FOUR STEP ITERATIVE SCHEME FOR VARIATIONAL INEQUALITIES AND FIXED POINT PROBLEMS

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Abstract. The aim of this paper is to introduce a new iterative scheme for finding solution of a variational inequality problem for inverse strongly monotone mapping and fixed point problem. The main result obtained in this paper extends the result of Shang et. al.

Key Words: Fixed points, Inverse strongly monotone mappings. Nonexpansive mappings, Variational inequalities.

Introduction: Variational inequalities were formulated between the end of 60' and the beginning of 70' of previous century by the italian mathematician G. Stampacchia [1].

The variational inequality problem is defined as follows:

Let H be a real Hilbert space with inner product $\langle .,. \rangle$ and norm $\|.\|$, respectively. Let C be a closed convex subset of H.The variational inequality problem is to find u ϵ C such that $< Au, v - u > \geq 0$, $\forall$ v ϵ C.

The set of solutions of variational inequality problem VI(C, A) is denoted by Ω. The variational inequality problem has been extensively studied in literature, see, for example,[2], [3], [4] and references therein.

Definitions. Let A: C → H be a mapping of C into H.A is called monotone if $< Au - Av, u - v > \geq 0$ $\forall$ u, v ϵ C.

A mapping A of C into H is called α-inverse-strongly-monotone [5, 6, 7] if there exists a positive real number α such that $< Au - Av, u - v > \geq \alpha\|Au - Av\|^2$, $\forall$ u, v ϵ C.

It is obvious that any α-inverse-strongly-monotone mapping A is monotone and Lipschitz continuous.

A mapping S : C→C is called nonexpansive [7] if

$\|Su - Sv\| \leq \|u - v\|$ $\forall$ u,v ϵ C.

We denote by F(S) the set of fixed points of S.Takahashi and Toyada [8] gave the following iterative scheme for finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of the variational inequality problem for an α-inverse-strongly-monotone mapping in a real Hilbert space.

Theorem.1.1. [8] Let C be a closed convex subset of a real Hilbert space H. Let A be an α-inverse-strongly-monotone mapping of C into H and let S be a nonexpansive mapping of C into itself such that F (S) ∩ VI(C,A) ≠ ϕ. Let $\{x_n\}$ be a sequence generated by $x_0 = x \in C$,

$x_{n+1} = \alpha_n x_n + (1-\alpha_n)SP_C(x_n - \lambda_n Ax_n)$, (1.1)

for every n = 0,1,2,........ where $\{\lambda_n\} \subset [a,b]$ for some a,b ϵ (0, 2α) and $\{\alpha_n\} \subset [c,d]$ for some c,d ϵ (0,1). Then, the sequence $\{x_n\}$ converges weakly to some point z ϵ F(S) ∩ VI(C,A), where

$z = \lim_{n\to\infty} P_{F(S)\cap VI(C,A)}x_n$.

In 2004, Xu [9] proved the following theorem:

Theorem 1.2. [9] Let H be a Hilbert space, C be a closed convex subset of H and T : C → C be a nonexpansive mapping with Fix(T) ≠ ϕ and f is a contraction on C. Let $\{x_n\}$ be given by

$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Tx_n$, $n \geq 0$. (1.2)

Then under the following hypotheses

(H1). $\lim_{n\to\infty} \alpha_n = 0$,

(H2). $\sum_{n=0}^{\infty} \alpha_n = \infty$,

(H3). either $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ or $\lim_{n\to\infty} \frac{\alpha_{n+1}}{\alpha_n} = 1$,

$x_n \to x$, where x is the unique solution of the variational inequality

$< (I - f)x, x - y > \leq 0$, y ϵ S.

In 2007, Junmin Chen, Lijuan Zhang and Tiegang Fan [10] studied viscosity approximation methods for nonexpansive mappings.

Theorem.1.3 [10] Let C be a closed convex subset of a real Hilbert space H. Let f : C → C be a contracton with coefficient k(0 < k < 1). Let A be an α-inverse-strongly-monotone mapping of C into H and let S be a nonexpansive mapping of C into itself such that F(S) ∩ VI(C,A) ≠ ϕ. Let $\{x_n\}$ be a sequence generated by

$x_0 = x \in C$,

$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)SP_C(x_n - \lambda_n Ax_n)$ (1.3)

for every n = 0,1,2,........ where $\{\lambda_n\} \subset [a, b]$ and $\{\alpha_n\}$ is a sequence in (0, 1). If $\{\alpha_n\}$ and $\{\lambda_n\}$ are chosen so that $\lambda_n \in [a,b]$ for some a, b with $0 < a < b < 2\alpha$,

$\lim_{n\to 0} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$,

$\sum_{n=0}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$.

Then $\{x_n\}$ converges strongly to q ϵ F(S) ∩ VI(C,A), which is the unique solution in the F(S) ∩ VI(C,A) to the following variational inequality

$< (I - f)q, q - p > \le 0$, $p \in F(S) \cap VI(C,A)$.

