GRAPH THEORY AND TRAVELLING PROBLEMS

PARVINDER SINGH

Abstract: Mathematics plays a vital role in various fields of knowledge. Graph theory is an important field in mathematics, used in various structural models. The field of graph theory started its journey from Konigsberg seven bridges problem in 1736.Graphs are mathematical structures used to model pairwise relations between objects from a certain collection. In case of Konigsberg seven bridges problem when situation was presented in terms of graphs, the case was simplified. So Euler concluded that these bridges cannot be traversed exactly once. Some important applications of graph theory are the Chinese Postman Problem and Travelling Salesman Problem.

Introduction:

Preliminaries:

1. A **walk** in a graph G is a finite sequence of edges $(e_1, e_2, \ldots\ldots, e_n)$ in which any two consecutive edges are adjacent and the **length of walk** is the number of edges.
2. A walk in which all edges are distinct is called a **trail.**
3. If the vertices of a trail are distinct (except the initial vertex and final vertex Possibly coincide), then the trail is called a **path.**
4. We say a walk/trail/path is **closed** if the initial vertex is also the final vertex.
5. A closed path with at least one edge is called a **cycle**.
6. A graph is **connected** if and only if there is a path between each pair of vertices.
7. If G is a simple graph with n vertices, m edges, and k components the n-k$\leq m \leq \binom{n-k+1}{2}$ and any simple graph with n vertices and more than $\binom{n-1}{2}$ edges must be connected.

Eulerian graph: A connected graph G is said to be **Eulerian** if there exists a closed trail containing every edge of G. A **Non-Eulerian** graph $G'$is said to be Semi-Eulerian if there exists a trail containing every edge of G'.

Königsberg bridges problem: The name **Eulerian** arises from the fact that Euler was the first person to solve the famous **Königsberg bridges problem.**

The problem asks whether you can cross each of the following seven bridges exactly once and return to your starting point.

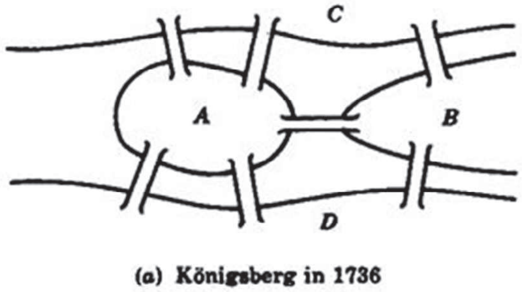


Fig.1.1

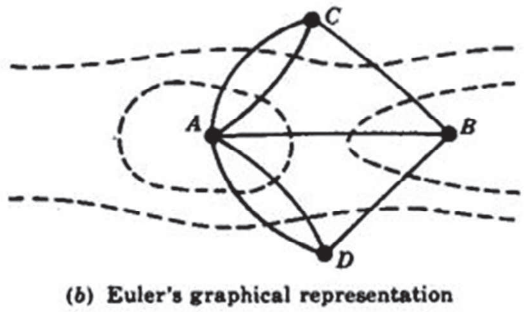


Fig.1.2

We will first present some definitions and then present a theorem that Euler used to show that it is in fact impossible to walk through the town and traverse all the bridges only once.

Eulerian trail: An Eulerian trail is a trail that visits every edge of the graph once and only once. It can end on a vertex different from the one on which it began. A graph of this kind is said to be **Traversable.**

Eulerian Circuit: An Eulerian circuit is an Eulerian trail that is a circuit. That is, it begins and ends on the same vertex.

Euler proved that the multigraph in fig 1.2 is not traversable and hence the walk in Konigsberg bridges problem is not possible. Suppose that a multigraph is traversable and a traversable trail does not begin or end at a vertex P. We claim that P is an even vertex. For whenever a traversable trail enters P by an edge there must always be an edge not previously used by which the trail can leave P. Thus the edge in the trail incident with P must appear in pairs and so P is an even vertex .Therefore if a vertex Q is odd the traversable trail must begin or end at Q Consequently a multigraph with more than two odd vertices cannot be traversable. Observe that the multigraph corresponding to the Konigsberg bridge problem has four odd vertices. Thus cannot walk through Konisberg so that each bridge is crossed exactly once.

