DUALITY FOR NONDIFFERENTIABLE MINIMAX MULTIOBJCTIVE FRACTIONAL OPTIMIZATION PROBLEMS WITH GENERALIZED INVEXITY

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Abstract: we introduce Kurush-Kuhn-Tucker sufficient optimality Mond-Weir type weak and strong duality theorems for a non-differentiable minimax fractional programming optimization problem in which numerator and denominator of multi objective each term consists of support function, under the assumptions of(V,σ,ρ) type-I invex functions.

Keywords: sufficient optimality conditions, duality theorems, generalized invexity, minimax fractional programminh problems.

Introduction: There are several chapters devoted to this topic by Mishra and Giorgio [6], Clarke [2] and Craven [3]. In the recent past, KUK et al. [5] have introduced the concept of (V,ρ)– Convexity, which is generalization of the V-convexity for vector valued functions and derived the generalized Karush – Kuhm – Tucker optimality conditions as well as weak and strong duality for non smooth multi objective fractional programs. Later, Kim et al. [4] extended their result in presence of support functions. Very recently, Kim et al [4] have introduced the assumption of (V,ρ) – convexity for (F.P) the following generalized non differentiable fractional programming problem (GFP);

Minimize

$$\max \left\{ \frac{f_i(x)+s(x/C_i)}{g_i(x)-s(x/D_i)} \,/\, i=1,....p \right\}$$

Subject to $h_j(x) \le 0,\ j=1,.....m$

Where

$f:=(f_1,.......f_p):R^n \to R^p,\ g:=(g_1,....g_p):R^n \to R^p$

and

$h:=(h_1,......h_m):R^n \to R^m$

are continuously differentiable and for each I = 1, ... P, C_i and D_i are compact convex sets of R^n

In this chapter, we introduce (V,P,σ) - convex function to derive the Kurush – Kuhn – Tucker Sufficient optimality and Mond – weir type weak and strong duality theorems for a generalized non differentiable minimax multi objective fractional optimization problem (GEP), in which numerator and denominator of each term consists of support function, and a constraint set defined by differentiable functions.

2 Preliminaries and definitions:-

In this chapter, we consider the following non differentiable multi objective fractional programming problem.

(GFP)Minimize

$$\max \left\{ \frac{f_i(x)+s(x/C_i)}{g_i(x)-s(x/D_i)} \,/\, i=1,....p \right\}$$

Subject to : $h_j(x) \le 0,\ j=1,.....m$

Where

$f:=(f_1,.......f_p):R^n \to R^p,\ g:=(g_1,....g_p):R^n \to R^p$

and

$h:=(h_1,......h_m):R^n \to R^m$

are continuously differentiable.

We assume that $g_i(x)-s(x/D_i)>0$, I = 1,....p. For each i = 1,....p, C_i and D_i are compact convex sets of R^n and define the support functions with respect to C_i and D_i as follows.

$$s(x/C_i) = \max \{ < x_1\ y_i > / \ y_i \in C_i \}$$

And

$$s(x/C_i) = \max \{ < x_1\ y_i > / \ y_i \in C_i \}$$

Further denote $I(x)=\{ j / hj(x)=0 \}$ for any $x \in R^n$.

Let $k_i(x)=s(x/C_i)$ and $\tilde{k}_i(x)=s(x/D_i)$, i = 1,......P.

Hence $k_i(x)$ and $\tilde{k}_i(x)$ are convex functions.

