tz_n$,

$z_n = (1-\gamma_n)x_n + \gamma_n Tx_n$, n = 0, 1, 2... (1.5)

where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ are real sequences in [0,1].

In 2007, Agarwal et al. [7] defined the S- iterative scheme as:

$x_{n+1} = (1-\alpha_n)Tx_n + \alpha_n Ty_n$,

$y_n = (1-\beta_n)x_n + \beta_n Tx_n$, , n = 0, 1, 2... (1.6)

where $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are sequences of positive numbers in [0,1].

Remarks:

(i) If $\alpha_n = 1$ for all n ∈ N, then Mann iteration (1.3) reduces to Picard iteration (1.2).

(ii) If $_n = 0$ for all n ∈ N then Ishikawa iteration (1.4) reduces to Mann iteration (1.3).

(iii) If $\gamma_n = 0$ for all n ∈ N then Noor iteration (1.5) reduces to Ishikawa iteration (1.4).

In 2011, Olatinwo [4] defined the concept of T-stability in convex metric space setting:

Definition 1.3: Let (X, d, W) be a convex metric space and T : X → X a selfmapping.

Let $\{x_n\}_{n=0}^{\infty} \subset X$ be the sequence generated by an iterative scheme involving T which is defined by

$x_{n+1} = f_{T,\alpha_n}^{x_n}$, $n = 0,1,2....$, (1.7)

where $x_o \in X$ is the initial approximation and $f_{T,\alpha_n}^{x_n}$ is some function having convex structure such that $\alpha_n \in [0, 1]$. Suppose that $\{x_n\}$ converges to a fixed point p of T. Let $\{y_n\}_{n=0}^{\infty} \subset X$ and set $\varepsilon_n = d(y_{n+1}, f_{T,\alpha_n}^{y_n}), (n = 0,1,2....)$. Then, the iterative scheme (1.7) is said to be T-stable with respect to T if and only if $\lim_{n\to\infty}\varepsilon_n = 0$, implies $\lim_{n\to\infty} y_n = p$.

The iterative schemes in terms of convex structure are as follows:

Let (X, d, W) be a convex metric space and T : X → X be a selfmap of X. For $x_0 \in X$,

(1.1.1) Picard iterative scheme:

$x_{n+1} = Tx_n$, n = 0, 1, 2....

(1.1.2) Mann iterative scheme:

$x_{n+1} = W(x_n, Tx_n, \alpha_n)$, n = 0, 1, 2....

where $\{\alpha_n\}_{n=0}^{\infty}$ is a real sequence in [0,1].

(1.1.3) Ishikawa iterative scheme:

$x_{n+1} = W(x_n, Ty_n, \alpha_n)$,

$y_n = W(x_n, Tx_n, \beta_n)$, n = 0, 1, 2....

where $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are real sequences in [0,1].

(1.1.4) Noor iterative scheme:

$x_{n+1} = W(x_n, Ty_n, \alpha_n)$,

$y_n = W(x_n, Tz_n, \beta_n)$

$z_n = W(x_n, Tx_n, \gamma_n)$, n = 0, 1, 2....

where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ are real sequences in [0,1].

(1.1.5) S-iterative scheme:

$x_{n+1} = W(Tx_n, Ty_n, \alpha_n)$,

$y_n = W(x_n, Tx_n, \beta_n)$, n = 0, 1, 2....

where $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are sequences of positive numbers in [0,1].

Imoru and Olatinwo [3] introduced a contractive condition as follows which we will use in our main result: if there exists a constant 0 ≤ δ <1 and a monotonically increasing and continuous function ϕ : [0, ∞) → [0, ∞) with ϕ(0) = 0, such that for all x, y ∈ X,

$$d(Tx, Ty) \le \delta d(x, y) + \phi(d(x, Tx)) \quad (1.8)$$

Now we give a Lemma which is used in our main results.

Lemma 1.4 [9, 10] If δ is a real number such that 0 ≤ δ <1 and $\{\varepsilon_n\}_{n=0}^{\infty}$ is a sequence of positive numbers such that $\lim_{n\to\infty}\varepsilon_n = 0$, then for any sequence of positive numbers $\{u_n\}_{n=0}^{\infty}$ satisfying $u_{n+1} \le \delta u_n + \epsilon_n$, n = 0, 1, 2.... ; we have $\lim_{n\to\infty} u_n = 0$.

2. Main Result

Theorem 2.1 Let (X, d, W) be a complete convex metric space and T: X → X a mapping satisfying contractive condition (1.8). Suppose that T has a fixed point p. For $x_0 \in X$, let S-iterative scheme $\{x_n\}_{n=0}^{\infty}$ be defined by (1.1.5), where $\alpha_n, \beta_n \in [0,1]$ such that $0 < \alpha \le \alpha_n$, $0 < \beta \le \beta_n$. Then the S-iteration is T-stable.

Proof: Suppose that $\{y_n\}_{n=0}^{\infty} \subset X$ is an arbitrary sequence in X and define

$\epsilon_n = d(y_{n+1}, W(Ty_n, Tq_n, \alpha_n))$, where $q_n = W(y_n, Ty_n, \beta_n)$.

Assume that $\lim_{n\to\infty}\varepsilon_n = 0$. Then by using contractive condition (1.8), we establish that $\lim_{n\to\infty} y_n = p$. Thus we have that,

$d(y_{n+1}, p) \le d(y_{n+1}, W(Ty_n, Tq_n, \alpha_n)) + d(W(Ty_n, Tq_n, \alpha_n), p)$

$\le \epsilon_n + (1-\alpha_n)d(Ty_n, Tp) + \alpha_n d(Tq_n, Tp)$

$\le \epsilon_n + (1-\alpha_n)\delta d(y_n, p) + \alpha_n \delta d(q_n, p)$ (2.1.1)

For the estimate of $d(q_n, p)$ in (2.1.1), we get

$d(q_n, p) = d(W(y_n, Ty_n, \beta_n), p)$

$\le (1-\beta_n)d(y_n, p) + \beta_n d(Ty_n, Tp)$

$\le (1-\beta_n)d(y_n, p) + \beta_n \delta d(y_n, p)$ (2.1.2)

Substituting (2.1.2) into (2.1.1) we get

$d(y_{n+1}, p) \le \epsilon_n + (1-\alpha_n)\delta d(y_n, p) + \alpha_n\delta[(1-\beta_n)d(y_n, p) + \beta_n \delta d(y_n, p)]$

$= \epsilon_n + (1-\alpha_n)\delta d(y_n, p) + \alpha_n\delta[(1-(1-\delta)\beta_n)d(y_n, p)]$

$= \delta(1-\alpha_n\beta_n(1-\delta))d(y_n,p) + \epsilon_n$

$\le \delta(1-\alpha\beta(1-\delta))d(y_n,p) + \epsilon_n$ (2.1.3)

Since $0 \le \delta(1-\alpha(1-\delta)) < 1$ and using Lemma 1.4 in (2.1.3) we get $\lim_{n\to\infty} d(y_n, p) = 0$, that is

$\lim_{n\to\infty} y_n = p$.

Conversely,

Let $\lim_{n\to\infty} y_n = p$. Then,

$\epsilon_n = d(y_{n+1}, W(Ty_n, Tq_n, \alpha_n))$

$\le d(y_{n+1}, p) + \delta(1-\alpha\beta(1-\delta))d(y_n,p) \to 0$ as $n\to\infty$.

This completes the proof.

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Research Scholar
Department of Mathematics, Maharshi Dayanand University, Rohtak-124001(INDIA)
Email: preety0709@gmail.com