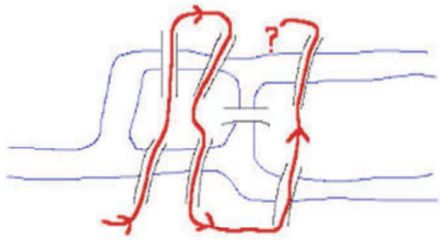


Fig.1.3

Theorem: A finite connected graph is Eulerian if and only if each vertex is of even degree.

Proof: let G is Eulerian graph and P is an Eulerian trail then there is a contribution of degree two towards each vertex therefore each vertex is of even degree.

Lemma: If each vertex of a graph G is of at least degree two then G contains a cycle.

On the other hand, suppose that each vertex of *G* has even degree and we need to construct an Eulerian trail.

We apply induction on the number of edges of *G*. Clearly when *G* contains only one edge the statement is true.

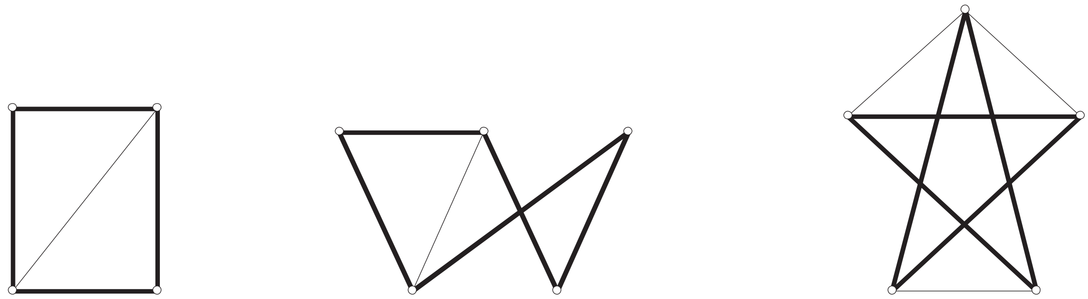
Assume that the statement is true for all graphs having fewer edges than *G*. Since *G* is connected, each vertex has degree at least two. By the lemma, there is a cycle *C* in *G*. Removing this cycle will produce a new graph *H*, which is possibly disconnected or even a null graph. By the induction hypothesis, each component of *H* has an Eulerian trail. Note that each component of H has an Eulerian trail. Note that each component of H has at least one vertex in common with C (otherwise, G should be disconnected).

Now, we obtain the required Eulerian trail (of *G*) by following the edges of *C*, if a non-isolated vertex of *H* is reached, tracing the Eulerian trail (of that Eulerian component of *H*), and then continuing along the edges of *C*, and so on. This process stops when we return to the starting vertex.

Corollaries:

(1) A connected graph is Eulerian if and only if its set of edges is union of disjoint cycles.

(2) A connected graph is semi-Eulerian if and only if it has exactly two vertices of odd degree.



Hamiltonian Graphs:

Hamiltonian Circuit: A Hamiltonian circuit in a graph is a closed path that visits every vertex in the graph exactly once. (Such a closed loop must be a cycle.)

A Hamiltonian circuit ends up at the vertex from where it started.

Hamiltonian graphs are named after the nineteenth-century Irish mathematician Sir William Rowan Hamilton(1805-1865). This type of problem is often referred to as the traveling salesman or postman problem.

Hamiltonian Graph: If a graph has a Hamiltonian circuit, then the graph is called a Hamiltonian graph. i.e. a closed path that visits every vertex in G exactly once.

Note that an Eulerian circuit traverses every edge exactly once but may repeat vertices, while a Hamiltonian circuit visits each vertex exactly once but may repeat edges.

Although it is clear that only connected graphs can be Hamiltonian, there is no simple criterion to tell us whether or not a graph is Hamiltonian as there is for Eulerian graphs.

