EULER VALUED INNER PRODUCT SPACE

S.VIJAYABALAJI, J.PARTHIBAN

Abstract: The main objective of this paper is to introduce the notion of Euler valued norm and inner product in a valued inner product space and derive some properties combined with the theory of numbers. We conclude with the existence of a vector which satisfying Pythagorean equation and construct a general formula using Gram - Schmidt process.

Keywords: Euler valued inner product, Euler valued norm, valued inner product, valued norm.

Introduction: Number theory may briefly be defined as the study of the properties of integers. It will be found later on that some properties of integers hold good only when the integers are positive and prime. Fermat is universally considered as the father of modern number theory [1].

The concept of valued inner product space was introduced by S. Vijayabalaji and J. Parthiban in [9], inner product of arithmetic mean as first and second kind were introduced. We prove Cauchy-Schwarz inequality obtained from the arithmetic mean of inner product of first and second kind. Scalar multiplication defined in vector space for magnification. Suppose that the scalar multiplication restricted to even numbers, magnification also restricted. The following questions arise in an Inner Product Space.

(1) Is an Inner product of a linearly independent vector and an orthogonal vector, an integer?

(2) In the second stage of Gram -Schmidt process, norm of a vector, subtracted from a linearly independent vector is an integer?

Our contribution is answering the above, using Euler function in Number theory and valuation in Algebraic Number theory. We conclude this paper by deriving a general formula for orthonormal vectors in R^2.

Preliminaries: This section recalls some basic definitions and results which will be needed in the sequel.

Definition2.1 [6] A **valuation** $[\![\cdot]\!]$ on a field K is a function defined on K with values in $R \geq 0$ satisfying the following axioms:

(1) $[\![x]\!] > 0$.

(2) $[\![x]\!] = 0 \; iff \; x = 0$.

(3) $[\![xy]\!] = [\![x]\!][\![y]\!]$,

(4) $[\![x + y]\!] \leq [\![x]\!] + [\![y]\!]$ for all $x, y \in K$.

Example2.2 [6] The real valuation on the rational Q is the absolute value on the real numbers, defined by $[\![x]\!] = \begin{cases} x, & if \; x \geq 0 \\ -x, & if \; x < 0 \end{cases}$

Definition2.3 [9] Let V be a real linear space over K. For every $u, v, w \in V$ and $\alpha, \beta \in K$, a real valued function $(\bullet,\bullet)$ on $V \times V$ satisfying the following:

1.$(u, u) \geq 0$

2.$(u, v) = (v, u)$

3.$(\alpha u + \beta v, w) = [\![\alpha]\!](u, w) + [\![\beta]\!](v, w)$ is called a valued inner product and the pair $(X, (\bullet,\bullet))$ is called the valued inner product space.

Definition2.4 [7]

The Euler valued function $\varphi(n)$

Let n be a positive integer. Then $\varphi(n)$ is defined as the number of positive integers relatively prime to n which do not exceed n.

Example2.5 [7] Let $n = 12$.

Positive integers relatively prime to n which do not exceed n are 1, 5, 7 and 11.

Hence $\varphi(n) = 4$

Theorem2.6 [7] If p is a prime, then $\varphi(p) = p - 1$.

Theorem2.7 [7] Let m be relatively prime to n. Then $\varphi(mn) = \varphi(m)\varphi(n)$.

Euler Valued Inner Product Space: Inspired by the theory of valued inner product space [9], we provide two equivalent definitions on valued norm and inner products namely Euler valued norm and inner product space.

Definition 3.1

Let V be a linear space over K. For every $x, y \in V$ and $\alpha(prime) \in K$, a real valued function $\|\bullet\|$ on V satisfying the following:

(1) $\|x\| \geq 0$.

(2) $\|(\alpha - 1)x\| = [\alpha]\|x\|$

(3) $\|x + y\| \leq \|x\| + \|y\|$ is called the Euler valued norm and the pair $(X, \|\bullet\|)$ is called the Euler valued normed linear space.

Definition 3.2

Let V be a real linear space over K. For every $u, v, w \in V$ and $\alpha(prime) \in K$, a real valued function $(\bullet,\bullet)$ on $V \times V$ satisfying the following:

(1) $(u, u) \geq 0$.

(2) $(u, v) = (v, u)$

(3) $((\alpha - 1)u, v) = [\alpha](u, v)$

(4) $(u + v, w) = (u, w) + (v, w)$

is called the Euler valued inner product and the pair $(X, (\bullet,\bullet))$ is called the Euler valued inner product space.