Choose $w_i \in dk_i(x)$ and $\tilde{w}_i \in d\tilde{k}_i(x)$ such that

$$dk_i(x) = \{ w_i \in C_i / < w_i, x >= s(x/C_i) \}$$

And $$d\tilde{k}_i(x) = \{ \tilde{w}_i \in C_i / < \tilde{w}_i, x >= s(x/D_i) \}$$

Where dk_i and $d\tilde{k}_i$ are the sub differential of k_i and $\tilde{k}$ respectively. Further

Let $S=\{ x \in R^n / h_j(x) \le 0, \quad j=1,.....m \}$

2.1 Definition:- A vector valued function $f : R^n \to R^p$ is said to be convex at $u \in R^n$ if for any $x \in R^n$ and for all i = 1, ...P one has

$$f_i(x) - f_i(u) \geq \nabla f_i(u)(x-u)^t$$

2.2:- Definition:- A vector valued function $f : R^n \to R^p$ is said to be (v,ρ)- Convex at $u \in R^n$ with respect to the function $\alpha_i : R^n \times R^n \to R_+ / \{0\}$ and $\rho_i \in R$, i =1,...p, such that for any $x \in R^n$ and for all i =1,...p it holds.

$$\alpha_i(x,u)\left[f_i(x) - f_i(u)\right] \geq \nabla f_i(u)(x-u)^t + \rho_i \left\|\theta_i(x,u)\right\|^2$$

The following Theorem from [ ] will be needed in the sequel.

2.3 Definition:-

Let $\left(\dfrac{f_i(x) + < w_i, x >}{g_i(x) - < \tilde{w}_i, x >}\right) = \theta_i\ (x)$, i=1,....P

$< w_i, x > = s(x/C_i)$, $< \tilde{w}_i, x > = s(x/D_i)$

The pair (ϕ_i, h_j) is called (V, ρ_i, σ_j) - convex at $u \in R^n$,

If there exist $\alpha_i : R^n \times R^n \to R_+ / \{0\}$

$$\bar{\alpha}_i(x,u) = \frac{g_i(x) + < \tilde{w}_i, x >}{g_i(x) - < \tilde{w}_i, x >} \quad \alpha_i\ (x,u) > 0$$

$$\bar{\theta}_i\ (x,u) = \left(\frac{1}{g_i(u) + < \tilde{w}_i, u >}\right)^{\frac{1}{2}} \theta_i(x,u)$$

$\beta_i : R^n \times R^n \to R_+ / \{0\}$,

$\rho_i \in R, i = 1,....P, : \sigma_j \in R, j = 1,...m$

Suchthat

$$\theta_i(x) - \theta_i(u) \geq \bar{\alpha}_i(x,u) \nabla \theta_i(u)(x-u)^t + \rho_i \left\|\bar{\theta}_i(x,u)\right\|^2$$

and

$$-h_j(u) \geq \beta_j(x,u) \nabla h_j(u)(x-u)^t + \sigma_j \left\|\bar{\theta}_i(x,u)\right\|^2$$

3 Optimality Conditions:-

The following Kurn –Tucker necessary optimality conditions for (GFP) from [ ] will be needed in the sequel.

3.1 Theorem : (Kuhn – Tucker necessary optimality condition) if x_0 is a solution of the problem (GFP) and under the assumption that one has $0 \notin C_0 \left\{\nabla h_j(x_0) / j = 1,....m\right\}$ then there exist $\lambda_i \geq 0$.

$$i \in I(x_0) := \left\{ i / \max_j \frac{f_j(x_0) + s(x_0/C_j)}{g_j(x_0) - s(x_0/D_j)} = \frac{f_i(x_0) + s(x_0/C_i)}{g_i(x_0) - s(x_0/D_i)} \right\},$$

$$\sum_{i=1}^{p} \lambda_i = 1, \mu_j \geq 0, \text{j=1,....m}$$

and $w_i \in C_i, \tilde{w} \in D_i, i = 1,...p,$

Such that

$$\sum_{i=1}^{p} \lambda_i \nabla \left(\frac{f_i(x_0) + < w_i, x_o >}{g_i(x_o) - (\tilde{w}_i, x_o)} \right) + \sum_{j=1}^{m} u_j \sigma h_i(x_o) = 0$$

$$< w_i, x_o > = S(x_o / c_i), < \tilde{w}_i, x_o > = s(x_o / D_i)$$

$$\sum_{j=1}^{m} \mu_j h_j(x_o) = 0$$

Theorem: 3.2: (KUHN-TUCKER type sufficient condition), support that there exist a feasible solution x_o for (GFP) and scalars