In 2009, Meijuan Shang, Yongfu SU, Xiaolong Qin [11], introduced a general three-step iterative scheme for finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of the variational inequality for an inverse-strongly monotone mapping.

Theorem.3.1.4. [11] Let C be a closed convex subset of a real Hilbert space H. Let $f : C \to C$ be a contracton with coefficient $k(0 < k < 1)$. Let A be an α-inverse-strongly-monotone mapping of C into H and let S be a nonexpansive mapping of C into itself such that $F(S) \cap VI(C,A) \neq \phi$. Let $\{x_n\}$ be a sequence generated by

$$x_0 = x \in C,$$
$$z_n = P_c(x_n - \tau_n A x_n),$$
$$y_n = P_c(z_n - \mu_n A z_n), \qquad (1.4)$$
$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) S P_c(y_n - \lambda_n A y_n),$$

where $\{\alpha_n\}$, $\{\lambda_n\}$, $\{\mu_n\}$ and $\{\tau_n\}$ satisfy the following conditions:

1. $\{\alpha_n\}$ is a sequence in (0, 1);
2. $\lim_{n\to 0} \alpha_n = 0, \sum_{n=0}^{\infty} \alpha_n = \infty, \sum_{n=0}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$
3. $\sum_{n=0}^{\infty} |\lambda_n - \lambda_{n-1}| < \infty$, $\sum_{n=0}^{\infty} |\mu_n - \mu_{n-1}| < \infty$, $\sum_{n=0}^{\infty} |\tau_n - \tau_{n-1}| < \infty$
4. $\{\lambda_n\}$, $\{\mu_n\}$ and $\{\tau_n\}$ are three sequences in [a, b] for some $a, b \in (0, 2\alpha)$.

Then $\{x_n\}$ converges strongly to $q \in F(S) \cap VI(C,A)$, which is the unique solution in the $F(S) \cap VI(C,A)$ to the following variational inequality

$< (I - f)q, q - p > \le 0$, $\forall p \in F(S) \cap VI(C,A)$.

In this paper, motivated by above results, we shall prove the strong convergence theorem for finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of the variational inequality problem for a monotone and Lipschitz continuous mapping in a real Hilbert space.

Preliminaries.

Let H be a real Hilbert space with inner product $< . , . >$ and norm $\|.\|$. Let C be a closed convex subset of H. We shall write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x. $x_n \to x$ implies that $\{x_n\}$ converges strongly to x. It is well known that for any $x \in H$, there exists a unique nearest point in C, such that

$\|u - P_C x\| = \inf\{\|u - y\| : y \in C\}$

P_C is called the metric projection of H onto C.

P_C is characterized by the properties:

$P_C x \in C$,

$< x - P_C x, y - P_C x > \le 0$, for all $x \in H, y \in C$,

$\|x - y\|^2 \ge \| x - P_C x \|^2 + \| y - P_C x\|^2$, for all $x \in H, y \in C$.

The metric projection P_C of H onto C satisfies $< x - y, P_C x - P_C y > \ge \| P_C x - P_C y \|^2$, for every $x,y \in H$,

Let A be a monotone mapping of C into H. In the context of the variational inequality problem, it is easy to see that

$u \in \Omega \Leftrightarrow u = P_C (u - \lambda A u)$, for any $\lambda > 0$.

It is known that H satisfies the Opial condition [43] that is, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality,

$\lim_{n\to\infty} inf \|x_n - x\| < \lim_{n\to\infty} inf \|x_n - y\|$

holds for every $y \in H$ with $y \neq x$. We also know that, if $\{x_n\}$ is sequence of H with $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$, then there holds that $x_n \to x$.

A set valued mapping $T : H \to 2^H$ is called monotone if for all $x,y \in H$, $f \in Tx$ and $g \in Ty$ imply $< x - y, f - g > \ge 0$. A monotone mapping $T: H \to 2^H$ is maximal if its graph G(T) is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping T is maximal if and only if for $(x, f) \in H \times H$, $< x - y, f - g > \ge 0$ for every $(y, g) \in G(T)$ implies $f \in Tx$.

Let $A : C \to H$ be a monotone, k – Lipschitz continuous mapping and $N_C v$ be the normal cone to C at $v \in C$, that is

$N_C v = \{w \in H : < v - u, w > \ge 0, \forall u \in C\}$

Define,

$$Tv = \begin{cases} Av + N_C v, & if\ v \in C \\ \phi, & if\ v \notin C \end{cases}$$

Then, T is maximal monotone [12-13] and $0 \in Tv$ if and only if $v \in VI(C,A)$.

Now we give a lemma, which will be used in the proof of result.

Lemma 2.1. [14] Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying

$$s_{n+1} = (1 - \lambda_n) s_n + \beta_n, \quad n \ge 0,$$

where $\{\lambda_n\}$ and $\{\beta_n\}$ satisfy the following conditions:

(i) $\{\lambda_n\} \subset [0, 1]$ and $\sum_{n=0}^{\infty} \lambda_n = \infty$ or equivalently, $\prod_{n=0}^{\infty} (1 - \lambda_n) = 0$;

(ii) $\limsup_{n\to\infty} \beta_n / \lambda_n \le 0$ or $\sum_{n=0}^{\infty} |\beta_n| < \infty$.