Example 3.3

We have $\varphi(p) = p - 1$.

Consider $\varphi(2) = 1$ and $\varphi(3) = 2$.

(1) $\varphi(p) > 0$.
(2) $\varphi(2 \times 3) = \varphi(6) = 2$.
Positive integers relatively prime to 6 which do not exceed 6 are 1 and 5.
Therefore $\varphi(6) = \varphi(2)\varphi(3)$
(3) $\varphi(2 + 3) = \varphi(5) = 4$.
But $\varphi(2) + \varphi(3) = 1 + 2 = 3$
Therefore $[\![x + y]\!] \nleq [\![x]\!] + [\![y]\!]$

Example 3.4

We have $\varphi(p) = p - 1$.
Consider $\varphi(3) = 2$ and $\varphi(5) = 4$.
(1) $\varphi(p) > 0$.
(2) $\varphi(3 \times 5) = \varphi(15) = 8$.
Positive integers relatively prime to 15 be 1, 2, 4, 7, 8, 11, 13 and 14.
Therefore $\varphi(6) = \varphi(3)\varphi(5)$
(3) $\varphi(3 + 5) = \varphi(8) = 4$.
Since, $\varphi(3) + \varphi(5) = 2 + 4 = 6$.
Therefore $[\![x + y]\!] \leq [\![x]\!] + [\![y]\!]$
Thus we conclude that the Euler function is a valuation for all odd primes.

Remark 3.5

Example 3.3 and Example 3.4 shows that the scalar multiplication on the Euler valued normed linear and Euler valued inner product space allows the even integers of the form $p - 1,\ p \geq 3$.

Proposition 3.6

Let x, y, z, m and n be the integers which satisfies the Pythagorean equation
$x^2 + y^2 = z^2$ as $x = m^2 - n^2$; $y = 2mn$ and $z = m^2 + n^2$ with $m - n = 1$. Then $x^2 = y + z$.

Proof

Now $x = m^2 - n^2$
$= (m + n)(m - n)$.
$= (m + n)$, here $m - n = 1$
Then $x^2 = (m + n)^2$
$= m^2 + n^2 + 2mn$.
$= 2mn + m^2 + n^2$.
$= y + z$.

Example 3.7

Let $m = 3$ and $n = 2$.
Then $x = m^2 - n^2 = 5$; $y = 2mn = 12$ and $z = m^2 + n^2 = 13$.
Then $5^2 = 12 + 13$.
Let $m = 4$ and $n = 3$.
Then $x = m^2 - n^2 = 7$; $y = 2mn = 24$ and $z = m^2 + n^2 = 25$.
Then $7^2 = 24 + 25$.

Remark 3.8

Example 3.7 shows that the scalar multiplication on the Euler valued normed linear and Euler valued inner product space allows the even integers of the form $p - 1,\ p \geq 3$. Then we should consider the Pythagorean triplets like (3,4,5), (5,12,13), (7,24,25) and so on.

Observation 3.9

(1) First element of a Pythagorean triplet must be an odd prime.
(2) $y = z - 1$ and $z = y + 1$.

Lemma 3.10

Let x, y, z, m and n be the positive integers which satisfies the Pythagorean equation
$x^2 + y^2 = z^2$ as $x = m^2 - n^2$; $y = 2mn$ and $z = m^2 + n^2$ with $m - n = 1$. Then the vectors (x, y) and $(z - x, 1)$ are linearly independent.

Proof

We have, any two vectors (x_1, y_1), (x_2, y_2) are linearly independent of R if and only if
$\alpha(x_1, y_1) + \beta(x_2, y_2) = (0,0)$.
This implies $\alpha = \beta = 0$.
In our problem, to show the vectors (x, y) and $(z - x, 1)$ are linearly independent if and only if
$\alpha(x, y) + \beta(z - x, 1) = (0,0)$.
This implies $\alpha = \beta = 0$.
That is $\left.\begin{array}{r}\alpha x + \beta(z - x) = 0\\ \alpha y + \beta(1) = 0\end{array}\right\} - -(1).$
This implies $\alpha = \beta = 0$.
The given vectors are linearly independent if and only if the system (1) of linear equations has trivial solution.
Therefore (1) has trivial solution if and only if $\begin{vmatrix} x & z - x \\ y & 1 \end{vmatrix} \neq 0$.
Since, $x < y < z$, then $x - y(z - x) \neq 0$.
Hence the vectors (x, y), $(z - x, 1)$ are linearly independent.
Hence the vectors form a basis of R^2.