$$\lambda_i > 0, i = 1,....p, \sum_{i=1}^{p} \lambda_i = 1, \mu_j \geq 0, j = 1,....m$$

and $w_i \in c_i, \tilde{w} \in D_i\ i = 1,....p$ such that

i)

$$\sum_{i=1}^{p} \lambda_i \nabla \left(\frac{f_i(x_o) + < w_i, x_o >}{g_i(x_o) - < \tilde{w}_i, x_o >} \right) + \sum_{j=1}^{m} \mu_j \nabla h_j(x_o) =$$

$$< w_i, x_o > = s(x_o / c_i), < \tilde{w}_i, x_o > = S(x_o / D_i)$$

$$\sum_{j=1}^{m} \mu_i h_j(x_o) = 0$$

ii) (θ_i, h_i) is (V, ρ_i, σ_j) - convex at x_0.

Then x_0 is an efficient solution for (GFP)

Proof: Hypothesis (i) implies that

$$0 = \sum_{i-1}^{p} \lambda_i \nabla \left(\frac{f_i(x_o) + < w_i, x_o >}{g_i(x_o) - < \tilde{w}_1, x_o >} \right) + \sum_{j-1}^{m} \mu_j \nabla h_j(x_o) \qquad (3.1)$$

Since (ϕ_i, h_j) is (v, ρ_i, σ_j) convex at x_o, we have for all $x \in S$

$$\frac{f_i(x) + cw_{i,} x}{g_i(x) - < \tilde{w}_i x >} - \frac{f_i(x_o) + < w_i, x_o >}{g_i(x_o) - < \tilde{w}_i, x_o >}$$

$$\geq \bar{\alpha}_i(x, x_o) \nabla \left(\frac{f_i(x_o) + < w_i, x_o >}{g_i(x_o) - < \tilde{w}_i, x_o >} \right) (x - x_o)^t + \rho_i \left| \bar{\theta}_i(x_, x_o) \right|^2$$

and o=

$$-h_j(x_0) \geq \beta_j(x,x_o) \nabla h_j(x_o)(x-x_o)^t + \sigma_j \left\|\theta_j(x,x_o)\right\|^2$$

By using $\bar{\alpha}_i(x,x_o) > 0, i=1,\ldots p$ and $\beta_j(x,x_o) > 0, j=1,\ldots.m$ we get

$$\frac{1}{\bar{\alpha}_i(x,x_o)}\left(\frac{f_i(x)+<w_i,x>}{g(x)-<\tilde{w}_i,x>}\right) - \frac{1}{\bar{\alpha}_i(x,x_o)}\left(\frac{f_i(x_o)+<w_i,x_o>}{g(x_o)<\tilde{w}_i,x_o>}\right)$$

$$\geq V\left(\frac{f_i(x_0)+<w_i,x_0>}{g_i(x_0)-<\tilde{w}_i,x_0>}\right)(x-x_0)^t + \frac{\rho_i\left\|\bar{\theta}_i(x,x_0)\right\|^2}{\bar{\alpha}_i(x,x_0)}$$

(3.2)

And $0 \geq \nabla h_j(x_0)(x-x_0)^t + \dfrac{\sigma_j\left\|\theta_j(x,x_0)\right\|^2}{\beta_j(x,x_0)}$

(3.3)

Adding (3.2) and (3.3), we get

$$\sum_{i=1}^{p}\frac{\lambda_i}{\bar{\alpha}_i(x,x_0)}\left(\frac{f_i(x)+<w_i,x>}{g(x)-<\tilde{w}_i,x>}\right) - \sum_{i=1}^{p}\frac{\lambda_i}{\bar{\alpha}_i(x,x_0)}\left(\frac{f_i(x_0)+<w_i,x_0>}{g(x_0)-<\tilde{w}_i,x_0>}\right)$$

$$\geq\left[\sum_{i=1}^{p}\lambda_i\nabla\left(\frac{f_i(x_0)+<w_i,x_0>}{g(x_0)-<\tilde{w}_i,x_0>}\right) + \sum_{j=1}^{m}\mu_j\nabla h_j(x_o)\right](x-x_0)^t$$

$$+\sum_{i=1}^{p}\lambda_i\frac{\rho_i\left\|\bar{\theta}_i(x,x_0)\right\|^2}{\bar{\alpha}_i(x,x_0)} + \sum_{j=1}^{m}\mu_j\frac{\sigma_j\left\|\theta_j(x,x_0)\right\|^2}{\beta_j(x,x_0)}$$