Then, $\lim_{n\to\infty} s_n = 0$.

Main Result: In this section, we give the main result.

Theorem.3.1 Let C be a closed convex subset of a real Hilbert space H. Let $f : C \to C$ be a contracton with coefficient $k(0 < k < 1)$. Let A be an α-inverse-strongly-monotone mapping of C into H and let S be a nonexpansive mapping of C into itself such that $F(S) \cap VI(C,A) \neq \phi$. Let $\{x_n\}$ be a sequence generated by

$x_0 = x \in C$,

$t_n = P_C(x_n - \upsilon_n Ax_n)$
$z_n = P_C(t_n - \tau_n At_n)$
$y_n = P_C(z_n - \mu_n Az_n)$
$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)SP_C(y_n - \lambda_n Ay_n)$, $n \geq 0$, where $\{\alpha_n\}$, $\{\lambda_n\}$,$\{\mu_n\}$,$\{\tau_n\}$,$\{\upsilon_n\}$ satisfy the following conditions:
(i). $\{\alpha_n\}$ is a sequence in (0, 1).
(ii). $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty, \sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$.
(iii). $\sum_{n=1}^{\infty} |\lambda_n - \lambda_{n-1}| < \infty, \sum_{n=1}^{\infty} |\mu_n - \mu_{n-1}| < \infty$, $\sum_{n=1}^{\infty} |\tau_n - \tau_{n-1}| < \infty$, $\sum_{n=1}^{\infty} |\upsilon_n - \upsilon_{n-1}| < \infty$.
(iv). $\{\lambda_n\}$,$\{\mu_n\}$,$\{\tau_n\}$,$\{\upsilon_n\}$ are four sequences in [a, b] for some a, b ∈ (0, 2α). Then $\{x_n\}$ converges strongly to q ∈ F(S) ∩ VI(C, A), which is the unique solution in F(S) ∩ VI(C, A) of variational inequality
$< (I - f)q, q - p > \leq 0, \forall p \in F(S) \cap VI(C, A)$.