Examples of Hamiltonian graphs:

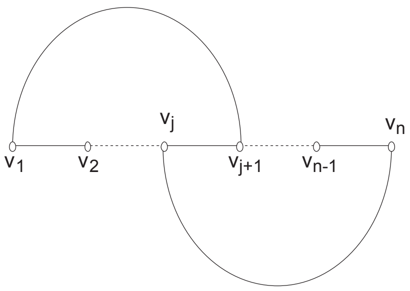
Theorem: (Ore,1960)

If G is a simple graph with n≥3 such that deg (u) +deg (v) ≥ n for each pair of non adjacent vertices u,v, then G is Hamiltonian.

Proof. Let G is not Hamiltonian. We may assume G is a maximal Non-Hamiltonian graph (in the sense that adding extra edge will give a Hamiltonian graph). Consider the longest path $v_1 \rightarrow v_2 \rightarrow \cdots \rightarrow v_n$ in the graph (note that this path must pass through every vertex). Clearly, v_1, v_n is not adjacent.

Case 1: When n=3, we have, so v_1, v_3 is adjacent. This is a contradiction.

Case 2: When $n > 3$, there are n-3 ordered pairs $(v_{i,} v_{i+1})$ where $i = 2,3, \dots n-2$. Let A be the set of the pairs such that $v_{1,} v_{i+1}$ are adjacent and B be the set of those that v_i, v_n are adjacent. Since deg(v_1) +deg(v_n)≥ n , we have $|A| + |B| \geq n-2$. But there is only n-3 pairs, one of the pair, says $(v_{j,} v_{j+1})$ with $2 \leq j \leq n-2$, must belong to both A and B. Therefore, v_1 is adjacent to v_{j+1} and v_j is adjacent to v_n, as shown in the following figure.



Now, $v_1 \to v_2 \dots . \to v_j \to v_n \to v_{n-1} \to \cdots . . \to v_{j+1} \to v_1$ is a Hamiltonian cycle.

In both cases, we lead to a contradiction by constructing a Hamiltonian cycle. We conclude that G is Hamiltonian.

Corollary: If G is a simple graph with $n \geq 3$ vertices such that $\deg(v) \geq \frac{n}{2}$ for each vertex v, then G is Hamiltonian.

Shortest path: For any type of graph whether directed, undirected or mixed we can define shortest path. But here we define shortest path only for undirected graph. In an undirected graph a path is a sequence of vertices P= $(v_1, v_2, \dots\dots, v_n) \in V \times V \times \dots . . \times V$. Such that v_i is adjacent to v_{i+1} for 1≤ $i \leq n$. P is called a path of length n. Let $e_{i,j}$ be an edge incedent on both the vertices v_i to v_j. Then a real valued weight function f: E→ R, on an undirected simple graph G, the shortest path between v_1 to v_n over all possible n minimizes the sum

$$\sum_{i=1}^{n-1} f(v_{i,i+1}).$$

When each edge of the graph G is of unit weight then f: E→ {1}.

There are various problems regarding shortest paths. Such as

1. **The single-source shortest path problem**: to find shortest path from a source vertex to all other vertices of the graph.
2. **The single-destination shortest path problem**: to find shortest path from all vertices in the directed graph to a single destination.
3. **The all-pairs shortest path problem:** to find shortest path between every pair of vertices in the graph.

There are various Algorithms to solve such problems as:

1. **Dijkstra's algorithms** to solve the single source shortest path problems.
2. **Bellman-Ford algorithm** to solve the single-source problem if the edge weights are negative.
3. **A^* Search algorithm** to solve single pair shortest path problem using heuristics.
4. **Floyd-Warshal algorithms** to solve all pair's shortest paths.
5. **Johnson's algorithm** to solve all pairs shortest path and may be faster than Floyd-Warshal on sparse graphs.

The Chinese postman problem:

This problem is discussed by Chinese mathematician Mei-Ku Kwan, asking the most efficient walk (with least repeat) for a postman to traverse each road in his route.

Problem.In a weighted simple graph find a closed walk to traverse each edge at least once with minimum total weight.

Note that if the graph is Eulerian, then the required walk is simply the Eulerian trail.