Using (3.1), we have

$$\sum_{i=1}^{p}\frac{\lambda_i}{\bar{\alpha}_i(x,x_0)}\left(\frac{f_i(x)+<w_i,x>}{g(x)-<\tilde{w}_i,x>}\right) - \sum_{i=1}^{p}\frac{\lambda_i}{\bar{\alpha}_i(x,x_0)}\left(\frac{f_i(x_0)+<w_i,x_0>}{g(x_0)-<\tilde{w}_i,x_0>}\right)$$

$$\geq\sum_{i=1}^{p}\lambda_i\frac{\rho_i\left\|\bar{\theta}_i(x,x_0)\right\|^2}{\bar{\alpha}_i(x,x_0)} + \sum_{j=1}^{m}\mu_j\frac{\sigma_j\left\|\theta_j(x,x_0)\right\|^2}{\beta_j(x,x_0)}$$

As $\sum_{i=1}^{p}\lambda_i\dfrac{\rho_i\left\|\bar{\theta}_i(x,x_0)\right\|^2}{\bar{\alpha}_i(x,x_0)} \geq 0$ and

$\sum_{j=1}^{m}\mu_j\dfrac{\sigma_j\left\|\theta_j(x,x_0)\right\|^2}{\beta_j(x,x_0)} \geq 0$ we get

$$\sum_{i=1}^{p}\frac{\lambda_i}{\bar{\alpha}_i(x,x_0)}\left(\frac{f_i(x)+<w_i,x>}{g(x)-<\tilde{w},x>}\right) - \sum_{i=1}^{p}\frac{\lambda_i}{\bar{\alpha}_i(x,x_0)}\left(\frac{f_i(x_0)+<w_i,x_0>}{g(x_0)-<\tilde{w}_i,x_0>}\right) \geq 0$$

Thus we have

$$\sum_{i=1}^{p}\frac{\lambda_i}{\bar{\alpha}_i(x,x_0)}\frac{f_i(x)+<w_i,x>}{g(x)-<\tilde{w}_i,x>} \geq \sum_{i=1}^{p}\frac{\lambda_i}{\bar{\alpha}_i(x,x_0)}\frac{f_i(x_0)+<w_i,x_0>}{g(x_0)-<\tilde{w}_i,x_0>}$$

Suppose that x_0 is not an efficient solution for (GEP), then there exist a feasible solution x for (GFP) and an index r such that $\theta_i(x) \leq \theta_i(x_0)$ for any $i \neq r$ and Φ, where

$$\theta_i(x) = \frac{f_i(x)+<w_i,x>}{g_i(x)-<\tilde{w}_i,x>} \qquad \text{for any i.}$$

Since $\lambda_i > 0$ and $\bar{\alpha}_i(x_1x_0) > 0$, i =1,.....P, we have

$$\sum_{i=1}^{p}\frac{\lambda_i}{\bar{\alpha}_i(x,x_0)}\Phi_i(x) < \sum_{i=1}^{p}\frac{\lambda_i}{\bar{\alpha}_i(x,x_0)}\Phi_i(x_0)$$

It follows that one has

$$\sum_{i=1}^{p}\frac{\lambda_i}{\bar{\alpha}_i(x,x_0)}\left(\frac{f_i(x)+<w_i,x>}{g(x)-<\tilde{w}_i,x>}\right) < \sum_{i=1}^{p}\frac{\lambda_i}{\bar{\alpha}_i(x,x_0)}\left(\frac{f_i(x_0)+<w_i,x_0>}{g(x_0)-<\tilde{w}_i,x_0>}\right)$$