Proof. For each x, y ∈ C and $\lambda_n \in [0, 2\alpha]$, we have,
$\|(I - \lambda_n)x - (I - \lambda_n)y\|^2 = \|(x - y) - \lambda_n(Ax - Ay)\|^2$
$= \|(x - y)\|^2 - 2\lambda_n\langle x - y, Ax - Ay \rangle + \lambda_n^2 \| Ax - Ay \|^2$
$\leq \|(x - y)\|^2 + \lambda_n(\lambda_n - 2\alpha) \| Ax - Ay \|^2$ (3.1)
$\Rightarrow I - \lambda_n A$ is nonexpansive. Letting u ∈ F(S) ∩ VI(C, A), we have, $u = P_C(u - \lambda Au)$, $\forall \in \lambda > 0$. Note that
$\| t_n - u \| = \| P_C(x_n - \upsilon_n Ax_n) - P_C(u - \upsilon_n Au) \|$
$\leq \| x_n - u \|$
$\| z_n - u \| = \| P_C(t_n - \tau_n At_n) - P_C(u - \tau_n Au) \|$
$\leq \| t_n - u \|$
$\leq \| x_n - u \|$
$\| y_n - u \| = \| P_C(z_n - \mu_n Az_n) - P_C(u - \tau_n Au) \|$
$\leq \| x_n - u \|$.
Now, $\| t_n - u \| = \| P_C(x_n - \upsilon_n Ax_n) - P_C(u - \upsilon_n Au) \|$
$\| x_{n+1} - u \| = \| \alpha_n(f(x_n) - u)$
$+ (1 - \alpha_n)(SP_C(y_n - \lambda_n Ay_n) - u) \|$
$\leq \alpha_n \| f(x_n) - f(u)\| + \alpha_n \| f(u) - u\|$
$+ (1 - \alpha_n)\|P_C(y_n - \lambda_n Ay_n) - u) \|$
$\leq \alpha_n k\| x_n - u \| + \alpha_n \|f(u) - u\|$
$+ (1 - \alpha_n)\| y_n - u \|$
$\leq \alpha_n k\| x_n - u \| + \alpha_n \|f(u) - u\|$
$+ (1 - \alpha_n) \| x_n - u \|$
$= (1 - (1 - k)\alpha_n)\| x_n - u \| + \alpha_n \|f(u) - u\|$
$\leq \max. \{\| x_n - u \|, \frac{1}{1-k} \|f(u) - u\|\}$.
By simple induction, we have,
$\| x_n - u \| \leq \max. \{\| x_0 - u \|, \frac{1}{1-k} \|f(u) - u\|\}$, $n \geq 0$.
So $\{x_n\}$ is bounded and hence $\{x_n\}$, $\{y_n\}$, $\{t_n\}$, $\{Ay_n\}$, $\{Ay_n\}$,$\{Az_n\}$,$\{At_n\}$ and $\{ SP_C(y_n - \lambda_n Ay_n\}$ are all bounded. Note that
$t_n = P_C(x_n - \upsilon_n Ax_n)$, $t_{n-1} = P_C(x_{n-1} - \upsilon_{n-1}Ax_{n-1})$
$\Rightarrow \| t_n - t_{n-1} \| = \| P_C(x_n - \upsilon_n Ax_n) - P_C(x_{n-1} - \upsilon_{n-1}Ax_{n-1}) \|$
$\leq \| (I - \upsilon_n A)(x_n - x_{n-1}) \| + \| (I - \upsilon_n A) x_{n-1} - (I - \upsilon_{n-1}A) x_{n-1} \|$
$\leq \| x_n - x_{n-1} \| + | \upsilon_n - \upsilon_{n-1} | \|Ax_{n-1} \|$
Similarly,
$\| z_n - z_{n-1} \|$
$\leq \| t_n - t_{n-1} \| + | \tau_n - \tau_{n-1} | \|At_{n-1} \|$
$\leq \| x_n - x_{n-1} \| + | \upsilon_n - \upsilon_{n-1} | \|Ax_{n-1} \| + | \tau_n - \tau_{n-1} | \|At_{n-1} \|$
Also,
$\| y_n - y_{n-1} \|$
$\leq \| z_n - z_{n-1} \| + | \mu_n - \mu_{n-1} | \|Az_{n-1} \|$
$\leq \| x_n - x_{n-1} \| + | \upsilon_n - \upsilon_{n-1} | \|Ax_{n-1} \| + | \tau_n - \tau_{n-1} | \|A\tau_{n-1} \| + | \mu_n - \mu_{n-1} | \|Az_{n-1} \|$.
Now,
$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)SP_C(y_n - \lambda_n Ay_n)$
$x_n = \alpha_{n-1}f(x_{n-1})+(1 - \alpha_{n-1})SP_C(y_{n-1} - \lambda_{n-1}Ay_{n-1})$
So,
$\| x_{n+1} - x_n \| \leq \| \alpha_n(f(x_n) - f(x_{n-1}))+(\alpha_n - \alpha_{n-1}) f(x_{n-1})\|+(1 - \alpha_n)\|SP_C((I - \lambda_n A)y_n) - SP_C((I - \lambda_{n-1}A)y_{n-1})\| +(\alpha_n - \alpha_{n-1})\| SP_C((I - \lambda_{n-1}A)y_{n-1}) \|$
$\leq k \alpha_n \| x_n - x_{n-1} \| + |\alpha_n - \alpha_{n-1}|\| f(x_{n-1})\| + \| SP_C((I - \lambda_{n-1}A)y_{n-1}) \| + (1 - \alpha_n)(\| y_n - y_{n-1}\| + | \lambda_n - \lambda_{n-1} | \|Ay_{n-1} \|)$
$\leq k \alpha_n \| x_n - x_{n-1} \| + (1 - \alpha_n) \| x_n - x_{n-1} \| + | \upsilon_n - \upsilon_{n-1} | \|Ax_{n-1} \| + | \tau_n - \tau_{n-1} | \|At_{n-1} \| + | \mu_n - \mu_{n-1} | \|Az_{n-1} \| + | \lambda_n - \lambda_{n-1} | \|Ay_{n-1} \| + |\alpha_n - \alpha_{n-1}|\| f(x_{n-1})\| + \| SP_C((I - \lambda_{n-1}A)y_{n-1}) \|$.
$\leq (1 - (1 - k)\alpha_n) \| x_n - x_{n-1} \|+(| \upsilon_n - \upsilon_{n-1} | + | \tau_n - \tau_{n-1} |+| \mu_n - \mu_{n-1} |+| \lambda_n - \lambda_{n-1} | + |\alpha_n - \alpha_{n-1}|)$. M,
where M is an appropriate constant such that
$M \geq \max\{\sup_{n\geq1} \|Ax_{n-1} \|, \sup_{n\geq1} \|At_{n-1} \|, \sup_{n\geq1} \|Az_{n-1} \|, \sup_{n\geq1} \|Ay_{n-1} \|, \sup_{n\geq1} \| f(x_{n-1})\| + \| SP_C((I - \lambda_{n-1}A)y_{n-1}) \|\}$.
By using lemma 2.1 and conditions (ii) and (iii), we get,
$\lim_{n\to\infty} \| x_{n+1} - x_n \| = 0$. (3.2)
So we obtain also $\| t_{n+1} - t_n \| \to 0$, $\| y_{n+1} - y_n \| \to 0$, $\| z_{n+1} - z_n \| \to 0$.
Let $w_n = P_C(y_n - \lambda_n Ay_n)$
$w_{n+1} = P_C(y_{n+1} - \lambda_{n+1}Ay_{n+1})$
So,
$\| w_{n+1} - w_n \| = \| P_C(y_{n+1} - \lambda_{n+1}Ay_{n+1}) - P_C(y_n - \lambda_n Ay_n) \|$
$\leq \| (y_{n+1} - \lambda_{n+1}Ay_{n+1}) - (y_n - \lambda_n Ay_n) \|$
$= \| (y_{n+1} - \lambda_{n+1}Ay_{n+1}) - (y_n - \lambda_{n+1}Ay_n)$
$+ (\lambda_n - \lambda_{n-1}) Ay_n \|$
$\leq \| (y_{n+1} - y_n \| + |\lambda_n - \lambda_{n-1}| \| Ay_n \|$, which together with $\| y_{n+1} - y_n \| \to 0$ and $\sum_{n=1}^{\infty} |\lambda_n - \lambda_{n-1}| < \infty$ implies that
$\lim_{n\to\infty} \| w_{n+1} - w_n \| = 0$. (3.3)
Now, we shall show that
$\| Sw_n - w_n \| \to 0$ as $n \to \infty$.From (3.1), we have
$\| x_{n+1} - u \|^2 = \| \alpha_n f(x_n) + (1 - \alpha_n)Sw_n - u) \|^2$
$\leq \alpha_n \| f(x_n) - u \|^2 + (1 - \alpha_n)\| w_n - u \|^2$
$\leq \alpha_n \| f(x_n) - u \|^2 + (1 - \alpha_n)(\| y_n - u \|^2 + \lambda_n(\lambda_n - 2\alpha)\| Ay_n - Au \|^2)$
$\leq \alpha_n \| f(x_n) - u \|^2 + \| x_n - u \|^2$