Example 3.11

Let $(x, y) = (3,4)$ and $z = 5$.
Then $(z - x, 1) = (2,1)$.
To show the linearly independent,
If $\alpha(3,4) + \beta(2,1) = (0,0)$,
then $\alpha = \beta = 0$.
That is $\left.\begin{array}{r}3\alpha + 2\beta = 0\\ 4\,\alpha + \beta = 0\end{array}\right\} - - - -(1).$
This implies $\alpha = \beta = 0$.
The given vectors (3,4) and (2,1) are linearly independent if and only if the system (1) of linear equations has trivial solution.
Therefore (1) has trivial solution if and only if $\begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix} \neq 0$.
Here $\begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix} = -5 \neq 0$.
Hence the vectors (3,4), (2,1) are linearly independent.
Hence the vectors form a basis of R^2.

Lemma 3.12

Let $v_1 = (x, y)$ and $v_2 = (z - x, 1)$ be the linearly independent of vectors R^2. Then
$\left(v_2, \frac{v_1}{\|v_1\|}\right) = [x]$.

Proof

We have $\|v_1\| = z$, since $\sqrt{x^2 + y^2} = z$.

Therefore $\frac{v_1}{\|v_1\|} = \left(\frac{x}{z}, \frac{y}{z}\right)$.

Then $\left(v_2, \frac{v_1}{\|v_1\|}\right) = \left((z - x, 1), \left(\frac{x}{z}, \frac{y}{z}\right)\right)$

$= (z - x)\frac{x}{z} + \frac{y}{z}$.

$= z\frac{x}{z} - \frac{x^2}{z} + \frac{y}{z}$.

$= x - \frac{(x^2 - y)}{z}$.

$= x - \frac{(y + z - y)}{z}$,

by proposition 3.6

$= x - 1$.

Therefore $\left(v_2, \frac{v_1}{\|v_1\|}\right) = [x]$.

Example 3.13

Let $v_1 = (x, y) = (11,60)$.

Then $\|v_1\| = z = 61$.

Therefore $v_1 = (z - x, 1) = (50,1)$.

$w_1 = \frac{v_1}{\|v_1\|} = \left(\frac{x}{z}, \frac{y}{z}\right) = \left(\frac{11}{61}, \frac{60}{61}\right)$.

$(v_2, w_1) = \left((50,1), \left(\frac{11}{61}, \frac{60}{61}\right)\right)$.

$= \frac{550}{61} + \frac{60}{61} = \frac{610}{61} = 10 = [11]$.

$= x - 1$.

Therefore $\left(v_2, \frac{v_1}{\|v_1\|}\right) = [x]$.

Theorem 3.14

Gram-Schmidt process for R²

Let $v_1 = (x, y)$ and $v_2 = (z - x, 1)$ be the linearly independent of vectors R^2. Then $w_1 = \frac{v_1}{\|v_1\|}$ and $w_2 = \frac{u_2}{\|u_2\|}$ are orthonormal.

Proof

We have $\|v_1\| = z$, since $\sqrt{x^2 + y^2} = z$.

$w_1 = \frac{v_1}{\|v_1\|} = \left(\frac{x}{z}, \frac{y}{z}\right)$.

Then $(v_2, w_1) = [x] = x - 1$.

$u_2 = v_2 - (v_2, w_1)w_1$.

$= (z - x, 1) - [x]\left(\frac{x}{z}, \frac{y}{z}\right)$.

$= (z - x, 1) - (x - 1)\left(\frac{x}{z}, \frac{y}{z}\right)$.

$= \left((z - x) - \frac{(x-1)x}{z}, 1 - \frac{(x-1)y}{z}\right)$.

$= \left(\frac{(z-x)z - (x-1)x}{z}, \frac{z - (x-1)y}{z}\right)$.

$= \left(\frac{z^2 - xz - x^2 + x}{z}, \frac{z - xy + y}{z}\right)$.

$= \left(\frac{x^2 + y^2 - xz - x^2 + x}{z}, \frac{z - xy + y}{z}\right)$.

$= \left(\frac{y^2 - xz + x}{z}, \frac{z + y - xy}{z}\right)$.

$= \left(\frac{y^2 - x(z-1)}{z}, \frac{x^2 - xy}{z}\right)$,

by proposition 3.6

$= \left(\frac{y^2 - xy}{z}, \frac{x^2 - xy}{z}\right)$,

by observation 3.9

$= \left(\frac{y(y-x)}{z}, \frac{x(x-y)}{z}\right)$.