Which contradicts the inequalities (34) and hence x_0 is an efficient solution for (GFP)

4 Mond – Weir type duality:-

We now consider the following Mond-Weir type dual for (GFP)

(DGFP)Maximize

$$\max\left\{\frac{f_i(u)+S(u/C_i)}{g_i(u)-S(u/D_i)} \;/\; i=1,\ldots.P\right\}$$

Subject to

$$\sum_{i=1}^{p}\lambda_i\nabla\left(\frac{f_i(u)+<w_i,u>}{g_i(u)-<\tilde{w}_i,u>}\right) + \sum_{j=1}^{m}\mu_j\nabla h_j(u) = 0$$

(4.1)

$$w_i \in C_i, <w_i,u> = S(u/C_i),\ \tilde{w}_i \in D_i,$$
$$<\tilde{w}_i,u> = S(u/D_i),\ i=1,\ldots m\tilde{w}_i$$

$$\lambda_i > 0, i=1,\ldots.P\ ,\sum_{i=1}^{p}\lambda_i = 1,\ \mu_j \geq 0, j=1,\ldots.m, \sum_{j=1}^{m}\mu_j h_j(u) = 0$$

Theorem: (4.1): (Weak Duality) Let x be a feasible solution for (GFP) and let $(u,\lambda,\mu,w,\tilde{w})$ be feasible for (DGFP) such that (Φ_i,h_j) is (Φ_i,h_i) is (V,ρ_i,σ_j) - convex at u. then the following cannot hold

$$\left(\frac{f_i(x)+S(x/C_i)}{g_i(x)-S(x/D_i)}\right) < \left(\frac{f_i(u)+S(u/C_i)}{g_i(u)-S(u/D_i)}\right)$$

(4.2)

Proof: Suppose that (4.2) holds that is

$$\left(\frac{f_i(x)+<w_i,x>}{g_i(x)-<\tilde{w}_i,x>}\right)<\left(\frac{f_i(u)+<w_i,u>}{g_i(u)-<\tilde{w}_i,u>}\right)$$

Using

$$\lambda_i>0, i=1,....P \ ,\sum_{i=1}^{p}\lambda_i=1 \ , \ \mu_j\ge 0, j=1,....m$$

we get

$$\sum_{i=1}^{p}\lambda_i\left(\frac{f_i(x)+<w_i,x>}{g_i(x)-<\tilde{w}_i,x>}\right)<\sum_{i=1}^{p}\lambda_i\left(\frac{f_i(u)+<w_i,u>}{g_i(u)-<\tilde{w}_i,u>}\right)$$

That is

$$\sum_{i=1}^{p}\lambda_i\left(\frac{f_i(x)+<w_i,x>}{g_i(x)-<\tilde{w}_i,x>}\right)-\sum_{i=1}^{p}\lambda_i\left(\frac{f_i(u)+<w_i,u>}{g_i(u)-<\tilde{w}_i,u>}\right)<0$$

(4.3)

and $-\sum_{j=1}^{m}\mu_j h_j(u)=0$ (4.4)

By (v,p,σ) convexity, we have

$$\sum_{i=1}^{p}\lambda_i\nabla\left(\frac{f_i(u)+<w_i,u>}{g_i(u)-<\tilde{w}_i,u>}\right)(x-u)^t+\sum_{i=1}^{p}\lambda_i\frac{\rho_i\left\|\theta_i(x,x_0)\right\|^2}{\alpha_i(x,u)}$$

$$\le\sum_{i=1}^{p}\frac{\lambda_i}{\alpha_i(x,u)}\left(\frac{f_i(x)+<w_i,x>}{g_i(x)-<\tilde{w}_i,x>}\right)-\sum_{i=1}^{p}\frac{\lambda_i}{\alpha_i(x,u)}\left(\frac{f_i(u)+<w_i,u>}{g_i(u)-<\tilde{w}_i,u>}\right)<0$$