$+ (1 - \alpha_n)a(b - 2\alpha)\| Ay_n - Au \|^2$
$\Rightarrow - (1 - \alpha_n)a(b - 2\alpha)\| Ay_n - Au \|^2$
$\leq \alpha_n \| f(x_n) - u \|^2 + \| x_n - u \|^2 - \| x_{n+1} - u \|^2$
$\leq \alpha_n \| f(x_n) - u \|^2 + (\| x_n - u \| - \| x_{n+1} - u \|)\| x_{n+1} - x_n \|$
Since $\alpha_n \to 0$ as $n \to \infty$, $a, b \in (0, 2\alpha)$ and by using (3.2), we get
$\lim_{n\to\infty} \| Ay_n - Au \| = 0.$ (3.4)
On the other hand, we have
$\| x_{n+1} - u \|^2$
$\leq \alpha_n \| f(x_n) - u \|^2 + (1 - \alpha_n)\| w_n - u \|^2$
$\leq \alpha_n \| f(x_n) - u \|^2 + (1 - \alpha_n)\| y_n - u \|^2$
$\leq \alpha_n \| f(x_n) - u \|^2 + (1 - \alpha_n)(\| z_n - u \|^2 + \mu_n(\mu_n - 2\alpha)\| Az_n - Au \|^2)$
$\leq \alpha_n \| f(x_n) - u \|^2 + \| x_n - u \|^2 + (1 - \alpha_n)a(b - 2\alpha)\| Az_n - Au \|^2$
$\Rightarrow - (1 - \alpha_n)a(b - 2\alpha)\| Az_n - Au \|^2 \leq \alpha_n \| f(x_n) - u \|^2 + \| x_n - u \|^2 - \| x_{n+1} - u \|^2$
Therefore we get
$\lim_{n\to\infty} \| Az_n - Au \| = 0.$ (3.5)
Similarly,
$\lim_{n\to\infty} \| Ax_n - Au \| = 0.$ (3.6)
$\lim_{n\to\infty} \| At_n - Au \| = 0.$ (3.7)
Using 2.1, we also obtain
$\| w_n - u \|^2 = \| P_C(y_n - \lambda_n Ay_n) - P_C(u - \lambda_n Au) \|^2$
$\leq \langle (y_n - \lambda_n Ay_n) - (u - \lambda_n Au), w_n - u \rangle$
$= \frac{1}{2}\{\|(y_n - \lambda_n Ay_n) - (u - \lambda_n Au)\|^2 + \|w_n - u\|^2 - \|(y_n - \lambda_n Ay_n) - (u - \lambda_n Au) - w_n - u \|^2 \}$
$\leq \frac{1}{2}\{\| y_n - u \|^2 + \| w_n - u \|^2 - \| (y_n - w_n) - \lambda_n(Ay_n - Au) \|^2 \}$
$= \frac{1}{2}\{\| y_n - u \|^2 + \| w_n - u \|^2 - \| (y_n - w_n) \|^2 + 2 \lambda_n\langle y_n - w_n, Ay_n - Au \rangle - \lambda_n^2\| (Ay_n - Au) \|^2\}.$
So, we get,
$\| w_n - u \|^2 \leq \frac{1}{2}\{\| y_n - u \|^2 - \| (y_n - w_n) \|^2 + 2 \lambda_n\langle y_n - w_n, Ay_n - Au \rangle - \lambda_n^2\| (Ay_n - Au) \|^2 \}.$
On the other hand, we have,
$\| x_{n+1} - u \|^2 = \| \alpha_n f(x_n) + (1 - \alpha_n)Sw_n - u) \|^2$
$\leq \alpha_n \| f(x_n) - u \|^2 + (1 - \alpha_n)\| w_n - u \|^2$
$\leq \alpha_n \| f(x_n) - u \|^2 + \| x_n - u \|^2 - \| (y_n - w_n) \|^2 + 2 \lambda_n\langle y_n - w_n, Ay_n - Au \rangle - \lambda_n^2\| (Ay_n - Au) \|^2 \}.$
From condition (ii), (3.2) and (3.4), we get
$\lim_{n\to\infty} \| y_n - w_n \| = 0.$ (3.8)
In the same way, from (2.1), we have,
$\| y_n - u \|^2 = \| P_C(z_n - \mu_n Az_n) - P_C(u - \mu_n Au) \|^2$
$\leq \langle (z_n - \mu_n Az_n) - (u - \mu_n Au), y_n - u \rangle$
$= \frac{1}{2}\{\|(z_n - \mu_n Az_n) - (u - \mu_n Au)\|^2 + \| y_n - u\|^2 - \| z_n - \mu_n Az_n) - (u - \mu_n Au) - y_n - u \|^2 \}$
$= \frac{1}{2}\{\| z_n - u\|^2 + \| y_n - u\|^2 - \| (z_n - y_n)\|^2 + 2 \mu_n\langle z_n - y_n, Az_n - Au \rangle - \mu_n^2\| (Az_n - Au)\|^2\}.$
So, we get,
$\| y_n - u \|^2 \leq \frac{1}{2}\{\| z_n - u\|^2 - \| z_n - y_n \|^2 + 2 \mu_n\langle z_n - y_n, Az_n - Au \rangle - \mu_n^2\| Az_n - Au \|^2 \}.$
Note that,
$\| x_{n+1} - u \|^2 = \| \alpha_n f(x_n) + (1 - \alpha_n)Sw_n - u) \|^2$
$\leq \alpha_n \| f(x_n) - u \|^2 + (1 - \alpha_n)\| y_n - u \|^2$
$\leq \alpha_n \| f(x_n) - u \|^2 + \| x_n - u \|^2 - \| z_n - y_n \|^2 + 2 \mu_n\langle z_n - y_n, Az_n - Au \rangle - \mu_n^2\| Az_n - Au\|^2 \}.$
From condition (ii), (3.2) and (3.5), we get
$\lim_{n\to\infty} \| z_n - y_n \| = 0.$ (3.9)
Also,
$\| t_n - u \|^2 = \| P_C(x_n - \upsilon_n Ax_n) - P_C(u - \upsilon_n Au) \|^2$
$\leq \langle (x_n - \upsilon_n Ax_n) - (u - \upsilon_n Au), t_n - u \rangle$
$= \frac{1}{2}\{\|(x_n - \upsilon_n Ax_n) - (u - \upsilon_n Au)\|^2 + \| t_n - u \|^2 - \| x_n - \upsilon_n Ax_n) - (u - \upsilon_n Au) - t_n - u \|^2 \}$
$= \frac{1}{2}\{\| x_n - u\|^2 + \| t_n - u\|^2 - \| x_n - t_n \|^2 + 2 \upsilon_n\langle x_n - t_n, Ax_n - Au \rangle - \upsilon_n^2\| Ax_n - Au\|^2\}.$
So, we get,
$\| t_n - u \|^2 \leq \frac{1}{2}\{\| x_n - u\|^2 - \| x_n - t_n \|^2 + 2 \upsilon_n\langle x_n - t_n, Ax_n - Au \rangle - \upsilon_n^2\| Ax_n - Au\|^2\}.$
Note that,
$\| x_{n+1} - u \|^2 = \| \alpha_n f(x_n) + (1 - \alpha_n)Sw_n - u) \|^2$
$\leq \alpha_n \| f(x_n) - u \|^2 + (1 - \alpha_n)\| t_n - u \|^2$
$\leq \alpha_n \| f(x_n) - u \|^2 + \| x_n - u \|^2 - \| x_n - t_n \|^2 + 2 \upsilon_n\langle x_n - t_n, Ax_n - Au \rangle - \upsilon_n^2\| Ax_n - Au\|^2\}.$
From condition (ii), (3.2) and (3.6), we get
$\lim_{n\to\infty} \| x_n - t_n \| = 0.$ (3.10) Now,
$\| z_n - u \|^2 = \| P_C(t_n - \tau_n At_n) - P_C(u - \tau_n Au) \|^2$
$\leq \langle (t_n - \tau_n At_n) - (u - \tau_n Au), z_n - u \rangle$
$= \frac{1}{2}\{\|(t_n - \tau_n At_n) - (u - \tau_n Au)\|^2 + \| z_n - u \|^2 - \| t_n - \tau_n At_n) - (u - \tau_n Au) - (z_n - u) \|^2 \}$
$= \frac{1}{2}\{\| t_n - u\|^2 + \| z_n - u\|^2 - \| t_n - z_n \|^2 + 2 \tau_n\langle t_n - z_n, At_n - Au \rangle - \tau_n^2\| At_n - Au\|^2\}.$
So, we get,
$\| z_n - u \|^2 \leq \frac{1}{2}\{\| t_n - u\|^2 - \| t_n - z_n \|^2 + 2 \tau_n\langle t_n - z_n, At_n - Au \rangle - \tau_n^2\| At_n - Au\|^2\}.$
Note that,
$\| x_{n+1} - u \|^2 = \| \alpha_n f(x_n) + (1 - \alpha_n)Sw_n - u) \|^2$
$\leq \alpha_n \| f(x_n) - u \|^2 + (1 - \alpha_n)\| z_n - u \|^2$
$\leq \alpha_n \| f(x_n) - u \|^2 + \| x_n - u \|^2 - \| t_n - z_n \|^2 + 2 \tau_n\langle t_n - z_n, At_n - Au \rangle - \tau_n^2\| At_n - Au\|^2\}.$