$\|u_2\| = \frac{1}{z}\sqrt{y^2(y - x)^2 + x^2(x - y)^2}$.

$= \frac{1}{z}\sqrt{(y - x)^2(x^2 + y^2)}$.

$= \frac{1}{z}\sqrt{(y - x)^2 z^2}$.

$\|u_2\| = y - x$.

$w_2 = \frac{u_2}{\|u_2\|} = \frac{1}{y-x}\left(\frac{y(y-x)}{z}, \frac{x(x-y)}{z}\right)$.

$w_2 = \frac{u_2}{\|u_2\|} = \left(\frac{y}{z}, \frac{-x}{z}\right)$.

Hence w_1 and w_2 are orthonormal.

Example 3.15

Let $v_1 = (x, y) = (5,12)$ and

$z = \sqrt{x^2 + y^2} = 13 = \|v_1\|$.

Then $v_2 = (z - x, 1) = (8,1)$

$w_1 = \frac{v_1}{\|v_1\|} = \left(\frac{x}{z}, \frac{y}{z}\right) = \left(\frac{5}{13}, \frac{12}{13}\right)$.

Then $(v_2, w_1) = [x] = [5] = 5 - 1 = 4$.

$u_2 = v_2 - (v_2, w_1)w_1$.

$= (8,1) - [5]\left(\frac{5}{13}, \frac{12}{13}\right)$.

$= (8,1) - 4\left(\frac{5}{13}, \frac{12}{13}\right)$.

$= \left(\frac{y(y-x)}{z}, \frac{x(x-y)}{z}\right) = \left(\frac{84}{13}, \frac{-35}{13}\right)$.

$\|u_2\| = y - x = 12 - 5 = 7$.

$w_2 = \left(\frac{y}{z}, \frac{-x}{z}\right) = \left(\frac{12}{13}, \frac{-5}{13}\right)$.

Hence w_1 and w_2 are orthonormal.

We shall calculate this orthonormal vectors and values lies in that process calculated easier, using the process describe above.

Example 3.16

Let $v_1 = (x, y) = (11,60)$

and $z = \sqrt{x^2 + y^2} = 61 = \|v_1\|$.

Then $v_2 = (z - x, 1) = (50,1)$

$w_1 = \left(\frac{x}{z}, \frac{y}{z}\right) = \left(\frac{11}{61}, \frac{60}{61}\right)$.

$(v_2, w_1) = [x] = [11] = 11 - 1 = 10$.

$u_2 = \left(\frac{y(y-x)}{z}, \frac{x(x-y)}{z}\right) = \left(\frac{2940}{61}, \frac{-539}{61}\right)$.

$\|u_2\| = y - x = 60 - 11 = 49$.

$w_2 = \left(\frac{y}{z}, \frac{-x}{z}\right) = \left(\frac{60}{61}, \frac{-11}{61}\right)$.

Hence w_1 and w_2 are orthonormal.

References :

1. Tom. M. Apostal, An introduction to analytic number theory, Springer, New York, 1976.
2. G.H. Hardy and E.M. Wright, An introduction to the theory of numbers, Oxford Univ. press, New York, 1980.
3. G.H. Hardy, J.E. Little wood, G. Polya, Inequalities, Cambridge Mathematical Library, Cambridge University Press, 1999.
4. J. S. Milne, Algebraic number theory, University of Michigan, 2011.

5. W. Rudin, Functional Analysis, McGraw-Hill Science, 1991.
6. Serge Lang, Algebraic number theory, Addison-Wesley publishing company, New York, 1970.
7. S. G. Telang, Number Theory, Tata McGraw-Hill Publishing Company limited, New Delhi, 1996.
8. S. Vijayabalaji and N. Thillaigovindan, Fuzzy n-inner product space, Bulletin of Korean Mathematical Society, 43, 2007, No.3, 447-459.
9. S. Vijayabalaji and J. Parthiban, Some inequalities in valued inner product space, International journal of pure and applied mathematics, Volume 86, No. 6, 2013, 983-992.
10. William J. Leveque, Elementary theory of numbers, Addison-Wesley publishing company, New York, 2008.

* * *

Department of Mathematics, University College of Engineering Panruti, Panruti- 607 106, Tamil Nadu, India. balaji1977harshini@gmail.com

Department of Mathematics, Kurinji College of Engineering and Technology, Manapparai-621 307, Tamil Nadu, India. mathparthiban@gmail.com