An obvious inequality about the minimum total weight d is

$$\sum w(e) \leq d \leq 2 \times \sum w(e)$$

because double each edge will give an Eulerian graph. If the graph is Semi-Eulerian, we can make it become Eulerian by double each edge in the shortest path joining two odd vertices. The Eulerian trail will be our required (most efficient) walk.

In general, the most efficient walk can be obtained by double some path(s), each path being the shortest path joining two odd vertices (remind that the number of odd vertices in any graph must be even, by hand-shaking lemma). The resulting graph is Eulerian and the required walk is then the Eulerian trail.

If the graph has 2k (where k≥ 1) odd vertices then there are (2k -1)! possible ways to pair up the odd vertices. We need to try all these possible ways.

Travelling Salesman Problem:

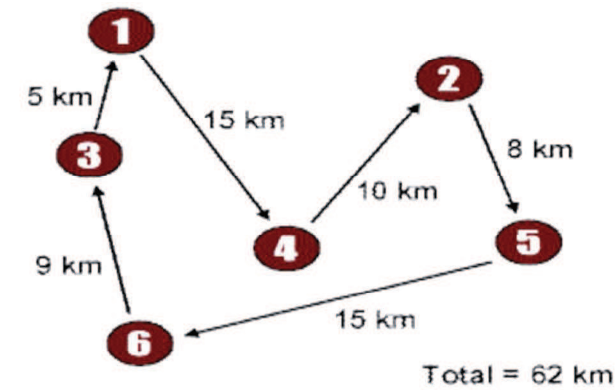
Definition: Given a set of cities and the distance between each possible pair, the **Travelling Salesman Problem** is to find the best possible way of 'visiting all the cities exactly once and returning to the starting point'.

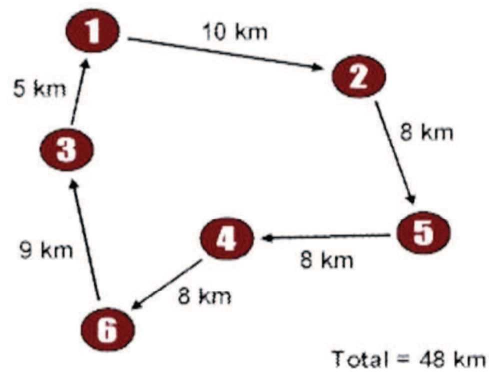
1. The problem was first defined in the 1800s by the Irish mathematician **W.R. Hamilton** and the British mathematician **Thomas Kirkman**.
2. It was, however, first formulated as a mathematical problem only in 1930 by **Karl Menger**.
3. The name **Travelling Salesman Problem** was introduced by American **Hassler Whiteney**.

In spite of the computational difficulty of the problem, a large number of heuristics and exact methods are known which can solve instances with thousands of cities

Problem: Given a complete undirected graph $G=(V,E)$ that has non-negative integer cost $c(u,v)$ associated with each edge (u, v) in E, the problem is to find a Hamiltonian cycle (tour) of G with minimum cost.

Example:





A salesman starts from the city 1 and has to visit six cities (1 through 6) and must come back to the starting city i.e., 1. The first route $1 \rightarrow 4 \rightarrow 2 \rightarrow 5 \rightarrow 6 \rightarrow 3 \rightarrow 1$ with the total length of 62 km is a relevant selection but is not the best solution. The second route $1 \rightarrow 2 \rightarrow 5 \rightarrow 4 \rightarrow 6 \rightarrow 3 \rightarrow 1$ represents the better solution as the total distance, 48 km, is less than for the first route.

Suppose $C(A)$ denote the total cost of the edges in the subset A subset of E i.e.,

$C(A) = \sum_{u,v \text{ in } A} C(u,v)$.

Moreover, the cost function satisfies the triangle inequality. That is, for all vertices u, v, w in V, we have

$$C(u, w) \leq C(u, v) + C(v, w).$$

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Head, Department of Mathematics,
S.G.G.S. Khalsa College, Mahilpur, (Hoshiarpur). Punjab. (India).Pin:146105
Email: parvinder070@gmail.com