From condition (ii), (3.2) and (3.7), we get
$\lim_{n\to\infty} \| t_n - z_n \| = 0.$ (3.11)
By (3.10) and (3.11), we have,
$\| t_n - z_n \| \le \| x_n - t_n \| + \| t_n - z_n \| \to 0$ as $n \to \infty$ (3.12)
By using (3.9), (3.12), we get,
$\| x_n - y_n \| \le \| x_n - z_n \| + \| z_n - y_n \| \to 0$ as $n \to \infty$ (3.13)
On the other hand, we have,
$\| x_n - Sw_n \| \le \| x_n - Sw_{n-1} \| + \| Sw_{n-1} - Sw_n \|$
$\le \alpha_{n-1} \| f(x_{n-1}) - Sw_{n-1} \| + \| w_{n-1} - w_n \|$
From condition (ii) and (3.3), we obtain,
$\lim_{n\to\infty} \| x_n - Sw_n \| = 0.$ (3.14)
From (3.13) and (3.14), we have,
$\| y_n - Sw_n \| \le \| y_n - x_n \| + \| x_n - Sw_n \| \to 0$ as $n \to \infty$ (3.15)
From (3.8) and (3.15), we have,
$\| w_n - Sw_n \| \le \| y_n - w_n \| + \| y_n - Sw_n \| \to 0$ as $n \to \infty$ (3.16)
Now, we show that
$\lim_{n\to\infty} \sup \langle f(q) - q, Sw_n - q \rangle \le 0$, where $q \in F(S) \cap VI(C, A)$ is the unique solution in $F(S) \cap VI(C, A)$ to the following variational inequality
$\langle (I - f)q, q - p \rangle \le 0$, $p \in F(S) \cap VI(C, A)$. To show it choose a subsequence $\{w_{n_i}\}$ of $\{w_n\}$ such that
$\lim_{n\to\infty} \sup \langle f(q) - q, Sw_n - q \rangle$
$\le \lim_{i\to\infty} \langle f(q) - q, Sw_{n_i} - q$.