And

$$\sum_{j=1}^{m}\mu_j\nabla h_j(u)(x-u)^t+\sum_{j=1}^{m}\mu_j\frac{\sigma_j\left\|\theta_j(x,u)\right\|^2}{\beta_j(x,u)}\le-\sum_{j=1}^{m}\frac{\mu_j}{\beta_j(x,u)}\nabla h_j(u)=0$$

That is,

$$\sum_{i=1}^{p}\lambda_i\nabla\left(\frac{f_i(u)+<w_i,u>}{g_i(u)-<\tilde{w}_i,u>}\right)(x-u)^t+\sum_{i=1}^{p}\lambda_i\frac{\rho_i\left\|\theta_i(x,u)\right\|^2}{\alpha_i(x,u)}<0$$

And

$$\sum_{j=1}^{m}\mu_j\nabla h_j(u)\,(x-u)^t+\sum_{j=1}^{m}\mu_j\frac{\sigma_j\left\|\theta_j(x_1x_0)\right\|^2}{\beta_j(x,u)}\le 0$$

By adding the above inequalities, we get

$$\left[\sum_{i=1}^{p}\lambda_i\nabla\left(\frac{f_i(u)+<w_i,u>}{g_i(u)-<\tilde{w}_i,u>}\right)+\sum_{j=1}^{m}\mu_j\nabla h_j(u)\right](x-u)^t<0$$

Which contradicty the dual constraints (4.1). Hence (4.2) cannot hold.

Theorem 4.2:- (Strong Duality) : Let $\bar{x}$ be a weakly efficient solution for (GFP). Then there exist $\bar{\lambda}\in R^P$, $\bar{\mu}\in R^m$ and $\bar{w}\in C$ such that $(\bar{x},\bar{\lambda},\bar{\mu},\bar{w},\bar{\tilde{w}})$ is feasible for (DGFP). Moreover, If the weak duality holds, then $(\bar{x},\bar{\lambda},\bar{\mu},\bar{w},\bar{\tilde{w}})$ is a weakly efficient solution for (DGFP)

Proof:-

Take $\bar{x}$ a wakly efficient solution for (GFP) and suppose that

$0\notin co\left\{\nabla h_j(\bar{x})/j=1,...m\right\}$.Then there exist $\bar{\lambda}\in R^P$, $\bar{\mu}\in R^m$ and

$\bar{w}_i\in C_i, \bar{\tilde{w}}\in D_i, i=1,....P$

Such that

$$\sum_{i=1}^{p}\lambda_i\nabla\left(\frac{f_i(\bar{x})+(w_i,\bar{x})}{g_i(\bar{x})-(\tilde{w}_i,\bar{x})}\right)+\sum_{j=1}^{m}\mu_j\nabla h_j(\bar{x})=0$$

$$(\bar{w}_i,\bar{x}) \ = \ S(\bar{x}/C_i)\ ,\left(\bar{\tilde{w}},\bar{x}\right)=S(\bar{x}/D_i)$$

$$\sum_{j=1}^{m}\bar{\mu}_j h_j(\bar{x})=0$$

$$\bar{\lambda}_i>0, i=1,....P, \sum_{i=1}^{p}\bar{\lambda}_i=1$$

Thus, $(\bar{x},\bar{\lambda},\bar{\mu},\bar{w},\bar{\tilde{w}})$ is a feasible solution for (DGFP). On the other hand, by weak duality (Theorem 4.1)

$$\max\left\{\frac{f_i(\bar{x})+S(\bar{x}/C_i)}{g_i(\bar{x})-S(\bar{x}/D_i)}/i=1,....P\right\}\ge\max\left\{\frac{f_i(u)+S(u/C_i)}{g_i(u)-S(u/D_i)}/i=1,....P\right\}$$

for any feasible solution $\left(x,\lambda,\mu,w,\bar{\tilde{w}}\right)$ of (DGFP). Hence $(\bar{x},\bar{\lambda},\bar{\mu},\bar{w},\bar{\tilde{w}})$ is a weakly efficient solution for (DGFP).

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