As $\{w_n\}$ is bounded, we have that a subsequence $\{w_{n_i}\}$ of $\{w_n\}$ that converges weakly to some z. From (3.13), we obtain $Sw_{n_i} \rightharpoonup z$.

We shall show that

$$z \in F(S)\cap VI(C,A).$$

Firstly, we shall show that $z \in VI(C,A)$.
Let

$$Tv = \begin{cases} Av + N_C v, & if\ v \in C \\ \phi, & if\ v \notin C \end{cases}$$

Then, T is maximal monotone and $0 \in Tv$ if and only if $v \in VI(C,A)$. Let $(v,w) \in G(T)$. Since $w - Av \in N_C v$ and $w_n \in C$, so we have

$$< v - w_n, w - Av > \ge 0.$$

On the other hand, from

$$w_n = P_C(y_n - \lambda_n A y_n),$$

we have,

$$< v - w_n, w_n - (y_n - \lambda_n A y_n) > \ge 0.$$

and hence,

$$< v - w_n, (w_n - y_n)/\lambda_n + Ay_n > \ge 0.$$

Therefore,

$< v - w_{n_i}, w > \ge < v - w_{n_i} Av >$

$\ge < v - w_{n_i} Av > - < v - w_{n_i} (w_{n_i} - y_{n_i})/\lambda_{n_i} + Ay_{n_i}$

$= < v - w_{n_i}, Av - Ay_{n_i} - (w_{n_i} - y_{n_i})/\lambda_{n_i} >$

$= < v - w_{n_i}, Av - Aw_{n_i} > + < v - w_{n_i}, Aw_{n_i} - Ay_{n_i} > - < v - w_{n_i}, (w_{n_i} - y_{n_i})/\lambda_{n_i} >$

$\ge < v - w_{n_i}, Aw_{n_i} - Ay_{n_i} > - < v - w_{n_i}, (w_{n_i} - y_{n_i})/\lambda_{n_i} >$.

Hence we obtain,

$$< v - z, w > \ge 0, \quad \text{as } i \to \infty.$$

Since T is maximal monotone, we have $z \in T^{-1}0$ and hence $z \in VI(C,A)$.
Next we show that $z \in F(S)$. Assume that $z \notin F(S)$.
From the Opial condition, we have $\lim_{i\to\infty} \inf \| w_{n_i} - z \| < \lim_{i\to\infty} \inf \| w_{n_i} - Sz \|$

$= \lim_{n\to\infty} \inf \| w_{n_i} - Sw_{n_i} + Sw_{n_i} - Sz \|$

$= \lim_{i\to\infty} \inf \| Sw_{n_i} - Sz \|$

$\le \lim_{i\to\infty} \inf \| w_{n_i} - z \|$, which is a contradiction.

So $z \in F(S)$.
Then we have,
$\lim_{n\to\infty} \sup \langle f(q) - q, Sw_n - q \rangle$
$= \lim_{i\to\infty} \langle f(q) - q, Sw_{n_i} - q$
$= \langle (I - f)q, z - q \rangle \le 0.$
Now, we compute
$\| x_{n+1} - q \|^2 = \| \alpha_n f(x_n) + (1 - \alpha_n) Sw_n - q) \|^2$
$\le \alpha_n^2 \| f(x_n) - u \|^2 + 2 \alpha_n (1 - \alpha_n) \langle f(x_n) - q, Sw_n - q \rangle + (1 - \alpha_n)^2 \| Sw_n - q \|^2$
$\le \alpha_n^2 \| f(x_n) - u \|^2 + 2 \alpha_n (1 - \alpha_n) \langle f(x_n) - f(q), Sw_n - q \rangle + 2 \alpha_n (1 - \alpha_n) \langle f(q) - q, Sw_n - q \rangle + (1 + 2\alpha_n - \alpha_n)^2 \| x_n - q \|^2$
$= (1 - \gamma_n) \| x_n - q \|^2 + \delta_n$, where $\gamma_n = \alpha_n(2 - \alpha_n - 2k(1 - \alpha_n))$ and $\delta_n = \alpha_n^2 \| f(x_n) - u \|^2 + 2 \alpha_n (1 - \alpha_n) \langle f(q) - q, Sw_n - q \rangle$. It is easy to see that $\sum_{n=0}^{\infty} \gamma_n = \infty$ and $\lim_{n\to\infty} \sup \frac{\delta_n}{\gamma_n} \le 0$, so by lemma 2.1, we obtain $x_n \rightharpoonup q$ as $n \to \infty$. This completes the proof.

References:

1. Stampaccha G., Formes bilineaires coercitives sur les ensembles convexes, C. R. Acad. Sci. Paris 258(1964), 4413-4416.
2. aiocchi C.- Capelo A.:Variational and Quasivariational Inequalities. Applications to Free-Boundary Problems. J. Wiley, New York, 1984. B

3. Kinderlehrer D.-Stampacchia G.:An Introduction to Variational Inequalities and their Applications,Academic Press, New York, 1980.
4. Nagurney A. : Network economics: a variational inequality approach, Kluwer Academic Publishers, Boston, 1993.
5. Browder, F. E. and Petryshyn, W. V., "Construction of Fixed Points of non-linear Mapping in Hilbert Space," Journal of Mathematics Analysis and Applications, Vol.20, pp. 197-228, 1967.
7. Liu F., and Nashed M. Z., "Regularization of Non-linear Ill- posed variational inequalities and convergence rates,"Set-valued Analysis, Vol. 6, pp. 313-344, 1998.
8. Takahashi, W., and Toyoda, M., Weak Convergence Theorems for Non-expansive Mappings and Monotone Mappings, Journal of Optimization Theory and applications, Vol. 118, pp. 417-428, 2003.
9. Xu H. K., Viscosity approximation methods for nonexpansive mappings, Journal of Mathematical Analysis and Applications, vol. 298, no. 1,pp. 279-291, 2004.
10. Chen J, Zhang L and Fan T, Viscosity approximation methods for nonexpansive mappings and monotone mappings, Journal of Mathematical Analysis and Applications, vol. 334, no. 2,pp. 1450-1461, 2007.
6. Iiduka H., Takahashi W. and Toyada M.," Approximation of solutions of variational inequalities for monotone mappings," Panamerican Mathematical Journal, vol. 14, no. 2, pp. 49-61, 2004.
11.
hang M., Yongfu S. U., Qin X., Three step iterations for nonexpansive mappings and inverse strongly monotone mappings, Jri Syst Sci and Complexity(2009), 333-344.
12. Rockafellar, R. T., "On the Maximality of Sums of nonlinear Monotone Operators," Transactions of the American Mathematical Society, Vol. 149, pp. 75-88, 1970.
13. Rockafellar, R. T.," Monotone operators and proximal point algorithm," SIAM Journal on Control and Optimization, vol. 14, no. 5, pp. 877-898, 1976.
14. Xu H. K.,"An iterative approach to quadratic optimization," Journal of Optimization Theory and Applications, vol. 116, pp. 659-678, 200.

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